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Bipolar Soft Generalized Topological Structures and Their Application in Decision Making

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Abstract. The basic of bipolar soft set theory stands for a mathematical instrument that brings together the soft set theory and bipolarity. Its definition is based on two soft sets, a set that provides positive information and other that gives negative. This paper mainly aims at defining a new bipolar soft generalized topological space; setting out of the point that the collection of bipolar soft sets forms the basis for the definition of the new concept is defined. Added to that, an investigation has been made of the four concepts of bipolar soft generalized, namely $\widetilde{\widetilde{\mathfrak{g}}}$-interior, $\widetilde{\widetilde{\mathfrak{g}}}$-closure, $\widetilde{\widetilde{\mathfrak{g}}}$-exterior and $\widetilde{\widetilde{\mathfrak{g}}}$-boundary. Furthermore, the main properties of bipolar soft generalized topological space ($\mathcal{BSGTS}$) are established. This paper also attends to the discussion of the relations between these new definitions and the application of the given bipolar soft generalized topological spaces in a decision-making problem where an algorithm for this application has been suggested. Finally, to clarify and substantiate what the current work subsumes, some examples have been provided.

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Key Words and Phrases: Soft set, $\mathcal{BSS}$, $\mathcal{BSGTS}$, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-open (bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed) set, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-interior, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure, decision making

1. Introduction

For formal modeling, reasoning, and computing, the majority of the traditional tools are characterized by being crisp, deterministic, and precise. Yet in the domains of economics, engineering, environment, social science, medical science, etc., many complicated problematic issues exist. As such, to solve or model them, the typical methods based on the case, in particular, may lack suitability. On this basis, a set of theories has been

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proposed to tackle these issues. In 1999, Molodtsov [27] adopted the soft set theory that was designed to solve sophisticated problems.

Molodtsov's theory has been implemented in several branches of mathematics. Examples are decision making problems, medical science, social sciences, operation research etc. The theory has been further improved by other researchers like Maji et al. [24] in terms of defining the operation family of special information systems. Based on this, the operations of the soft set were redefined by Çağman and Enginoglu [14] who further constructed, by using the soft set theory, a uni-int decision making method. Finally, the soft sets were compared to both fuzzy and rough sets by Aktas and Çağman [1].

Later on, some properties and applications of the soft set theory have been investigated by many researchers (see [9], [13], [32], [34], [35], [37], [41], [42]). For the time being, two definitions of the soft topological spaces exist. The concept of soft topological spaces on a universe set was first defined by Shabir and Naz [37]. Likewise, for the demonstration of the notion of soft topological spaces, the soft sets were also, Çağman [15]. This was followed by a excess of researches that tackled the soft topological spaces (see [2], [3], [4], [5], [6], [7], [8], [10], [11], [12], [18], [19], [22], [23], [25], [26], [33], [38], [40]).

The notion of generalized neighborhood system and generalized topological spaces was defined by Császár ([16],[17]) who further studied a set of its basic properties, namely continuous functions, associated by interior and closure operations on generalized topological spaces, and compared his findings to those of the usual topology. On their part, Thomas and John [39] constructed the notion of soft generalized topological spaces ($\mathcal{SGTS}s$) via soft generalized open sets over an initial universe with a fixed set of parameters, and studied some of their properties such as compactness and separation axioms. The generalized topology differs from that based on its axioms. According to Császár, a family of subsets of Ω stands for a generalized topology on Ω when the empty set and arbitrary union of its members are included. It is worth noting that soft sets theory, not sets, forms the basis for the soft generalized topological spaces. In 2013, the bipolar soft set structure which may form the source of more general and clear results than the soft set structure was investigated by Shabir and Naz [38]. Varied definitions of the bipolar soft set and the basic operations such as intersection, union and complementation were put forward by Shabir and Naz [38] and Karaaslan and Karatas [22]. Based on Dubois and Prada [18], decision making is constructed on two sides, namely negative and positive. A number of definitions, operations, and applications on bipolar soft sets have been investigated in ([4], [22], [23], [40]). Added to that, Öztürk [31] studied the concepts of closure and interior operations, basis and subspace in bipolar soft topological spaces.

There has been an expansion of the definition of bipolar soft topological spaces defined in [36] by Fadel and Dzul-Kifli [20] who have attended to the key concepts and properties and put forward some illustrative examples. Additionally, there has been further works on the topological structures on bipolar soft sets, (see [21],[22]). For instance, Musa and Asaad ([28], [29], [30]) introduced a new idea concerning bipolar soft sets by extending the hypersoft sets named bipolar hypersoft sets. They further investigated bipolar hypersoft topological spaces and some of their operations and properties.

The coming parts are organized as follows: In Section 2, some related preliminaries are briefly recalled. In Section 3, the new concept of bipolar soft topological spaces called bipolar soft generalized topological spaces have been firstly defined. This has been followed by the presentation of the basic properties of bipolar soft generalized topological spaces on an initial bipolar soft set. In addition, the definitions of the notions of bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-open sets, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-close sets, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-interior, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-exterior and bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-boundary have been provided along the study of their properties and the investigation of the relation between such concepts. In Section 4, this has been followed by presenting the application of bipolar soft generalized topological spaces in a decision making problem. Finally, a binary information table has been utilized in our attempt to analogously represent the bipolar soft generalized topological spaces. Section 5 concludes this paper.

2. Preliminaries

In this section, we introduce some basic concepts about bipolar soft sets. In this paper, let Ω be an initial universe, $\Upsilon(\Omega)$ be denoted the collection of all subsets of Ω and ϖ be a set of parameters. Let $\varsigma, \sigma \subseteq \varpi$ and $\mathcal{BSS}(\Omega)$ be the set of all bipolar soft sets over Ω with parameters ϖ. Now, we mention the main definitions of bipolar soft sets and its related topics that we need through the paper.

Definition 1. *[24] Let $\varsigma = \{\varrho_1, \varrho_2, ..., \varrho_n\}$ be a set of parameters. The **Not** set of ς denoted by $\neg\varsigma = \{\neg\varrho_1, \neg\varrho_2, ..., \neg\varrho_n\}$ for all i, $\neg\varrho_i =$ Not ϱ_i.*

Definition 2. *[38] A triple $(\Theta, \Lambda, \varsigma)$ is called a bipolar soft set on Ω, where Θ and Λ are mappings defined by $\Theta : \varsigma \longrightarrow \Upsilon(\Omega)$ and $\Lambda : \neg\varsigma \longrightarrow \Upsilon(\Omega)$ such that $\Theta(\varrho) \cap \Lambda(\neg\varrho) = \phi$ for all $\varrho \in \varsigma$ and $\neg\varrho \in \neg\sigma$.*
In other words, a bipolar soft set $(\Theta, \Lambda, \varsigma)$ can be written as

$$(\Theta, \Lambda, \varsigma) = \{(\varrho, \Theta(\varrho), \Lambda(\neg\varrho)) : \varrho \in \varsigma, \Theta(\varrho) \cap \Lambda(\neg\varrho) = \phi\}.$$

Definition 3. *[38] For any two bipolar soft sets $(\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_2, \Lambda_2, \sigma)$, we say that $(\Theta_1, \Lambda_1, \varsigma)$ is a bipolar soft subset of $(\Theta_2, \Lambda_2, \sigma)$ if:*

(i) $\varsigma \subseteq \sigma$ and,

(ii) $\Theta_1(\varrho) \subseteq \Theta_2(\varrho)$ and $\Lambda_2(\neg\varrho) \subseteq \Lambda_1(\neg\varrho)$ for all $\varrho \in \varsigma$ and $\neg\varrho \in \neg\varsigma$.

This relationship is denoted by $(\Theta_1, \Lambda_1, \varsigma) \widetilde{\widetilde{\subseteq}} (\Theta_2, \Lambda_2, \varsigma)$. Similarly, we say that $(\Theta_1, \Lambda_1, \varsigma)$ is a bipolar soft superset of $(\Theta_2, \Lambda_2, \sigma)$, denoted by $(\Theta_1, \Lambda_1, \varsigma) \widetilde{\widetilde{\supseteq}} (\Theta_2, \Lambda_2, \sigma)$, if $(\Theta_2, \Lambda_2, \sigma)$ is a bipolar soft subset of $(\Theta_1, \Lambda_1, \varsigma)$.

Definition 4. *[38] Two bipolar soft sets $(\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_2, \Lambda_2, \sigma)$ are said to be equal, denoted by $(\Theta_1, \Lambda_1, \varsigma) = (\Theta_2, \Lambda_2, \sigma)$, if $(\Theta_1, \Lambda_1, \varsigma)$ is a bipolar soft subset of $(\Theta_2, \Lambda_2, \sigma)$ and $(\Theta_2, \Lambda_2, \sigma)$ is a bipolar soft subset of $(\Theta_1, \Lambda_1, \varsigma)$.*

Definition 5. *[38] The complement of a bipolar soft set* $(\Theta, \Lambda, \varsigma)$ *is denoted by* $(\Theta, \Lambda, \varsigma)^c$ *and defined by* $(\Theta, \Lambda, \varsigma)^c$=$(\Theta^c, \Lambda^c, \varsigma)$ *where* Θ^c *and* Λ^c *are mappings given by* $\Theta^c(\varrho) = \Lambda(\neg\varrho)$ *and* $\Lambda^c(\neg\varrho) = \Theta(\varrho)$ *for all* $\varrho \in \varsigma$ *and* $\neg\varrho \in \neg\varsigma$.

Definition 6. *[38] A relative null bipolar soft set* $(\Phi, \tilde{\tilde{\Omega}}, \varsigma)$ *is a bipolar soft set* $(\Theta, \Lambda, \varsigma)$ *if* $\Theta(\varrho) = \phi$ *for all* $\varrho \in \varsigma$ *and* $\Lambda(\neg\varrho) = \Omega$ *for all* $\neg\varrho \in \neg\varsigma$.

Definition 7. *[38] A relative absolute bipolar soft set* $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$ *is a bipolar soft set* $(\Theta, \Lambda, \varsigma)$ *if* $\Theta(\varrho) = \Omega$ *for all* $\varrho \in \varsigma$ *and* $\Lambda(\neg\varrho) = \phi$ *for all* $\neg\varrho \in \neg\varsigma$.

Definition 8. *[38] The bipolar soft intersection between two bipolar soft sets* $(\Theta_1, \Lambda_1, \varsigma)$ *and* $(\Theta_2, \Lambda_2, \sigma)$ *is the bipolar soft set* (χ, Ψ, κ) *where* $\kappa = \varsigma \cap \sigma$ *is a nonempty set and for all* $\varrho \in \kappa$,

$$\chi(\varrho) = \Theta_1(\varrho) \cap \Theta_2(e) \text{ and } \Psi(\neg\varrho) = \Lambda_1(\neg\varrho) \cup \Lambda_2(\neg\varrho).$$

It is denoted by $(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cap}} (\Theta_2, \Lambda_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 9. *[38] The bipolar soft union between two bipolar soft sets* $(\Theta_1, \Lambda_1, \varsigma)$ *and* $(\Theta_2, \Lambda_2, \sigma)$ *is the bipolar soft set* (χ, Ψ, κ) *where* $\kappa = \varsigma \cup \sigma$ *is a nonempty set and for all* $\varrho \in \kappa$,

$$\chi(\varrho) = \Theta_1(\varrho) \cup \Theta_2(e) \text{ and } \Psi(\neg\varrho) = \Lambda_1(\neg\varrho) \cap \Lambda_2(\neg\varrho).$$

It is denoted by $(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} (\Theta_2, \Lambda_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 10. *[38] The bipolar soft union between two bipolar soft sets* $(\Theta_1, \Lambda_1, \varsigma)$ *and* $(\Theta_2, \Lambda_2, \sigma)$ *is the bipolar soft set* (χ, Ψ, κ) *where* $\kappa = \varsigma \cup \sigma$ *is a nonempty set and for all* $\varrho \in \kappa$,

$$\chi(\varrho) = \begin{cases} \Theta_1(\varrho), & \varrho \in \varsigma - \sigma, \\ \Theta_2(\varrho), & \varrho \in \sigma - \varsigma, \\ \Theta_1(\varrho) \cup \Theta_2(\varrho), & \varrho \in \varsigma \cap \sigma. \end{cases}$$

$$\Psi(\neg\varrho) = \begin{cases} \Theta_1(\neg\varrho), & \neg\varrho \in \neg\varsigma - \neg\sigma, \\ \Theta_2(\neg\varrho), & \neg\varrho \in \neg\sigma - \neg\varsigma, \\ \Theta_1(\neg\varrho) \cap \Theta_2(\neg\varrho), & \neg\varrho \in \neg\varsigma \cap \neg\sigma. \end{cases}$$

It is denoted by $(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} (\Theta_2, \Lambda_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 11. *[38] The bipolar soft intersection between two bipolar soft sets* $(\Theta_1, \Lambda_1, \varsigma)$ *and* $(\Theta_2, \Lambda_2, \sigma)$ *is the bipolar soft set* (χ, Ψ, κ) *where* $\kappa = \varsigma \cup \sigma$ *is a nonempty set and for all* $\varrho \in \kappa$,

$$\chi(\varrho) = \begin{cases} \Theta_1(\varrho), & \varrho \in \varsigma - \sigma, \\ \Theta_2(\varrho), & \varrho \in \sigma - \varsigma, \\ \Theta_1(\varrho) \cap \Theta_2(\varrho), & \varrho \in \varsigma \cap \sigma. \end{cases}$$

$$\Psi(\neg\varrho) = \begin{cases} \Theta_1(\neg\varrho), & \neg\varrho \in \neg\varsigma - \neg\sigma, \\ \Theta_2(\neg\varrho), & \neg\varrho \in \neg\sigma - \neg\varsigma, \\ \Theta_1(\neg\varrho) \cup \Theta_2(\neg\varrho), & \neg\varrho \in \neg\varsigma \cap \neg\sigma. \end{cases}$$

It is denoted by $(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 12. *[38] Let* $\widetilde{\mathfrak{g}}$ *be the family of soft sets of* Ω*, then* $\widetilde{\mathfrak{g}}$ *is said to be soft generalized topology* ($\mathcal{SGT}$) *on* Ω *if*

(i) (Φ, ς) *belong to* $\widetilde{\mathfrak{g}}$.

(ii) The union of any members of soft sets in $\widetilde{\mathfrak{g}}$ *belongs to* $\widetilde{\mathfrak{g}}$.

Then $(\Omega, \widetilde{\mathfrak{g}}, \varsigma)$ *is called a soft generalized topological space* ($\mathcal{SGTS}$) *over* Ω*. Every member of* $\widetilde{\mathfrak{g}}$ *is called a soft* $\widetilde{\mathfrak{g}}$*-open set. The complement of a soft* $\widetilde{\mathfrak{g}}$*-open set is soft* $\widetilde{\mathfrak{g}}$*-closed.*

Proposition 1. *[38] If* $(\Theta_1, \Lambda_1, \varsigma)$,$(\Theta_2, \Lambda_2, \sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$*, then:*

(i) $((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \sigma))^c = (\Theta_1, \Lambda_1, \varsigma)^c \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \sigma)^c$.

(ii) $((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \sigma))^c = (\Theta_1, \Lambda_1, \varsigma)^c \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \sigma)^c$.

(iii) $((\Theta_1, \Lambda_1, \varsigma)^c)^c = (\Theta_1, \Lambda_1, \varsigma)$.

(iv) $(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, (\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \sigma)^c \,\widetilde{\widetilde{\subseteq}}\, (\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \sigma)^c \,\widetilde{\widetilde{\subseteq}}\, (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$.

Definition 13. *[20] The bipolar soft difference between two bipolar soft sets* $(\Theta_1, \Lambda_1, \varsigma)$ *and* $(\Theta_2, \Lambda_2, \sigma)$ *is the bipolar soft set* $(\Theta, \Lambda, \kappa)$ *where* $\kappa = \varsigma \cup \sigma$ *is defined as*

$$(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\setminus}}\, (\Theta_2, \Lambda_2, \sigma) = (\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \sigma)^c.$$

3. Main Results

In this section, we introduce the bipolar soft generalized topological spaces and we investigate some concepts and properties such as bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-interior, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-exterior and bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-boundary.

Definition 14. *Let* $\widetilde{\widetilde{\mathfrak{g}}}$ *be the collection of bipolar soft subsets over* Ω*, then* $\widetilde{\widetilde{\mathfrak{g}}}$ *is said to be a bipolar soft generalized topology* ($\mathcal{BSGT}$) *on* Ω *if it satisfies the following conditions:*

(i) $(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$.

(ii) If $(\Theta_j, \Lambda_j, \varsigma) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$ *for all* $j \in \mathcal{J}$*, then* $\widetilde{\widetilde{\bigcup}}_{j\in\mathcal{J}}(\Theta_j, \Lambda_j, \varsigma) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$.

Then $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *is called a bipolar soft generalized topological space* ($\mathcal{BSGTS}$) *over* Ω.

Definition 15. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$*, if* $\tilde{\tilde{\mathfrak{g}}}$ *is the collection of all possible bipolar soft sets which can be defined over* Ω*, then* $\tilde{\tilde{\mathfrak{g}}}$ *is called the discrete* $\mathcal{BSGT}$ *on* Ω*.*

Definition 16. *A* $\mathcal{BSGT}$ $\tilde{\tilde{\mathfrak{g}}}$ *is said to be strong if* $(\tilde{\tilde{\Omega}}, \Phi, \varsigma) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$*.*

Definition 17. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$*, then the members of* $\tilde{\tilde{\mathfrak{g}}}$ *are said to be bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open sets in* Ω*.*

Clearly $(\Phi, \tilde{\tilde{\Omega}}, \varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open.

Definition 18. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \varsigma, \neg\varsigma)$ *and let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}s$*. Then:*

(i) If $\tilde{\tilde{\mathfrak{g}}}_1 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{g}}}_2$ *then* $\tilde{\tilde{\mathfrak{g}}}_2$ *is bipolar soft finer than* $\tilde{\tilde{\mathfrak{g}}}_1$*.*

(ii) If either $\tilde{\tilde{\mathfrak{g}}}_1 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{g}}}_2$ *or* $\tilde{\tilde{\mathfrak{g}}}_2 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{g}}}_1$*, then* $\tilde{\tilde{\mathfrak{g}}}_1$ *is bipolar soft comparable with* $\tilde{\tilde{\mathfrak{g}}}_2$*.*

Example 1. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$ *be the universal set which are eight categories of people that are living in Duhok city. It can be defined by:*

$$\Omega = \{Syrian\,Refugees, Turkish\,Refugees, Iranian\,Refugees, Host\,Community, IDPs, Residents, Returnees, Foreigners\}.$$

Let $\varsigma = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_8\}$ *be the set of parameters, where* ϱ_i*,* $i = 1, 2, ...7, 8$*, stands for parameters "Hard Working", "Negligent", "Flexibility", "Rigidity", "Self-Confidence", "Shyness", "Skillful" and "Unskillful" respectively. It is regarded as positive description and non positive description which belong to each category. The eight categories of people wish to find a job, to employ in government institute or work in a company in Duhok city. Now, we can divide the set* ς *into two parts* $\varsigma_1 = \{\varrho_1, \varrho_3, \varrho_5, \varrho_7\}$ *and* $\varsigma_2 = \{\varrho_2, \varrho_4, \varrho_6, \varrho_8\}$ *and the bijective function* $f \colon \varsigma_1 \to \varsigma_2$ *can be defined as*

$$f(\varrho_i) = \neg\varrho_i = \varrho_{i+1} \text{ for } i = 1, 3, 5, 7.$$

Here the notion $\neg\varrho_i$ *means Not* ϱ_i *for all* $i = 1, 3, 5, 7$*. Now, we can describe the following* $\mathcal{BSGTS}$ $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \varsigma), (\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma), (\Theta_3, \Lambda_3, \varsigma)\}$ *offers to select some workers and employ them in tourism companies in Duhok city, where*

$$\begin{aligned}
(\Theta_1, \Lambda_1, \varsigma) = &\{(\varrho_1, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_6\}), (\varrho_3, \{\omega_2, \omega_5, \omega_7\}, \{\omega_1, \omega_3, \omega_8\}),\\
&(\varrho_5, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_5, \omega_8\}), (\varrho_7, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_2, \omega_3\})\},\\
(\Theta_2, \Lambda_2, \varsigma) = &\{(\varrho_1, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_5, \omega_6, \omega_7\}), (\varrho_3, \{\omega_2, \omega_5\}, \{\omega_1, \omega_3, \omega_4, \omega_8\}),\\
&(\varrho 5, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_5, \omega_7, \omega_8\}), (\varrho_7, \{\omega_5\}, \{\omega_2, \omega_3, \omega_4\})\} and\\
(\Theta_3, \Lambda_3, \varsigma) = &\{(\varrho_1, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_6\}), (\varrho_3, \{\omega_2, \omega_5, \omega_7\}, \{\omega_1, \omega_3, \omega_8\}),\\
&(\varrho_5, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_5, \omega_8\}), (\varrho_7, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_2, \omega_3\})\}.
\end{aligned}$$

Each $(\Theta, \Lambda, \varsigma)$ *in* $\tilde{\tilde{\mathfrak{g}}}$ *can be depicted as a table. Each category includes positive description* α_i *and negative description* β_j *and it can be represented by* (α_i, β_j)*. If the description exists in a category, then it is considered as 1, otherwise, it is 0.*
Tabular representation of bipolar soft sets $(\Theta_1, \Lambda_1, \varsigma)$*,* $(\Theta_2, \Lambda_2, \varsigma)$ *and* $(\Theta_3, \Lambda_3, \varsigma)$ *are given in Tables 1,2 and 3.*

Table 1: Tabular form of the bipolar soft set $(\Theta_1, \Lambda_1, \varsigma)$

$(\Theta_1, \Lambda_1, \varsigma)$	$(\Theta_1, \Lambda_1)(\varrho_1)$	$(\Theta_1, \Lambda_1)(\varrho_3)$	$(\Theta_1, \Lambda_1)(\varrho_5)$	$(\Theta_1, \Lambda_1)(\varrho_7)$
ω_1	(1,0)	(0,1)	(0,1)	(0,0)
ω_2	(0,1)	(1,0)	(0,1)	(0,1)
ω_3	(1,0)	(0,1)	(1,0)	(0,1)
ω_4	(1,0)	(0,0)	(1,0)	(0,0)
ω_5	(0,0)	(1,0)	(0,1)	(1,0)
ω_6	(0,1)	(0,0)	(0,0)	(1,0)
ω_7	(0,0)	(1,0)	(0,0)	(0,1)
ω_8	(0,0)	(0,1)	(0,1)	(1,0)

Table 2: Tabular form of the bipolar soft set $(\Theta_2, \Lambda_2, \varsigma)$

$(\Theta_2, \Lambda_2, \varsigma)$	$(\Theta_2, \Lambda_2)(\varrho_1)$	$(\Theta_2, \Lambda_2)(\varrho_3)$	$(\Theta_2, \Lambda_2)(\varrho_5)$	$(\Theta_2, \Lambda_2)(\varrho_7)$
ω_1	(1,0)	(0,1)	(1,0)	(0,0)
ω_2	(1,0)	(1,0)	(0,1)	(0,1)
ω_3	(0,1)	(0,1)	(1,0)	(0,1)
ω_4	(1,0)	(0,1)	(1,0)	(0,1)
ω_5	(0,1)	(0,0)	(0,1)	(1,0)
ω_6	(0,1)	(0,0)	(0,0)	(0,0)
ω_7	(0,1)	(0,0)	(0,1)	(0,0)
ω_8	(0,0)	(0,1)	(0,1)	(0,0)

Table 3: Tabular form of the bipolar soft set $(\Theta_3, \Lambda_3, \varsigma)$

$(\Theta_3, \Lambda_3, \varsigma)$	$(\Theta_3, \Lambda_3)(\varrho_1)$	$(\Theta_3, \Lambda_3)(\varrho_3)$	$(\Theta_3, \Lambda_3)(\varrho_5)$	$(\Theta_3, \Lambda_3)(\varrho_7)$
ω_1	(1,0)	(0,1)	(1,0)	(0,0)
ω_2	(1,0)	(1,0)	(0,1)	(0,1)
ω_3	(1,0)	(0,1)	(1,0)	(0,1)
ω_4	(1,0)	(0,0)	(1,0)	(0,0)
ω_5	(0,0)	(1,0)	(0,1)	(1,0)
ω_6	(0,1)	(0,0)	(0,0)	(1,0)
ω_7	(0,0)	(1,0)	(0,0)	(1,0)
ω_8	(0,0)	(0,1)	(0,1)	(1,0)

Theorem 1. *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$*, then* $\widetilde{\mathfrak{g}} = \{(\Theta, \varsigma) : (\Theta, \Lambda, \varsigma) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}\}$ *is* $\mathcal{SGT}$*.*

Proof. Suppose that $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ is a $\mathcal{BSGTS}$. Then $(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}$ implies that $(\Phi, \varsigma) \widetilde{\in} \widetilde{\mathfrak{g}}$. Let $\{(\Theta_i, \varsigma) : i \in \mathcal{I}\}$ belongs to $\widetilde{\mathfrak{g}}$. Since $(\Theta_i, \Lambda_i, \varsigma) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}$ for all $i \in \mathcal{I}$, so that $\widetilde{\widetilde{\bigcup}}_{i\in\mathcal{I}}(\Theta_i, \Lambda_i, \varsigma)$ $\widetilde{\widetilde{\in}}$ $\widetilde{\widetilde{\mathfrak{g}}}$. Thus, $\widetilde{\bigcup}_{i\in\mathcal{I}}(\Theta_i, \varsigma) \mathrel{\widetilde{\in}} \widetilde{\mathfrak{g}}$. Hence $\widetilde{\mathfrak{g}}$ defines a $\mathcal{SGT}$.

The following example shows that the converse of Theorem 1 is not true.

Example 2. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and* $\varsigma = \{\varrho_1, \varrho_2\}$.
Suppose that $\widetilde{\mathfrak{g}} = \{(\Phi, \varsigma), (\Theta_1, \varsigma), (\Theta_2, \varsigma), (\Theta_3, \varsigma), (\Theta_4, \varsigma)\}$ *and*

$\neg\widetilde{\mathfrak{g}} = \{(\Phi,\varsigma),(\Lambda_1,\neg\varsigma),(\Lambda_2,\neg\varsigma),(\Lambda_3,\neg\varsigma),(\Lambda_4,\neg\varsigma)\}$ *are two soft generalized topologies defined on* Ω, *where*

$$\begin{aligned}(\Theta_1,\varsigma) &= \{(\varrho_1,\{\omega_2\}),(\varrho_2,\{\omega_1\})\},\\ (\Theta_2,\varsigma) &= \{(\varrho_1,\{\omega_1\}),(\varrho_2,\{\omega_3\})\},\\ (\Theta_3,\varsigma) &= \{(\varrho_1,\{\omega_2\}),(\varrho_2,\{\omega_1,\omega_3\})\},\\ (\Theta_4,\varsigma) &= \{(\varrho_1,\{\omega_1,h_2\}),(\varrho_2,\{\omega_1,\omega_3\})\},\end{aligned}$$

and

$$\begin{aligned}(\Lambda_1,\neg\varsigma) &= \{(\neg\varrho_1,\{\omega_1,\omega_3,\omega_4\}),(\neg\varrho_2,\{\omega_2,\omega_4\})\},\\ (\Lambda_2,\neg\varsigma) &= \{(\neg\varrho_1,\{\omega_3\}),(\neg\varrho_2,\{\omega_4\})\},\\ (\Lambda_3,\neg\varsigma) &= \{(\neg\varrho_1,\{\omega_3,\omega_4\}),(\neg\varrho_2,\{\omega_2\})\},\\ (\Lambda_4,\neg\varsigma) &= \{(\neg\varrho_1,\{\omega_3,\omega_4\}),(\neg\varrho_2,\{\omega_2,\omega_4\})\}.\end{aligned}$$

Then $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi,\widetilde{\widetilde{\Omega}},\varsigma),(\Theta_1,\Lambda_1,\varsigma),(\Theta_2,\Lambda_2,\varsigma),(\Theta_3,\Lambda_3,\varsigma),(\Theta_4,\Lambda_4,\varsigma\}$, *where* $(\Theta_1,\Lambda_1,\varsigma)$, $(\Theta_2,\Lambda_2,\varsigma)$, $(\Theta_3,\Lambda_3,\varsigma)$ *and* $(\Theta_4,\Lambda_4,\varsigma)$ *are bipolar soft sets defined as follows*

$$\begin{aligned}(\Theta_1,\Lambda_1,\varsigma) &= \{(\varrho_1,\{\omega_2\},\{\omega_1,\omega_3,\omega_4\}),(\varrho_2,\{\omega_1\},\{\omega_2,\omega_4\})\},\\ (\Theta_2,\Lambda_2,\varsigma) &= \{(\varrho_1,\{\omega_1\},\{\omega_3\}),(\varrho_2,\{\omega_3\},\{\omega_4\})\},\\ (\Theta_3,\Lambda_3,\varsigma) &= \{(\varrho_1,\{\omega_2\}.\{\omega_3,\omega_4\}),(\varrho_2,\{\omega_1,\omega_3\},\{\omega_2\})\},\\ (\Theta_4,\Lambda_4,\varsigma) &= \{(\varrho_1,\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}),(\varrho_2,\{\omega_1,\omega_3\},\{\omega_2,\omega_4\})\}.\end{aligned}$$

Thus, $(\Theta_1,\Lambda_1,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta_2,\Lambda_2,\varsigma) = \{(\varrho_1,\{\omega_1,\omega_2\},\{\omega_3\}),(\varrho_2,\{\omega_1,\omega_3\},\{\omega_4\})\}\ \widetilde{\widetilde{\notin}}\ \widetilde{\widetilde{\mathfrak{g}}}$. *Therefore,* $\widetilde{\widetilde{\mathfrak{g}}}$ *is not* $\mathcal{BSGT}$.

The following theorem shows when that converse of Theorem 1 is true.

Theorem 2. *Let* $(\Omega,\widetilde{\mathfrak{g}},\varsigma)$ *be a* $\mathcal{SGT}$. *Then the collection* $\widetilde{\widetilde{\mathfrak{g}}}$ *consisting of bipolar soft sets* $(\Theta,\Lambda,\varsigma)$ *such that* $(\Theta,\varsigma)\ \widetilde{\in}\ \widetilde{\mathfrak{g}}$ *and* $\Lambda(\neg\varrho) = \Omega\setminus\Theta(\varrho)$ *for all* $\neg\varrho\in\neg\varsigma$, *defines a* $\mathcal{BSGT}$ *on* Ω.

Proof.

(i) Since $(\Phi,\varsigma)\ \widetilde{\in}\ \widetilde{\mathfrak{g}}$, then $\Omega(\neg\varrho) = \Omega\setminus\Phi(\varrho) = \Omega\setminus\phi = \Omega$ and hence $(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$.

(ii) Let $\{(\Theta_i,\Lambda_i,\varsigma) : i\in\mathcal{I}\}\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$. Then $\{(\Theta_i,\varsigma) : i\in\mathcal{I}\}\ \widetilde{\in}\ \widetilde{\mathfrak{g}}$ and $\Lambda_i(\neg\varrho) = \Omega\setminus\Theta_i(\varrho)$. Now, since $\widetilde{\mathfrak{g}}$ is a $\mathcal{SGT}$, then $\widetilde{\bigcup}_{i\in\mathcal{I}}(\Theta_i,\varsigma)\ \widetilde{\in}\ \widetilde{\mathfrak{g}}$. Let $(\Theta,\varsigma) = \widetilde{\bigcup}_{i\in\mathcal{I}}(\Theta_i,\varsigma)\ \widetilde{\in}\ \widetilde{\mathfrak{g}}$, then $\varsigma(\neg\varrho) = \Omega\setminus(\bigcup_{i\in\mathcal{I}}\Theta_i(\varrho)) = \bigcap_{i\in\mathcal{I}}\Lambda_i(\neg\varrho)$. Thus, $\widetilde{\widetilde{\bigcup}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$. Therefore, the proof is completed.

Theorem 3. *Let* $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $\{\widetilde{\widetilde{\mathfrak{g}}}_i\}_{i\in\mathcal{I}}$ *be an indexed family of* $\mathcal{BSGT}$*s. Then* $\widetilde{\widetilde{\bigcap}}_{i\in\mathcal{I}}\widetilde{\widetilde{\mathfrak{g}}}$ *is a* $\mathcal{BSGT}$, *where each* $\widetilde{\widetilde{\mathfrak{g}}}_i$ *is bipolar soft finer than* $\widetilde{\widetilde{\bigcap}}_{i\in\mathcal{I}}\widetilde{\widetilde{\mathfrak{g}}}_i$ *for each* i.

Proof. Since each $\{\tilde{\tilde{\mathfrak{g}}}_i\}$, $i \in \mathcal{I}$ is a $\mathcal{BSGT}$ over Ω, the bipolar soft set $(\Phi, \tilde{\tilde{\Omega}}, \varsigma) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{g}}}_i$, $i \in \mathcal{I}$ and hence $(\Phi, \tilde{\tilde{\Omega}}, \varsigma) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}\tilde{\tilde{\mathfrak{g}}}_i$. Let $\{(\Theta_j, \Lambda_j, \varsigma) : j \in \mathcal{J}\}$ be a family of bipolar soft sets in $\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}\, \tilde{\tilde{\mathfrak{g}}}_i$. Then each $(\Theta_j, \Lambda_j, \varsigma)$ belongs to each $\tilde{\tilde{\mathfrak{g}}}_i$. But $\tilde{\tilde{\mathfrak{g}}}_i$ being $\mathcal{BSGT}$ is closed under arbitrary bipolar soft unions. So $\tilde{\tilde{\bigcup}}_{j\in\mathcal{J}}(\Theta_j, \Lambda_j, \varsigma) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}\, \tilde{\tilde{\mathfrak{g}}}_i$. Hence $\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}\tilde{\tilde{\mathfrak{g}}}_i$ is a $\mathcal{BSGT}$ define on Ω. Clearly each $\tilde{\tilde{\mathfrak{g}}}_i$, $i \in \mathcal{I}$, is bipolar soft finer than $\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}\tilde{\tilde{\mathfrak{g}}}_i$.

Remark 1. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \varsigma, \neg\varsigma)$ *and* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \varsigma, \neg\varsigma)$ *be* $\mathcal{BSGTS}$*s. Then* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cap}}\, \tilde{\tilde{\mathfrak{g}}}_2, \varsigma, \neg\varsigma)$ *may not be a* $\mathcal{BSGTS}$ *as shown by the following example.*

Example 3. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ *and* $\varsigma = \{\varrho_1, \varrho_2\}$*. Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \varsigma, \neg\varsigma)$, $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \varsigma, \neg\varsigma)$ $\tilde{\tilde{\in}}$ $\mathcal{BSGTS}$*s where*

$$\tilde{\tilde{\mathfrak{g}}}_1 = \{(\Phi, \tilde{\tilde{\Omega}}, \varsigma), (\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma)\},$$

and

$$\tilde{\tilde{\mathfrak{g}}}_2 = \{(\Phi, \tilde{\tilde{\Omega}}, \varsigma), (\Theta_3, \Lambda_3, \varsigma), (\Theta_4, \Lambda_4, \varsigma)\},$$

where

$$\begin{aligned}
(\Theta_1, \Lambda_1, \varsigma) &= \{(\varrho_1, \{\omega_1, \omega_2\}, \{\omega_5\}), (\varrho_2, \{\omega_2\}, \{\omega_1, \omega_4, \omega_5\})\},\\
(\Theta_2, \Lambda_2, \varsigma) &= \{(\varrho_1, \{\omega_1, \omega_2\}, \{\omega_5\}), (\varrho_2, \{\omega_2\}, \{\omega_1, \omega_4\})\},\\
(\Theta_3, \Lambda_3, \varsigma) &= \{(\varrho_1, \{\omega_3, \omega_4\}, \{\omega_5\}), (\varrho_2, \{\omega_3\}, \{\omega_4\})\} and\\
(\Theta_4, \Lambda_4, \varsigma) &= \{(\varrho_1, \{\omega_3, \omega_4\}, \{\omega_5\}), (\varrho_2, \{\omega_3\}, \{\omega_4, \omega_5\})\}.
\end{aligned}$$

Now, $\tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cup}}\, \tilde{\tilde{\mathfrak{g}}}_2 = \{(\Phi, \tilde{\tilde{\Omega}}, \varsigma), (\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma), (\Theta_3, \Lambda_3, \varsigma), (\Theta_4, \Lambda_4, \varsigma)\}$*, then* $\tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cup}}\, \tilde{\tilde{\mathfrak{g}}}_2$ *is not* $\mathcal{BSGTS}$ *since* $(\Theta_2, \Lambda_2, \varsigma) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cup}}\, \tilde{\tilde{\mathfrak{g}}}_2$ *and* $(\Theta_3, \Lambda_3, \varsigma) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cup}}\, \tilde{\tilde{\mathfrak{g}}}_2$*, but*
$(\Theta_2, \Lambda_2, \varsigma) \,\tilde{\tilde{\cup}}\, (\Theta_3, \Lambda_3, \varsigma) = \{(\varrho_1, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5\}), (\varrho_2, \{\omega_2, \omega_3\}, \{\omega_4\})\} \,\tilde{\tilde{\notin}}\, \tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\cup}}\, \tilde{\tilde{\mathfrak{g}}}_2$.

Theorem 4. *Let* $\tilde{\tilde{\zeta}}$ *be a family of bipolar soft sets defined on a universal set* Ω*, then there exists a unique* $\mathcal{BSGT}$ $\tilde{\tilde{\mathfrak{g}}}$ *such that it is the smallest* $\mathcal{BSGT}$ *containing* $\tilde{\tilde{\zeta}}$*.*

Proof. Consider the collection of all $\mathcal{BSGT}s$ on Ω which contains $\tilde{\tilde{\zeta}}$ (as subset of $\Upsilon(\Omega)$) surely contains $\tilde{\tilde{\zeta}}$. Now, let $\tilde{\tilde{\mathfrak{g}}}$ be the intersection of the members of this collection. By Theorem 3, $\tilde{\tilde{\mathfrak{g}}}$ is a $\mathcal{BSGT}$, it contains $\tilde{\tilde{\zeta}}$ and clearly it is the smallest $\mathcal{BSGT}$ containing $\tilde{\tilde{\zeta}}$, for any such $\mathcal{BSGT}$ will be member of the collection of $\mathcal{BSGT}s$ just considered, and hence bipolar soft finer than its intersections $\tilde{\tilde{\mathfrak{g}}}$. Uniqueness of $\tilde{\tilde{\mathfrak{g}}}$ is trivial.

Definition 19. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\in}}\, \mathcal{BSS}(\Omega)$*. Then the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-interior of* $(\Theta, \Lambda, \varsigma)$*, denoted by* $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$*, is the bipolar soft union of all bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open subsets of* $(\Theta, \Lambda, \varsigma)$*.*
In other words, $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$ *is a largest bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open set contained in* $(\Theta, \Lambda, \varsigma)$*, so, we can write as*

$$i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)=\tilde{\tilde{\bigcup}}\{(\chi,\psi,\varsigma)\colon (\chi,\psi,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}},\ (\chi,\psi,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta,\Lambda,\varsigma)\}.$$

Here, are some properties of $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.

Theorem 5. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta_1,\Lambda_1,\varsigma)$,$(\Theta_2,\Lambda_2,\varsigma)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then*

(i) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_1,\Lambda_1,\varsigma)$.

(ii) $(\Theta_1,\Lambda_1,\varsigma)$ *is bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open if and only if* $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)=(\Theta_1,\Lambda_1,\varsigma)$.

(iii) $i_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)=i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$.

(iv) *If* $(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_2,\Lambda_2,\varsigma)$, *then* $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2,\Lambda_2,\varsigma)$.

(v) $i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\cap}}\ (\Theta_2,\Lambda_2,\varsigma))\ \tilde{\tilde{\subseteq}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\cap}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2,\Lambda_2,\varsigma)$.

(vi) $i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\cup}}\ (\Theta_2,\Lambda_2,\varsigma))\ \tilde{\tilde{\supseteq}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\cup}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2,\Lambda_2,\varsigma)$.

(vii) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Phi,\tilde{\tilde{\Omega}},\varsigma)=(\Phi,\tilde{\tilde{\Omega}},\varsigma)$.

Proof.

(i) Since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)=\tilde{\tilde{\bigcup}}\{(\Theta_j,\Lambda_j,\varsigma):(\Theta_j,\Lambda_j,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}},\ (\Theta_j,\Lambda_j,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_1,\Lambda_1,\varsigma),j\in\mathcal{J}\}$. Then $\Theta_j(\varrho)\subseteq\Theta_1(\varrho)$ and $\Lambda_1(\neg\varrho)\subseteq\Lambda_{\mathcal{J}}(\neg\varrho)$ for all $j\in\mathcal{J}$. So, $\bigcup_{j\in\mathcal{J}}\Theta_j(\varrho)\subseteq\Theta_1(\varrho)$ and $\Lambda_1(\neg\varrho)\subseteq\bigcap_{j\in\mathcal{J}}\Lambda_j(\neg\varrho)$. Therefore $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_1,\Lambda_1,\varsigma)$.

(ii) Let $(\Theta_1,\Lambda_1,\varsigma)$ be a bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open set. Then $(\Theta_1,\Lambda_1,\varsigma)$ is the largest bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open set contained in $(\Theta_1,\Lambda_1,\varsigma)$. From (i), we have $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_1,\Lambda_1,\varsigma)$. Therefore, $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)=(\Theta_1,\Lambda_1,\varsigma)$.
Conversely, assume that $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)=(\Theta_1,\Lambda_1,\varsigma)$. Since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$ is a bipolar soft union of all bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open subsets of $(\Theta_1,\Lambda_1,\varsigma)$ and $\tilde{\tilde{\mathfrak{g}}}$ is closed under arbitrary bipolar soft union, then $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open. Thus, $(\Theta_1,\Lambda_1,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open.

(iii) Since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$ is a bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open set. Thus by (ii), $i_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma))=i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$.

(iv) Suppose $(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_2,\Lambda_2,\varsigma)$. From (i), $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_1,\Lambda_1,\varsigma)$. Therefore, $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta_2,\Lambda_2,\varsigma)$. Now, $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)$ is a bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open set contained in $(\Theta_2,\Lambda_2,\varsigma)$. So, it is contained in bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-interior, and since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2,\Lambda_2,\varsigma)$ is the largest bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-open set contained in $(\Theta_2, \Lambda_2, \varsigma)$. Therefore, $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$ $\widetilde{\widetilde{\subseteq}}$ $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(v) Since $(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, (\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, (\Theta_2, \Lambda_2, \varsigma)$. So, by (iv), $i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \varsigma)) \,\widetilde{\widetilde{\subseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$ and $i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \varsigma))$ $\widetilde{\widetilde{\subseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$. Hence, $i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\Theta_2, \Lambda_2, \varsigma)) \,\widetilde{\widetilde{\subseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vi) We know that $(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, (\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \varsigma)$ and $(\Theta_2, \Lambda_2, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, (\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}$ $(\Theta_2, \Lambda_2, \varsigma)$. Then by (v), $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\subseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \varsigma))$ and $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$ $\widetilde{\widetilde{\subseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \varsigma))$. So, $i_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}\, (\Theta_2, \Lambda_2, \varsigma)) \,\widetilde{\widetilde{\supseteq}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \,\widetilde{\widetilde{\cup}}$ $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vii) The proof is trivial.

In the next example, we will show that the equality of parts (v) and (vi) in Theorem 5 do not hold.

Example 4. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, $\varsigma = \{\varrho_3, \varrho_4\}$ *and* $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma), (\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma), (\Theta_3, \Lambda_3, \varsigma), (\Theta_4, \Lambda_4, \varsigma), (\Theta_5, \Lambda_5, \varsigma)\}$, *where*

$$
\begin{aligned}
(\Theta_1, \Lambda_1, \varsigma) &= \{(\varrho_3, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2\}), (\varrho_4, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2\})\},\\
(\Theta_2, \Lambda_2, \varsigma) &= \{(\varrho_3, \{\omega_4\}, \{\omega_1, \omega_2\}), (\varrho_4, \{\omega_4\}, \{\omega_1, \omega_2\})\},\\
((\Theta_3, \Lambda_3, \varsigma) &= \{(\varrho_3, \{\omega_4\}, \{\omega_2, \omega_3\}), (\varrho_4, \{\omega_4\}, \{\omega_2, \omega_3\})\},\\
(\Theta_4, \Lambda_4, \varsigma) &= \{(\varrho_3, \{\omega_1, \omega_3, \omega_4, \omega_5\}, \{\omega_2\}), (\varrho_4, \{\omega_1, \omega_3, \omega_4, \omega_5\}, \{\omega_2\})\} \textit{ and}\\
(\Theta_5, \Lambda_5, \varsigma) &= \{(\varrho_3, \{\omega_4\}, \{\omega_2\}), (\varrho_4, \{\omega_4\}, \{\omega_2\})\}.
\end{aligned}
$$

To show the converse of (v). Let
$(\chi_1, \psi_1, \varsigma) = \{(\varrho_3, \{\omega_4, \omega_5\}, \{\omega_1, \omega_2\}), (\varrho_4, \{\omega_4, \omega_5\}, \{\omega_1, \omega_2\})\}$ *and*
$(\Theta_3, \Lambda_3, \varsigma) = (\chi_2, \psi_2, \varsigma) = \{(\varrho_3, \{\omega_4\}, \{\omega_2, \omega_3\}), (\varrho_4, \{\omega_4\}, \{\omega_2, \omega_3\})\}$.
So, $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1, \psi_1, \varsigma) = (\Theta_2, \Lambda_2, \varsigma)$ *and* $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2, \psi_2, \varsigma) = (\Theta_3, \Lambda_3, \varsigma)$,
Hence, $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1, \psi_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2, \psi_2, \varsigma) = \{(e\varrho_3, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}), (\varrho_4, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\})\}$.
Also,

$$
\begin{aligned}
i_{\widetilde{\widetilde{\mathfrak{g}}}}((\chi_1, \psi_1, \varsigma) \widetilde{\widetilde{\cap}} (\chi_2, \psi_2, \varsigma)) &= i_{\widetilde{\widetilde{\mathfrak{g}}}}\{(\varrho_3, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}), (\varrho_4, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\})\}\\
&= (\Phi, \widetilde{\widetilde{\Omega}}, \varsigma).
\end{aligned}
$$

Therefore,

$$
i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1, \psi_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2, \psi_2, \varsigma) \neq i_{\widetilde{\widetilde{\mathfrak{g}}}}((\chi_1, \psi_1, \varsigma) \,\widetilde{\widetilde{\cap}}\, (\chi_2, \psi_2, \varsigma)).
$$

Now, to show the converse of (vi). Let
$(\chi_1,\psi_1,\varsigma)=\{(\varrho_3,\{\omega_1,\omega_3\},\{\omega_2\}),(\varrho_4,\{\omega_1,\omega_3\},\{\omega_2\})\}$ *and*
$(\chi_2,\psi_2,\varsigma)=\{(e_3,\{\omega_4,\omega_5\},\{\omega_1,\omega_2,\omega_3\}),(\varrho_4,\{\omega_4,\omega_5\},\{\omega_1,\omega_2,\omega_3\})\}$.
Then $i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)=(\Phi,\tilde{\tilde{\Omega}},\varsigma)$ *and* $i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)=(\Phi,\tilde{\tilde{\Omega}},\varsigma)$.
Hence,

$$i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)\tilde{\tilde{\cup}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)=(\Phi,\tilde{\tilde{\Omega}},\varsigma).$$

But $(\chi_1,\psi_1,\varsigma)\tilde{\tilde{\cup}}(\chi_2,\psi_2,\varsigma)=\{(\varrho_3,\{\omega_1,\omega_3,\omega_4,\omega_5\},\{\omega_2\}),(\varrho_4,\{\omega_1,\omega_3,\omega_4,\omega_5\},\{\omega_2\})\}$.
While $i_{\tilde{\tilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varsigma)\tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\varsigma))=(\Theta_4,\Lambda_4,\varsigma)$.
So, $i_{\tilde{\tilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varsigma)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\varsigma))\neq i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)\ \tilde{\tilde{\cup}}\ i_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)$.

Definition 20. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then* $(\Theta,\Lambda,\varsigma)$ *is said to be bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed if its bipolar soft complement* $(\Theta,\Lambda,\varsigma)^c$ *is bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open.*

Theorem 6. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$*, then* $\neg\tilde{\mathfrak{g}}=\{(\Lambda,\neg\varsigma):(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}}\}$ *is* $\mathcal{SGT}$ *in terms of soft* $\tilde{\mathfrak{g}}$*-closed.*

Proof. Similar to Theorem 1.

Theorem 7. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *. Then the following properties hold*

(i) $(\tilde{\tilde{\Omega}},\Phi,\varsigma)$ *is bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed.*

(ii) Arbitrary bipolar soft intersections of the bipolar soft $\tilde{\tilde{\mathfrak{g}}}$*-closed sets are bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed.*

Proof.

(i) Since the complement of the absolute bipolar soft set $(\tilde{\tilde{\Omega}},\Phi,\varsigma)$ is the relative null bipolar soft set $(\Phi,\tilde{\tilde{\Omega}},\varsigma)$, and $(\Phi,\tilde{\tilde{\Omega}},\varsigma)\tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}}$. Thus, $(\tilde{\tilde{\Omega}},\Phi,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed.

(ii) Let $\{(\Theta_i,\Lambda_i,\varsigma)\}_{i\in\mathcal{I}}$ be a given collection of bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed sets. To show $\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed. Now $(\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}((\Theta_i,\Lambda_i,\varsigma))^c=\tilde{\tilde{\bigcup}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma)^c$. Since $(\Theta_i,\Lambda_i,\varsigma)$ is a bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed for each $i\in I$. So $(\Theta_i,\Lambda_i,\varsigma)^c$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open sets and hence $\tilde{\tilde{\bigcup}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma)^c$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open. Therefore, $(\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma))^c$ is also bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open. This means that, $\tilde{\tilde{\bigcap}}_{i\in\mathcal{I}}(\Theta_i,\Lambda_i,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed.

Definition 21. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closure of* $(\Theta,\Lambda,\varsigma)$*, denoted by* $c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$*, is the bipolar soft intersection of all*

bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed sets containing $(\Theta, \Lambda, \varsigma)$.
In other words, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$ *is the smallest bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set containing* $(\Theta, \Lambda, \varsigma)$*, so, we can write as*

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = \widetilde{\widetilde{\bigcap}} \{(\chi, \psi, \varsigma) : (\chi, \psi, \varsigma) \text{ is bipolar soft } \widetilde{\widetilde{\mathfrak{g}}}\text{-closed; } (\chi, \psi, \varsigma) \, \widetilde{\widetilde{\supseteq}} \, (\Theta, \Lambda, \varsigma)\}.$$

Theorem 8. *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma) \, \widetilde{\widetilde{\in}} \, \mathcal{BSS}(\Omega)$. *Then*

(i) $(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(ii) $(\Theta_1, \Lambda_1, \varsigma)$ *is bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed if and only if* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) = (\Theta_1, \Lambda_1, \varsigma)$.

(iii) $c_{\widetilde{\widetilde{\mathfrak{g}}}}(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)) = c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(iv) If $(\Theta_1, \Lambda_1, \varsigma) \widetilde{\widetilde{\subseteq}} (\Theta_2, \Lambda_2, \varsigma)$, *then* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(v) $c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \widetilde{\widetilde{\cap}} (\Theta_2, \Lambda_2, \varsigma)) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\cap}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vi) $c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\cup}} \, (\Theta_2, \Lambda_2, \varsigma) \, \widetilde{\widetilde{\supseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\cup}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vii) $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma) = (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$.

Proof.

(i) Since $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) = \widetilde{\widetilde{\bigcap}}\{(\Theta_i, \Lambda_i, \varsigma) : (\Theta_i, \Lambda_i, \varsigma)^c \, \widetilde{\widetilde{\in}} \, \widetilde{\widetilde{\mathfrak{g}}}, (\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, (\Theta_i, \Lambda_i, \varsigma), i \in \mathcal{I}\}$. Then $\Theta_1(\varrho) \subseteq \Theta_i(\varrho)$ and $\Lambda_i(\neg\varrho) \subseteq \Lambda_1(\neg\varrho)$ for all $i \in \mathcal{I}$. So, $\Theta_1(\varrho) \subseteq \bigcap_{i \in \mathcal{I}} \Lambda_i(\varrho)$ and $\bigcup_{i \in \mathcal{I}} \Lambda_i(\neg\varrho) \subseteq \Lambda_1(\neg\varrho)$. Thus, $(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(ii) Let $(\Theta_1, \Lambda_1, \varsigma)$ be a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set. Then $(\Theta_1, \Lambda_1, \varsigma)$, is the smallest bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set containing $(\Theta_1, \Lambda_1, \varsigma)$. From (i), we have $(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$. Therefore, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) = (\Theta_1, \Lambda_1, \varsigma)$.

(iii) Since $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$ is a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set. Thus by (ii), it is equal to its bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure. Therefore $c_{\widetilde{\widetilde{\mathfrak{g}}}}(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)) = c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(iv) Suppose $(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, (\Theta_2, \Lambda_2, \varsigma)$. From (i), $(\Theta_2, \Lambda_2, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$. Thus, $(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$. Now, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$ is a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set containing $(\Theta_1, \Lambda_1, \varsigma)$. So it is containing its bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure, and since $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$ is the smallest bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closed set containing $(\Theta_1, \Lambda_1, \varsigma)$. Therefore, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \, \widetilde{\widetilde{\subseteq}} \, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(v) Since $(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; (\Theta_2, \Lambda_2, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; (\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; (\Theta_2, \Lambda_2, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}(\Theta_2, \Lambda_2, \varsigma)$. So by (iv), we get

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; (\Theta_2, \Lambda_2, \varsigma)) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$$

and

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; (\Theta_2, \Lambda_2, \varsigma)) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma).$$

Hence, $c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; (\Theta_2, \Lambda_2, \varsigma)) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcap}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vi) Since $(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; (\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; (\Theta_2, \Lambda_2, \varsigma)$ and $(\Theta_2, \Lambda_2, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; (\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; (\Theta_2, \Lambda_2, \varsigma)$. Then by (iv), we have $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; (\Theta_2, \Lambda_2, \varsigma))$ and $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; (\Theta_2, \Lambda_2, \varsigma))$. Thus,

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma) \;\widetilde{\widetilde{\sqsubseteq}}\; c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \;\widetilde{\widetilde{\sqcup}}\; (\Theta_2, \Lambda_2, \varsigma)).$$

(vii) The proof is trivial.

In the next example, we will show that the equality of parts (v) and (vi) in Theorem 8 does not hold.

Example 5. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, $A = \{\varrho_3, \varrho_4\}$ *and* $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma), (\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma), (\Theta_3, \Lambda_3, \varsigma), (\Theta_4, \Lambda_4, \varsigma), (\Theta_5, \Lambda_5, \varsigma), (\Theta_6, \Lambda_6, \varsigma)\}$ *where*

$$\begin{aligned}
(\Theta_1, \Lambda_1, \varsigma) &= \{(\varrho_3, \{\omega_2, \omega_4\}, \{\omega_1, \omega_3, \omega_5\}), (\varrho_4, \{\omega_2, \omega_4\}, \{\omega_1, \omega_3, \omega_5\})\},\\
(\Theta_2, \Lambda_2, \varsigma) &= \{(\varrho_3, \{\omega_4, \omega_5\}, \{\omega_2, \omega_3\}), (\varrho_4, \{\omega_4, \omega_5\}, \{\omega_2, \omega_3\})\},\\
(\Theta_3, \Lambda_3, \varsigma) &= \{(\varrho_3, \{\omega_1, \omega_5\}, \{\omega_2, \omega_4\}), (\varrho_4, \{\omega_1, \omega_5\}, \{\omega_2, \omega_4\})\},\\
(\Theta_4, \Lambda_4, \varsigma) &= \{(\varrho_3, \{\omega_2, \omega_4, \omega_5\}, \{\omega_3\}), (\varrho_4, \{\omega_2, \omega_4, \omega_5\}, \{\omega_3\}),\\
(\Theta_5, \Lambda_5, \varsigma) &= \{(\varrho_3, \{\omega_1, \omega_2, \omega_4, \omega_5\}, \phi), (\varrho_4, \{\omega_1, \omega_2, \omega_4, \omega_5\}, \phi)\} \textit{ and}\\
(\Theta_6, \Lambda_6, \varsigma) &= \{(\varrho_3, \{\omega_1, \omega_4, \omega_5\}, \{\omega_2\}), (\varrho_4, \{\omega_1, \omega_4, \omega_5\}, \{\omega_2\})\}.
\end{aligned}$$

Then

$$\widetilde{\widetilde{\mathfrak{g}}}^{c} = \{(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma),\ (\Theta_1, \Lambda_1, \varsigma)^c,\ (\Theta_2, \Lambda_2, \varsigma)^c,\ (\Theta_3, \Lambda_3, \varsigma)^c,\ (\Theta_4, \Lambda_4, \varsigma)^c,\ (\Theta_5, \Lambda_5, \varsigma)^c,\ (\Theta_6, \Lambda_6, \varsigma)^c\},$$

where

$$(\Theta_1, \Lambda_1, \varsigma)^c = \{(\varrho_3, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4\}), (\varrho_4, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4\})\},$$

$$(\Theta_2,\Lambda_2,\varsigma)^c=\{(\varrho_3,\{\omega_2,\omega_3\},\{\omega_4,\omega_5\}),(\varrho_4,\{\omega_2,\omega_3\},\{\omega_4,\omega_5\})\},$$
$$(\Theta_3,\Lambda_3,\varsigma)^c=\{(\varrho_3,\{\omega_2,\omega_4\},\{\omega_1,\omega_5\}),(\varrho_4,\{\omega_2,\omega_4\},\{\omega_1,\omega_5\})\}$$
$$(\Theta_4,\Lambda_4,\varsigma)^c=\{(\varrho_3,\{\omega_3\},\{\omega_2,\omega_4,\omega_5\}),(\varrho_4,\{\omega_3\},,\{\omega_2,\omega_4,\omega_5\})\},$$
$$(\Theta_5,\Lambda_5,\varsigma)^c=\{(\varrho_3,\phi,\{\omega_1,\omega_2,\omega_4,\omega_5\}),(\varrho_4,\phi,\{\omega_1,\omega_2,\omega_4,\omega_5\})\}\ and$$
$$(\Theta_6,\Lambda_6,\varsigma)^c=\{(\varrho_3,\{\omega_2\},\{\omega_1,\omega_4,\omega_5\}),(\varrho_4\{\omega_2\},\{\omega_1,\omega_4,\omega_5\})\}.$$

To show the converse of (v). Let
$(\chi_1,\psi_1,\varsigma)=\{(\varrho_3,\{\omega_2,\omega_4,\omega_5\},\{\omega_1,\omega_3\}),(\varrho_4,\{\omega_2,\omega_4,\omega_5\},\{\omega_1,\omega_3\})\}$ *and*
$(\chi_2,\psi_2,\varsigma)=\{(\varrho_3,\{\omega_1,\omega_2,\omega_4\},\{\omega_5\}),(\varrho_4,\{\omega_1,\omega_2,\omega_4\},\{\omega_5\})\}$.
So, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ *and* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$.
Hence,

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)\ \widetilde{\widetilde{\cap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma).$$

Also,

$$\begin{aligned}c_{\widetilde{\widetilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varsigma)\widetilde{\widetilde{\cap}}(\chi_2,\psi_2,\varsigma))&=c_{\widetilde{\widetilde{\mathfrak{g}}}}(\{(\varrho_3,\{\omega_2,\omega_4\},\{\omega_1,\omega_3,\omega_5\}),(\varrho_4,\{\omega_2,\omega_4\},\{\omega_1,\omega_3,\omega_5\})\})\\&=(\Theta_3,\Lambda_3,\varsigma)^c.\end{aligned}$$

Thus,

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)\ \widetilde{\widetilde{\cap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)\neq c_{\widetilde{\widetilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varsigma)\ \widetilde{\widetilde{\cap}}\ (\chi_2,\psi_2,\varsigma)).$$

Now, to show the converse of (vi). Let

$$(\chi_1,\psi_1,\varsigma)=\{(\varrho_3,\{\omega_1,\omega_5\},\{\omega_2,\omega_4\}),(\varrho_4,\{\omega_1,\omega_5\},\{\omega_2,\omega_4\})\},$$

and

$$(\chi_2,\psi_2,\varsigma)=\{(\varrho_3,\{\omega_2,\omega_3\},\{\omega_1,\omega_4,\omega_5\}),(\varrho_4,\{\omega_2,\omega_3\},\{\omega_1,\omega_4,\omega_5\})\}.$$

Then $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)=(\Theta_1,\Lambda_1,\varsigma)^c$ *and* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)=(\Theta_2,\Lambda_2,\varsigma)^c$.
Thus,

$$\begin{aligned}c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varsigma)\,\widetilde{\widetilde{\cup}}\,c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varsigma)&=(\Theta_1,\Lambda_1,\varsigma)^c\,\widetilde{\widetilde{\cup}}\,(\Theta_2,\Lambda_2,\varsigma)^c\\&=\{(\varrho_3,\{\omega_1,\omega_2,\omega_3,\omega_5\},\{\omega_4\}),(\varrho_4,\{\omega_1,\omega_2,\omega_3,\omega_5\},\{\omega_4\})\}.\end{aligned}$$

But

$$(\chi_1,\psi_1,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\chi_2,\psi_2,\varsigma)=\{(\varrho_3,\{\omega_1,\omega_2,\omega_3,\omega_5\},\{\omega_4\}),(\varrho_4,\{\omega_1,\omega_2,\omega_3,\omega_5\},\{\omega_4\})\}.$$

Hence, $c_{\widetilde{\widetilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\chi_2,\psi_2,\varsigma))=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. *Therefore,*

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}((\Theta_1,\Lambda_1,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta_2,\Lambda_2,\varsigma)\neq c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_1,\Lambda_1,\varsigma)\ \widetilde{\widetilde{\cup}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta_2,\Lambda_2,\varsigma).$$

Proposition 2. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$*. Then* $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\subseteq}} (\Theta, \Lambda, \varsigma) \tilde{\tilde{\subseteq}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

Theorem 9. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma), (\chi, \psi, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$*. Then*

(i) $c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c = (i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c$.

(ii) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c = (c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c$.

(iii) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = (c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c)^c$.

(iv) $c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = (i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c)^c$.

(v) $i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta, \Lambda, \varsigma) \tilde{\tilde{\setminus}} (\chi, \psi, \varsigma)) \tilde{\tilde{\subseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\setminus}} i_{\tilde{\tilde{\mathfrak{g}}}}(\chi, \psi, \varsigma)$.

Proof. It is enough to prove only parts (i) and (v) since the proof of other parts are similar.

(i) Since $(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c = (\tilde{\tilde{\bigcup}}\{(\Theta_i, \Lambda_i, \varsigma) : (\Theta_i, \Lambda_i, \varsigma) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}},\ (\Theta_i, \Lambda_i, \varsigma) \tilde{\tilde{\subseteq}} (\Theta, \Lambda, \varsigma),\ i \in \mathcal{I}\})^c$ $= \tilde{\tilde{\bigcap}} \{(\Theta_i, \Lambda_i, \varsigma)^c\text{: } (\Theta_i, \Lambda_i, \varsigma) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}},\ (\Theta, \Lambda, \varsigma)^c \tilde{\tilde{\subseteq}} (\Theta_i, \Lambda_i, \varsigma)^c,\ i \in \mathcal{I}\} = c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c$.

(v) Since

$$\begin{aligned} i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta, \Lambda, \varsigma)\tilde{\tilde{\setminus}}(\chi, \psi, \varsigma)) &= i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta, \Lambda, \varsigma)\tilde{\tilde{\cap}}(\chi, \psi, \varsigma)^c) \\ &\tilde{\tilde{\subseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)\tilde{\tilde{\cap}} i_{\tilde{\tilde{\mathfrak{g}}}}(\chi, \psi, \varsigma)^c \,(by\ Theorem\ 5(v)) \\ &= i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)\tilde{\tilde{\cap}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\chi, \psi, \varsigma))^c \\ &\tilde{\tilde{\subseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)\tilde{\tilde{\cap}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\chi, \psi, \varsigma))^c \\ &= i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)\tilde{\tilde{\setminus}} i_{\tilde{\tilde{\mathfrak{g}}}}(\chi, \psi, \varsigma). \end{aligned}$$

Definition 22. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$*. Then the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-boundary of* $(\Theta, \Lambda, \varsigma)$*, denoted by* $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$*, is defined as*

$$b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c.$$

Proposition 3. *It is clear that* $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c$.

Theorem 10. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$*. Then*

(i) $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\subseteq}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

(ii) $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\subseteq}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

(iii) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\setminus}}\ b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.

(iv) $b_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.

(v) $b_{\tilde{\tilde{\mathfrak{g}}}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.

Proof.

(i) Since $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cap}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c$. Then $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.

(ii)

$$\begin{aligned}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cup}}\, b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) &= (\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cup}}\,(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)\\ &= ((\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cup}}\, c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}}((\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cup}}\, c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)\\ &= c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}((\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cup}}\, c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)\\ &\widetilde{\widetilde{\subseteq}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma).\end{aligned}$$

(iii)

$$\begin{aligned}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\setminus}} b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) &= (\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}(b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c\\ &= (\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}\, c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)^c\\ &= (\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c\widetilde{\widetilde{\cup}}\, i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\\ &= ((\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)\widetilde{\widetilde{\cup}}\,((\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\\ &= (\Phi,\Lambda,\varsigma)\widetilde{\widetilde{\cup}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\\ &\widetilde{\widetilde{\supseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma).\end{aligned}$$

(iv)

$$\begin{aligned}b_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)) &= c_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c\\ &= c_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c)\\ &\widetilde{\widetilde{\subseteq}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c\\ &= b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma).\end{aligned}$$

(v)

$$b_{\tilde{\tilde{\mathfrak{g}}}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)) = c_{\tilde{\tilde{\mathfrak{g}}}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c$$

$$\widetilde{\widetilde{\subseteq}}(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}}c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c$$
$$= b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma).$$

The following example shows that the equality of (ii), (iii), (iv) and (v) in Theorem 10 does not hold in general.

Example 6. *Let* $\Omega = \{\omega_1,\omega_2,\omega_3,\omega_4\}$, $\varsigma = \{\varrho_1,\varrho_2\}$ *and* $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi,\widetilde{\widetilde{\Omega}},\varsigma),(\Theta_1,\Lambda_1,\varsigma),(\Theta_2,\Lambda_2,\varsigma),(\Theta_3,\Lambda_3,\varsigma)\}$*, where*

$$(\Theta_1,\Lambda_1,\varsigma) = \{(\varrho_1,\{\omega_3\},\{\omega_1\}),(\varrho_2,\{\omega_3\},\{\omega_1,\omega_2\})\},$$
$$(\Theta_2,\Lambda_2,\varsigma) = \{(\varrho_1,\phi,\{\omega_2,\omega_3\}),(\varrho_2,\{\omega_1\},\{\omega_3\})\} \, and$$
$$(\Theta_3,\Lambda_3,\varsigma) = \{(\varrho_1,\{\omega_3\},\phi),(\varrho_2,\{\omega_1,\omega_3\},\phi)\}.$$

Let $(\Theta,\Lambda,\varsigma) = \{(\varrho_1,\{\omega_1\},\{\omega_3\}),(\varrho_2,\{\omega_1\},\{\omega_3\})\}$. *Now,* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Theta_1,\Lambda_1,\varsigma)^c$, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c = (\Theta_2,\Lambda_2,\varsigma)^c$ *and* $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Theta_2,\Lambda_2,\varsigma)$. *Thus,* $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Theta_3,\Lambda_3,\varsigma)^c$. *Hence*

$$b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta,\Lambda,\varsigma) = \{(\varrho_1,\{\omega_1\},\{\omega_3\}),(\varrho_2,\{\omega_1\},\{\omega_3\})\} = (\Theta,\Lambda,\varsigma).$$

Therefore, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) \neq b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta,\Lambda,\varsigma)$.
Also, $(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\setminus}}b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = \{(\varrho_1,\phi,\{\omega_3\}),(\varrho_2,\{\omega_1\},\{\omega_3\})\}$.
Thus, $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) \neq (\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\setminus}}b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$.
Now, if we take $(\chi,\psi,\varsigma) = \{(\varrho_1,\phi,\{\omega_2\}),(\varrho_2,\phi,\{\omega_3\})\}$ *is a bipolar soft set. Then* $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma) = (\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$. *Hence*

$$b_{\widetilde{\widetilde{\mathfrak{g}}}}(i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)) = \{(\varrho_1,\phi,\{\omega_3\}),(\varrho_2,\phi,\{\omega_1,\omega_3\})\} = (\Theta_3,\Lambda_3,\varsigma)^c$$

and

$$b_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma) = (\widetilde{\widetilde{\Omega}},\Phi,\varsigma).$$

Thus, $b_{\widetilde{\widetilde{\mathfrak{g}}}}(i_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)) \neq b_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)$. *Also,* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma) = (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$.
Hence, $b_{\widetilde{\widetilde{\mathfrak{g}}}}(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)) = (\Theta_3,\Lambda_3,\varsigma)^c$. *Therefore,* $b_{\widetilde{\widetilde{\mathfrak{g}}}}(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)) \neq b_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi,\psi,\varsigma)$.

Theorem 11. *Let* $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then* $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cap}}\ i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Phi,\Lambda,\varsigma)$.

Proof. We start by

$$b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$$
$$= (c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\setminus}}i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\cap}}i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$$

$$= c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\tilde{\tilde{\cap}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c\tilde{\tilde{\cap}}i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$$
$$= c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\tilde{\tilde{\cap}}(\Phi,\Lambda,\varsigma)$$
$$= (\Phi,\Lambda,\varsigma).$$

Theorem 12. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$*. Then the following hold:*

(i) If $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}}$*, then* $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\cap}}\ b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Phi,\Lambda,\varsigma)$*.*

(ii) If $(\Theta,\Lambda,\varsigma)$ *is bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed, then* $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta,\Lambda,\varsigma)$*.*

Proof.

(i) Suppose $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}}$. Then by Theorem 5 (ii), we have $(\Theta,\Lambda,\varsigma) = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$. Since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\subseteq}}\ (b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c$. That means $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\subseteq}}\ (b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c$. Therefore, $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\cap}}\ b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Phi,\Lambda,\varsigma)$.

(ii) By Theorem 10 (i), we have $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\subseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)$. Since $(\Theta,\Lambda,\varsigma)$ is a bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed set, then $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\tilde{\tilde{\subseteq}}(\Theta,\Lambda,\varsigma)$.

The following example shows that the converse of Theorem 12 does not hold in general.

Example 7. *Take the bipolar soft set* $(\Theta,\Lambda,\varsigma)$ *as in Example 6. Then*

$$b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\cap}}\ (\Theta,\Lambda,\varsigma) = \{(\varrho_1,\phi,\{\omega_3\}),(\varrho_2,\phi,\{\omega_1,\omega_3\})\},$$

and

$$b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\subseteq}}\ (\Theta,\Lambda,\varsigma).$$

While $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\notin}}\ \tilde{\tilde{\mathfrak{g}}}$ *and* $(\Theta,\Lambda,\varsigma)$ *is not bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed.*

Theorem 13. *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varsigma,\neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$ *If* $(\Theta,\Lambda,\varsigma)$ *is both bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open and bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed. Then* $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) = (\Phi,\Lambda,\varsigma)$*.*

Proof. Assume that $(\Theta,\Lambda,\varsigma)$ is bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-open and bipolar soft $\tilde{\tilde{\mathfrak{g}}}$-closed. Thus

$$\begin{aligned} b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma) &= c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\tilde{\tilde{\cap}}c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)^c \\ &= c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma)\tilde{\tilde{\cap}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta,\Lambda,\varsigma))^c \\ &= (\Theta,\Lambda,\varsigma)\tilde{\tilde{\cap}}(\Theta,\Lambda,\varsigma)^c \\ &= (\Phi,\Lambda,\varsigma). \end{aligned}$$

The following example shows that the converse of Theorem 13 does not hold in general.

Example 8. *Take the bipolar soft set* $(\Theta, \Lambda, \varsigma)$ *as in Example 6. Then* $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = (\Phi, \Lambda, \varsigma)$, *but* $(\Theta, \Lambda, \varsigma)$ *is neither bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open nor bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-closed.*

Definition 23. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$. *Then the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-exterior of* $(\Theta, \Lambda, \varsigma)$ *denoted by* $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$, *is the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-interior of the bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-complement of* $(\Theta, \Lambda, \varsigma)$.
In the other words, $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c$.

Proposition 4. *The following statements are true for any* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$:

(i) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = (c_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c$.

(ii) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c = i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta, \Lambda, \varsigma)^c)^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

(iii) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cap}} (\Theta, \Lambda, \varsigma) = (\Phi, \Lambda, \varsigma)$.

(iv) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$ *is the largest bipolar soft* $\tilde{\tilde{\mathfrak{g}}}$*-open set contained in* $(\Theta, \Lambda, \varsigma)^c$ *and hence* $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$.

Proposition 5. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$. *Then*

$$e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta, \Lambda, \varsigma) = \tilde{\tilde{\bigcup}}\{(\chi, \psi, \varsigma) : (\chi, \psi, \varsigma) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}, (\chi, \psi, \varsigma) \tilde{\tilde{\subseteq}} (\Theta, \Lambda, \varsigma)^c\}.$$

Proof. It is obvious.

Theorem 14. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$. *Then*

(i) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Phi, \tilde{\tilde{\Omega}}, \varsigma) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$ *and* $e_{\tilde{\tilde{\mathfrak{g}}}}(\tilde{\tilde{\Omega}}, \Phi, \varsigma) = (\Phi, \tilde{\tilde{\Omega}}, \varsigma)$.

(ii) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\subseteq}} (\Theta_1, \Lambda_1, \varsigma)^c$.

(iii) $e_{\tilde{\tilde{\mathfrak{g}}}}(e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma))^c = e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(iv) If $(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\subseteq}} (\Theta_2, \Lambda_2, \varsigma)$, *then* $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma) \tilde{\tilde{\subseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(v) $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\subseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma))$.

(vi) $e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} (\Theta_2, \Lambda_2, \varsigma)) \tilde{\tilde{\subseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cap}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vii) $e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cap}} (\Theta_2, \Lambda_2, \varsigma)) \tilde{\tilde{\supseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

Proof.

(i) $e_{\tilde{\tilde{\mathfrak{g}}}}(\Phi, \tilde{\tilde{\Omega}}, \varsigma) = i_{\tilde{\tilde{\mathfrak{g}}}}(\Phi, \tilde{\tilde{\Omega}}, \varsigma)^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\tilde{\tilde{\Omega}}, \Phi, \varsigma) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$ and $e_{\tilde{\tilde{\mathfrak{g}}}}(\tilde{\tilde{\Omega}}, \Phi, \varsigma) = i_{\tilde{\tilde{\mathfrak{g}}}}(\Phi, \tilde{\tilde{\Omega}}, \varsigma) = (\Phi, \tilde{\tilde{\Omega}}, \varsigma)$.

(ii) Since $i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)^c \tilde{\tilde{\subseteq}} (\Theta_1, \Lambda_1, \varsigma)^c$. Then $e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\subseteq}} (\Theta_1, \Lambda_1, \varsigma)^c$.

(iii) $e_{\tilde{\tilde{\mathfrak{g}}}}(e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma))^c = i_{\tilde{\tilde{\mathfrak{g}}}}(e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)) = i_{\tilde{\tilde{\mathfrak{g}}}}(i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)^c) = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)^c = e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)$.

(iv) Obvious.

(v) Follows directly from (ii) and (iv).

(vi)

$$\begin{aligned} e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cup}}(\Theta_2, \Lambda_2, \varsigma)) &= i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cup}}(\Theta_2, \Lambda_2, \varsigma))^c \\ &= i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma)^c \tilde{\tilde{\cap}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)^c) \\ &\tilde{\tilde{\subseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)^c \tilde{\tilde{\cap}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma))^c \,(by\ Theorem\, 5(v)) \\ &= e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cap}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma). \end{aligned}$$

Thus, $e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} (\Theta_2, \Lambda_2, \varsigma)) \tilde{\tilde{\subseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cap}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

(vii)

$$\begin{aligned} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma) &= i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)^c \tilde{\tilde{\cup}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)^c \\ &\tilde{\tilde{\subseteq}} i_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma)^c \tilde{\tilde{\cup}} (\Theta_2, \Lambda_2, \varsigma)^c) \,(by\ Theorem\, 5(vi)) \\ &= i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cap}} (\Theta_2, \Lambda_2, \varsigma)^c \\ &= e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma)\tilde{\tilde{\cap}} (\Theta_2, \Lambda_2, \varsigma)). \end{aligned}$$

Thus, $e_{\tilde{\tilde{\mathfrak{g}}}}((\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cap}} (\Theta_2, \Lambda_2, \varsigma)) \tilde{\tilde{\supseteq}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_1, \Lambda_1, \varsigma) \tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta_2, \Lambda_2, \varsigma)$.

Theorem 15. *Let* $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varsigma, \neg\varsigma)$ *be a* $\mathcal{BSGTS}$ *and* $(\Theta, \Lambda, \varsigma) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$. *Then*

(i) $(b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

(ii) $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$.

Proof.

(i) Since $(b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \tilde{\tilde{\cup}} e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.

(ii) From (i), we have $(b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma))^c = i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\cup}}\, e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma)$.
Therefore, $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\cup}}\, i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\cup}}\, e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\subseteq}}\, (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$.

The following example shows that the converse of Theorem 15 does not hold in general.

Example 9. *Take the bipolar soft set* $(\Theta, \Lambda, \varsigma)$ *as in Example 6. Then*

$$i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = \{(\varrho_1, \phi, \{\omega_2, \omega_3\}), (\varrho_2, \{\omega_1\}, \{\omega_3\})\},$$
$$b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = \{(\varrho_1, \phi, \{\omega_3\}), (\varrho_2, \phi, \{\omega_1, \omega_3\})\} \textit{ and}$$
$$e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) = \{(\varrho_1, \{\omega_3\}, \{\omega_1\}), (\varrho_2, \{\omega_3\}, \{\omega_1, \omega_2\})\}.$$

Thus, $b_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\cup}}\, i_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \,\tilde{\tilde{\cup}}\, e_{\tilde{\tilde{\mathfrak{g}}}}(\Theta, \Lambda, \varsigma) \neq (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$.

4. An Application on $\mathcal{BSGTS}$

The present section gives the application of $\mathcal{BSGTS}s$ and investigates some of its properties.

Definition 24. *Let* $\varsigma = \{\varrho_1, \varrho_2, \ldots, \varrho_n\}$ *be a parameters set,* $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ *be an initial universe and* $(\Theta, \Lambda, \varsigma)$ *be a* $\mathcal{BSS}$ *over* Ω*. Then the score of an object by* κ_i*, is computed as* $\kappa_i = p_i - n_i$ *where* p_i *represents the set of positive description* (ϱ_i) *which is available for those who are applying for a job and it is computed as* $p_i = \sum_{j=1}^{n} \alpha_{ij}$*. Whereas,* n_i *represents the set of negative description* $(\neg\varrho_i)$ *which is available for those who are applying for a job and it is computed as* $n_i = \sum_{j=1}^{n} \beta_{ij}$*.*

This means that κ_i *is the different point between the scores of positive descriptions except the scores of negative descriptions to get the highest score for their selection to own job.*

Now , we can depend on the following algorithm to select a sample among those who applying for a job in vacancy jobs.

Algorithm 1: *The algorithm for the selection of a preferable choice is given as the following steps:*

Step 1. Input the $\mathcal{BSS}(\Theta, \Lambda, \varsigma)$.
Step 2. Write the $\mathcal{BSS}$ in the tabular form.
Step 3. Compute the score κ_i of ω_i , $\forall \omega_i \in \Omega$.
Step 4. Find $\kappa_S = \max \kappa_i$.
Step 5. If s has more than one value, then one of ω_i or all of ω_i could be preferable choice.
Step 6. To select new κ_s. Go to **Step 4**.

Example 10. *We consider the problem in Example 1 to select the most suitable people who are offered by Mr. Ibrahim. According to tourism companies' conditions, people who own specific description will be selected from* ς *and* n *selection of people in*

Duhok city. Suppose $\varsigma_1 = \{\varrho_1 = "Hard\ Working", \varrho_3 = "Flexibility", \varrho_5 = "Self\text{-}Confidence", \varrho_7 = "Skillful"\}$ *and* $\varsigma_2 = \{\varrho_2 = "Negligent", \varrho_4 = "Rigidity", \varrho_6 = "Shyness", \varrho_8 = "Unskillful"\}$. *Here, we will given* $\varsigma = \varsigma_1 \cup \varsigma_2$.
Now, we can use the above algorithm to select employees that are come for a job in tourism.

Table 4: Tabular form the score of the bipolar soft set $(\Theta_1, \Lambda_1, \varsigma)$

$(\Theta_1, \Lambda_1)(\varrho_i)$	p_i	n_i	κ_i
ω_1	1	2	-1
ω_2	1	3	-2
ω_3	2	2	0
ω_4	2	0	2
ω_5	2	1	1
ω_6	1	1	0
ω_7	2	0	2
ω_8	1	2	-1

Table 5: Tabular form the score of the bipolar soft set $(\Theta_2, \Lambda_2, \varsigma)$

$(\Theta_2, \Lambda_2)(\varrho_i)$	p_i	n_i	κ_i
ω_1	2	1	1
ω_2	2	2	0
ω_3	1	3	-2
ω_4	2	2	0
ω_5	1	2	-1
ω_6	0	1	-1
ω_7	0	2	-2
ω_8	0	2	-2

Table 6: Tabular form the score of the bipolar soft set $(\Theta_3, \Lambda_3, \varsigma)$

$(\Theta_3, \Lambda_3)(\varrho_i)$	p_i	n_i	κ_i
ω_1	2	1	1
ω_2	2	2	0
ω_3	2	2	0
ω_4	2	0	2
ω_5	2	1	1
ω_6	1	1	0
ω_7	2	0	2
ω_8	1	2	-1

Clearly, the maximum of $(\Theta_1, \Lambda_1, \varsigma), (\Theta_2, \Lambda_2, \varsigma)$ *and* $(\Theta_3, \Lambda_3, \varsigma)$ *are 2, 1, and 2 respectively. The optimal elements of* Ω *are* ω_4, ω_1 *and* ω_7.

5. Conclusions

We have introduced bipolar soft generalized topological spaces via bipolar soft sets. The bipolar soft sets $\widetilde{\widetilde{\mathfrak{g}}}$-interior, $\widetilde{\widetilde{\mathfrak{g}}}$-closure, $\widetilde{\widetilde{\mathfrak{g}}}$-exterior and $\widetilde{\widetilde{\mathfrak{g}}}$-boundary have been investigated and some results among them are obtained. Furthermore, the application of bipolar soft generalized topological spaces in a decision making problem has been presented. In the future work, we will construct bipolar soft connectedness, bipolar soft compactness, bipolar soft separation axioms and bipolar soft mappings using bipolar soft generalized topological spaces.

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