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On Weakly Connected Closed Geodetic Domination in Graphs Under Some Binary Operations

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Abstract. Let G be a simple connected graph. For $S \subseteq V(G)$, the weakly connected closed geodetic dominating set S of G is a geodetic closure $I_G[S]$ which is between S and is the set of all vertices on geodesics (shortest path) between two vertices of S. We select vertices of G sequentially as follows: Select a vertex v_1 and let $S_1 = \{v_1\}$. Select a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$. Then successively select vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, ..., v_i\}$ for $i = 1, 2, ..., k$ until we select a vertex v_k in the given manner that yields $I_G[S_k] = V(G)$. Also, the subgraph weakly induced $\langle S\rangle_w$ by S is connected where $\langle S\rangle_w = \langle N[S], E_w\rangle$ with $E_w = \{u, v \in E(G) : u \in S \text{ or } v \in S\}$ and S is a dominating set of G. The minimum cardinality of weakly connected closed geodetic dominating set of G is denoted by $\gamma_{wcg}(G)$. In this paper, the authors show and investigate the concept weakly connected closed geodetic dominating sets of some graphs and the join, corona, and Cartesian product of two graphs are characterized. The weakly connected closed geodetic domination numbers of these graphs are determined. Also, some relationships between weakly connected closed geodetic dominating set, weakly connected closed geodetic set, geodetic dominating set, and geodetic connected dominating set are established.

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Key Words and Phrases: weakly connected closed geodetic dominating set, and weakly connected closed geodetic domination number.

1. Introduction

In this paper we explore a parameter that is, defined in the same manner that the well-known weakly connected closed geodetic number of a graph G is. Indeed, while a weakly connected closed geodetic set of a graph G necessitates a geodetic closure $I_G[S]$ which is between S and is the set of all vertices on geodesics (shortest path) between two vertices of S and the subgraph weakly induced $\langle S\rangle_w$ by S is connected where $\langle S\rangle_w = \langle N[S], E_w\rangle$ with

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$E_w = \{u, v \in E(G) : u \in S \text{ or } v \in S\}$. The motivation of introducing the concept is to give a further investgation on weakly connected domination, closed geodetic domination and some of its variations. In fact, it can be shown that every weakly connected closed geodetic dominating set is a weakly connected closed geodetic set of a graph G. Thereupon, the weakly connected closed geodetic number of a graph G is at most equal to the weakly connected closed geodetic domination number of a graph G.

The concept of weakly connected closed geodetic nmbers was introduce and studied by Patangan, et. al [12]. Some concept and its number are also introduced by Aniversario, et.al [1], Chellathurai, et. al [6], Dunbar, et.al [7], Jamil, et.al [11], and Sandueta, et.al [13]. Furthermore, the weakly connected closed geodetic number of a graph may be used to give bounds on some weakly connected closed geodetic domination related parameters. Moreover, this newly concept may be applied to introduce some concepts (say, a variant of weakly connected closed geodetic domination) in the future.

2. Terminology and Notation

A set is a *dominating set* of G if $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among the dominating sets of G. A dominating set S with $|S| = \gamma(G)$ is said to be γ*-set* of G. A *connected dominating set* S of a graph G is a dominating set such that the subgraph $\langle S \rangle$ induced by S in G is connected. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G, denoted by $\gamma_c(G)$. A connected dominating set S with $|S| = \gamma_c(G)$ is called γ_c*-set* of G Tarr, et.al [14], and Duckworth, et.al [8].

Let $S \subseteq V(G)$. The *subgraph weakly induced* by S is the graph $\langle S \rangle_w = (N_G[S], E_w)$, where $E_w = \{uv \in E(G) : u \in S \text{ or } v \in S\}$. The symbol $E_w(S)$ means E_w, Patangan, et.al [12]. A dominating set $S \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph $\langle S \rangle_w$ weakly induced by S is connected. The *weakly connected domination number* $\gamma_w(G)$ of G is the minimum cardinality among all weakly connected dominating sets of G. A weakly connected dominating set S with $|S| = \gamma_w(G)$ is said to be γ_w*-set* of G, Sandueta, et.al [13].

Let $u, v \in V(G)$. A shortest path from u to v in G is called a *u-v geodesic* of G. The set $I_G[u, v]$ consists of u, v, and all vertices lying in some u-v geodesic of G. For a nonempty subset S of $V(G)$, $I_G[S] = \bigcup_{u,v \in S} I[u, v]$, Chartrand, et.al [4]. Let G be a connected graph, then set $S \subseteq V(G)$ is a *geodetic set* of G if $I_G[S] = V(G)$. The set $I_G[S]$ is called the *geodetic closure of* G. The minimum cardinality of a geodetic set is the geodetic number of G, and is denoted by $g(G)$. The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a *g-set*. Henceforth, the set $I_G(u, v)$ denotes the set $I_G[u, v] \setminus \{u, v\}$. A set $S \subseteq V(G)$ is called a *geodetic dominating set* of G if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is the *geodetic domination number* of G, and is denoted by $\gamma_g(G)$. A geodetic dominating set S with $|S| = \gamma_g(G)$ is said to be a γ_g*-set* G.A set of vertices in S in a graph G is said to be *geodetic connected dominating set* of G if S is both a geodetic set and connected dominating set. The minimum cardinality of a geodetic connected dominating set of G is called a *geodetic connected domination number* of G, denoted by $\gamma_{gc}(G)$. A geodetic connected dominating set S with $|S| = \gamma_{gc}(G)$ is said to be a γ_{gc}-*set* of G. The geoodesic set, geodetic dominating set and geodetic connected dominating set are studied by Escuadro, et.al [9], Patangan et.al, [12], and Tejaswini, et.al [15]. The set S is a *closed geodetic cover* of a graph G if $S = \{v_1, v_2, ..., v_k\}$ and is obtained by choosing the vertices $v_1, v_2, ..., v_k$ such that the following hold:

(i) $v_1 \neq v_2$;

(ii) $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$; and

(ii) $I_G[S_k] = V(G)$, where $S_i = \{v_1, v_2, ..., v_i\}$ for all $i = 1, 2, ..., k$

If $S \subseteq V(G)$ satisfies (i) and (ii) of the definition above, then S is a *closed geodetic subset* of $V(G)$. The collection of all closed geodetic covers of G is denoted by $C^*(G)$. The closed geodetic number of G, is given by $cgn(G) = min\{|S| : S \in C^*(G)\}$. A set $S \in C^*(G)$ with $|S| = cgn(G)$ is called the *closed geodetic basis* of G and is denoted by $cgb(G)$ Aniversario, et.al [1], and Patangan, et.al [12].

A vertex v in a connected G is an *extreme vertex* if the neighborhood $N(v)$ of v induces a complete subgraph of G. The set of all extreme vertices in G is denoted by $Ext(G)$. By a neighborhood $N(v)$ of a vertex v in G is the set of all vertices x in G suh that $d_G(v, x) \leq 1$. A set $S \subseteq V(G)$ is said to be a closure absorbing set in G if for every $v \in V(G) \setminus S$, there exist $u, w \in N(v) \cap S$ with $d_G(u, w) = 2$, Cagaanan [3], and Aniversario, et.al [1]. Let G be the connected graph and $S \subseteq V(G)$. The 2 - path closure $P_2[S]_G$ of S is that set $P_2[S]_G = S \cup \{w \in V(G) : w \in I_G[u, v]$ for some u, $v \in S$ with $d_G(u, v) = 2\}$. The set S is called 2 - *path closure absorbing set* if $P_2[S]_G = V(G)$, Canoy, et.al [5], and Aniversario, et.al [1]. A set S is called a *weakly connected closed geodetic set* of G, if it satisfies the following properties:

(i) $S \in C^*(G)$; and

(ii) $\langle S \rangle_w$ is connected.

The minimum cardinality of a weakly connected closed geodetic set is called the *weakly connected closed geodetic number* of G, denoted by $wcgn(G)$. In a weakly connected closed geodetic set S, for every $v \in S$, there exists $u \in S$ such that $d_G(u, v) \leq 2$. Moreover, if $\{S_1, S_2, ..., S_k\}$ is the sequence corresponding to the weakly connected geodetic set S, $\langle S_i \rangle_w$ is connected for each $i = 1, 2, ..., k$, Patangan, et.al [12].

3. Results

Definition 1. *A weakly connected closed geodetic set of G which is dominating is called a weakly connected closed geodetic dominating set of G. The minimum cardinality of a weakly* *connected closed geodetic dominating set is called weakly connected closed geodetic domination number of G, denoted by $\gamma_{wcg}(G)$. A weakly connected closed geodetic dominating set S with $|S| = \gamma_{wcg}(G)$ is said to be a γ_{wcg}-set of G.*

Example 1. *Let G be the graph in Figure 1 and $S = \{u_2, u_4, u_6\}$. Then $u_2 \neq u_4$ with $u_6 \notin I_G[u_2, u_4]$ and*

$$I_G[u_2, u_4] = \{u_2, u_4\},$$
$$I_G[u_2, u_6] = \{u_2, u_1, u_6\} \cup \{u_2, u_5, u_6\} = \{u_1, u_2, u_5, u_6\} \text{ and}$$
$$I_G[u_4, u_6] = \{u_4, u_5, u_6\} \cup \{u_4, u_3, u_6\} = \{u_3, u_4, u_5, u_6\}.$$

Thus, $I_G[S] = \{u_1, u_2, u_3, u_4, u_5, u_6\} = V(G)$. Since $u_6 \notin I_G[u_2, u_4]$, $I_G[S]$ is a geodetic closure of S. Also, $N_G[S] = V(G)$, $\langle S\rangle_w$ is connected. In fact, it can be verified that there is no set of lesser cardinality than S that is a weakly connected. Note that $u_6 \notin I_G[u_2, u_4]$. Thus, $S = \{u_2, u_4, u_6\}$ is a weakly connected closed geodetic set and dominating.

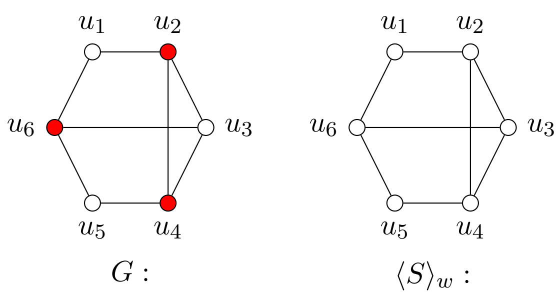

Figure 1: Graph G with $\gamma_{wcg}(G) = 3$

Remark 1. *Every weakly connected closed geodetic dominating set of a graph G is weakly connected closed geodetic set. So, $wcgn(G) \leq \gamma_{wcg}(G)$.*

Remark 2. *For any nontrivial connected graph G of order n,*

$$2 \leq \max\{\gamma(G), wcg(G)\} \leq \gamma_{wcg}(G) \leq n.$$

Remark 3. *Every superset of a weakly connected closed geodetic dominating set is weakly connected closed geodetic dominating set.*

Lemma 1. Aniversario, et al [1] Every geodetic cover of a connected graph G contains all its extreme vertices.

Theorem 1. *Let G be a connected graph of order n. Then*

(*i*) *every weakly connected closed geodetic dominating set of G contains its extreme vertices.*

(*ii*) *if the set S of extreme vertices of G is a weakly connected closed geodetic dominating set of G. Then S is a unique minimum weakly connected closed geodetic dominating set of G and $\gamma_{wcg}(G) = |S|$.*

Proof.

(i) Let S be a weakly connected closed geodetic dominating set and let v be an extreme vertex of G. Assume that $v \notin S$. Then by Lemma 1, S is not a geodetic cover of G. Thus, S is not a closed geodetic dominating set of G. Hence, S is not weakly connected closed geodetic dominating set of G, which is a contradiction. Therefore, each extreme vertex of G belongs to every weakly connected closed geodetic dominating set of G.

(ii) Let S be a set of extreme vertices of G. Suppose S is a weakly connected closed geodetic dominating set of G and let v be an extreme vertex of G.
Claim 1: S is a minimum weakly connected closed geodetic dominating set of G.
Suppose S is not γ_{wcg}-*set* of G. Then there exists $v \in S$ such that $S \setminus \{v\}$ is a weakly connected closed geodetic dominating set of G. So, $v \notin I_G[u, w]$ for some u, $w \in S$ and $v \neq x$, y for all x, $y \in V(G)$ since v is an extreme vertex of G. Then $v \notin V(G)$, which is a contradiction. Consequently, S is a minimum weakly connected closed geodetic dominating set of G.

Claim 2: S is unque minimum weakly connected closed geodetic dominating set of G.
Let S be a unique weakly connected closed geodetic dominating set of G. Then since S contains its extreme vertices and is a minimum weakly connected closed geodetic dominating set of G. Therefore, S is unque minimum weakly connected closed geodetic dominating set of G. Furthermore, unique.

□

The next result immediately follows from Theorem 1.

Corollary 1. *Every weakly connected closed geodetic dominating set of G contains its extreme vertices, then*

(*i*) *the complete graph K_n has $\gamma_{wcg}(K_n) = n$ for $n \geq 2$.*

(*ii*) *the path P_n of order n has $\gamma_{wcg}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

(*iii*) *the cycle C_n of order n has $\gamma_{wcg}(C_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 4$.*

(*iv*) *the complement of a cycle C_n of order n has $\gamma_{wcg}(\overline{C}_n) = 3$ for $n \geq 5$.*

(*v*) *the fan F_n of order n has $\gamma_{wcg}(F_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 4$.*

(*vi*) *the wheel W_n of order n has $\gamma_{wcg}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil$ for $n \geq 5$.*

(*vii*) *the Petersen graph G has $\gamma_{wcg}(G) = 4$.*

Theorem 2. *Let G be a connected graph of order n. Then $\gamma_{wcg}(G) = n$ if and only if $G = K_n$.*

Proof. Suppose that $\gamma_{wcg}(G) = n$. Assume that $G \neq K_n$. Then there exist $x, y \in V(G)$ such that $d_G(x, y) = 2$. Now, construct a set $S = \{v_1, v_2, ..., v_k\}$ where $S \in C^*(G)$ and $x = v_1$ and $y = v_2$. Since $I_G[v_1, v_2] \neq \{v_1, v_2\}$ and $v_i \in I_G[S_{i-1}]$ for all $i = 3, 2, 4, ..., k$, we have $I_G[S] \neq S$. In fact, $I_G[S] = V(G)$. Thus, $k < n$. Moreover, Since $N_G[S] = V(G)$ and $E_w(S)$ of G is induces a connected subgraph, it follows that $\langle S \rangle_w$ is also connected, Furthermore, since $I_G[S] = V(G)$, it follows that S is a dominating set of G. Therefore. $\gamma_{wcg}(G) = n$ which is a contradiction to the assumption. Consequently, $G = K_n$. The converse follows from Corollary 1 (i).

□

Lemma 2. *Let $m, n \geq 2$ and let U and W be the partite sets of $K_{m,n}$. A subset S of $V(K_{m,n})$ is a weakly connected closed geodetic dominating set of $K_{m,n}$ if and only if S is any of the following:*

(i) $S = U$;

(ii) $S = W$;

(iii) $S = U \cup \{w\}$ *for some* $w \in W$;

(iv) $S = W \cup \{u\}$ *for some* $u \in U$.

Theorem 3. *Let $m, n \geq 2$ and let U and W be the partite sets of $K_{m,n}$. Then $\gamma_{wcg}(K_{m,n}) = \min\{|S| : S \in \mathcal{W}(K_{m,n})\}$.*

Proof. Let $m, n \geq 2$ and let U and W be the partite sets of $K_{m,n}$. By Lemma 2, $\gamma_{wcg}(K_{m,n}) = min\{|U|, |W|, |U \cup \{w\}|$ for some $w \in W$ and $|W \cup \{u\}|$ for some $u \in U\}$.

□

Corollary 2. *Let $m, n \geq 2$ and let U and W be the partite sets of $K_{m,n}$. Then $\gamma_{wcg}(K_{m,n}) = min\{m, n\}$.*

Theorem 4. *Let $m, n \geq 2$ and $S \subseteq V(K_{m,n})$. Then S is a γ_{wcg}-set of $K_{m,n}$ if and only if S is wcg-set of $K_{m,n}$.*

Theorem 5. *For the complete bipartite $K_{m,n}$,*

(i) $\gamma_{wcg}(K_{m,n}) = 2$, *for* $m = n = 1$.

(ii) $\gamma_{wcg}(K_{m,n}) = n$, *for* $n \geq 2$, $m = 1$.

(iii) $\gamma_{wcg}(K_{m,n}) = m$, *for* $m \geq 2$, $n = 1$.

Corollary 3. *The star $K_{1,n-1}$ of order n has $\gamma_{wcg}(K_{1,n-1}) = n - 1$.*

Theorem 6. *For a helm H_n, $\gamma_{wcg}(H_n) = n + 1$ for $n \geq 3$.*

Proof. Let u_i be the vertices of a cycle C_n where $n = 1, 2, 3, ..., n$, $v^{'}$ be the center vertex of H_n and v_i be the pendant vertices of H_n where $n = 1, 2, 3, ..., n$. Then every v_i is connected to each vertex u_i. Let $S_1 = \{v_1\}$ and $S_2 = \{v_1, v_3\}$ where $I_{H_n}[S_2] = \{v_1, v_2, v_3\}$. Continuing this process we obtain a set $S_n \in C^*(G)$ where $I_{H_n}[S_n] = V(H_n)$. Therefore S_n is a closed geodetic cover of H_n. However, S_n is not a dominating set of H_n since S_n does not dominate the vertex $v^{'}$. Now, we need to pick $v^{'} \notin S_n$ for $S_{n+1} = \{v^{'}, v_1, ..., v_n\}$. Let $S_1 = \{v^{'}\}$, $S_2 = \{v^{'}, v_2\}$ where $I_{H_n}[S_2] = \{v^{'}, v_1, v_2\}$. Continuing this process we obtain a set $S_{n+1} \in C^*(G)$ where $I_{H_2}[S_{n+1}] = V(H_n)$. Thus, S_{n+1} is both closed geodetic cover and dominating set of H_n. Clearly, $N_G[S_{n+1}] = V(H_n)$ and $E_w(S)$ induces a connected subgraph. It follows that $\langle S \rangle_w$ is connected. Therefore, S_{n+1} is weakly connected closed geodetic dominating set of H_n. Furthermore, $\gamma_{wcg}(H_n) = |S_{n+1}| = n + 1$.

□

The next results present some relationships between $\gamma_{wcg}(G)$, $wcgn(G)$, $\gamma_w(G)$, $\gamma_g(G)$, $\gamma_{gc}(G)$ and $\gamma(G)$.

Theorem 7. *Let G be any connected graph of order $n \geq 2$. Then $\gamma_{wcg}(G) = 2$ if and only if $wcgn(G) = 2$.*

Theorem 8. *If G is a connected graph with $\gamma(G) = 1$, then $\gamma_{wcg}(G) = wcgn(G)$.*

Proposition 1. *For a complete bipartite graph $K_{m,n}$ with integers $m, n \geq 2$,*

$$\gamma_{wcg}(K_{m,n}) = \min\{m, n\} = wcgn(K_{m,n}).$$

Theorem 9. *Let G be a connected graph of order n. Then,*

$$\gamma_g(G) \leq \gamma_{wcg}(G).$$

Proof. Let G be a connected graph. Suppose that $\gamma_{wcg}(G) < \gamma_g(G)$. Let $S = \{v_1, v_2, v_3, ..., v_i\}$ is a γ_g-set of G. Then $\gamma_{wcg}(G) < |S| = \gamma_g(G)$. Hence, by removing an element in S, say v_1 we have $\gamma_{wcg}(G) \leq |S|$. If $|S \setminus \{u_1\}| = \gamma_{wcg}(G)$, then $S \setminus \{u_1\}$ is a geodetic dominating set of G, a contradiction. If $\gamma_{wcg}(G) < |S \setminus \{u_1\}|$, then repeat the process above until we get $|S \setminus \{u_i^{\prime}s\}| = \gamma_{wcg}(G)$. Therefore, $S \setminus \{u_i^{\prime}s\}$ is a geodetic dominating set of G, which is a contradiction. Consequently, we have $\gamma_g(G) \leq \gamma_{wcg}(G)$ in any case.

□

Theorem 10. *Let G be a complete graph K_n for $n \geq 2$, if $G = K_n$, then $\gamma_g(K_n) = \gamma_{wcg}(K_n)$.*

Corollary 4. *The $\gamma_{wcg}(G) = \gamma_w(G)$ for some special graphs given as follows:*

(*i*) *The complement of a cycle C_n of order n has*

$$\gamma_{wcg}(\overline{C}_n) = 3 = \gamma_g(\overline{C}_n) \text{ for } n \geq 5.$$

(ii) The star graph $K_{1,n-1}$ of order n has

$$\gamma_{wcg}(K_{1,n-1}) = n - 1 = \gamma_g(K_{1,n-1}).$$

(iii) The wheel graph W_n has

$$\gamma_{wcg}(W_n) = \lceil \tfrac{n-1}{2} \rceil = \gamma_g(W_n) \text{ for } n \geq 5.$$

Theorem 11. *The complete bipartite $K_{m,n}$ has $\gamma_g(K_{m,n}) \leq \gamma_{wcg}(K_{m,n})$, for $m, n \geq 2$.*

Proposition 2. *Let G be a complete graph K_n for $n \geq 2$ vertices. Then*

$$\gamma_{wcg}(K_n) = \gamma_{gc}(K_n).$$

Proposition 3. *Let G be a path P_n, then $\gamma_{wcg}(P_n) < \gamma_{gc}(P_n)$.*

Theorem 12. *Let G be a cycle C_n, then $\gamma_{wcg}(C_n) \leq \gamma_{gc}(C_n)$ for $n \geq 4$.*

Theorem 13. *Let G be a complete bipartite $K_{m,n}$ for $2 \leq m, n \leq 4$. Then*

$$\gamma_{wcg}(K_{m,n}) \leq \gamma_{gc}(K_{m,n}).$$

Corollary 5. *If G is a complete bipartite $K_{m,n}$ for m, $n \geq 5$. Then*

$$\gamma_{wcg}(K_{m,n}) \geq \gamma_{gc}(K_{m,n}).$$

The join of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G+H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$, Harary [2].

Lemma 3. Aniversario, et.al [1] If G is a connected graph and $diam(G) = 2$, then every geodetic cover of G is a 2-path closure absorbing set in G.

Theorem 14. Aniversario, et.al. [1] Let H be a connected noncomplete graph, and let $G = H + K_p$. Let $S \subseteq V(H)$. If S is a 2-path closure absorbing set in H and $S \in C^*(H)$, then $S \in C^*(G)$.

Theorem 15. *Let H be a connected noncomplete graph and let $G = H + K_p$. If S is a 2-path closure absorbing in H and $S \in \mathcal{W}(H)$, then $S \in \mathcal{W}(G)$.*

Proof. Let H be a connected noncomplete graph and let $G = H + K_p$. Suppose that S is a 2 - path closure absorbing in H and $S \in \mathcal{W}(H)$. Then by Theorem 14, $S \in C^*(G)$. To show that S is a weakly connected dominating set of G. Since G is connected, for every u, $v \in S$, $d_G(u, v) = 2$ and for all $y \in V(K_p)$, y is in u-v geodesic. Thus, $V(\langle S \rangle_w) = V(G)$. It remains to show that for every u, $v \in S$ there is an edge mu or nv with m,$n \in V(G) \setminus S$.

Suppose there exists $x \in S$ such that mx, $nx \notin E_w(S)$ for any m, $n \in V(G) \setminus S$. If m, $n \in V(K_p)$, then mx, $nx \in E_w(S)$. However, if m, $n \in V(H) \setminus S$, then mx, $nx \in E_w(S)$. Further, if without loss of generality, $m \in V(K_p)$ and $n \in V(H) \setminus S$, then mx, $nx \in E_w(S)$. In either case, mx, $nx \in E_w(S)$. Hence, $\langle S\rangle_w$ is connected. It follows that S is a weakly connected set of G. Since every vertex in H is adjacent to every vertex in K_p, there exist $u, v \in S$ such that $d_H(u, v) = 2$ such that every vertex in K_p lies in the $u - v$ geodesic of G and $d_G(u, v) = 2$. It follows that $V(K_p) \subseteq N[S]$. Hence, $V(G) = V(H) \cup V(K_p) \subseteq N[S]$. Hence, S is a dominating set of G. Thus, S is a γ_{wcg} - set, that is $S \in \mathcal{W}(G)$.

□

Theorem 16. *Let H be a connected noncomplete graph and $G = H+K_p$. If S is a γ_{wcg}-set of G, then $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H.*

Corollary 6. *Let H be a connected noncomplete graph and $G = H + K_p$, then*

$$\gamma_{wcg}(H + K_p) = \min\{|S|: S \subseteq V(H), S \in \mathcal{W}(G) \text{ and } P_2[S]_H = V(H)\}.$$

Proof. Define $\omega = \min\{[S] : S \subseteq V(H), S \in \mathcal{W}(G) \text{ and } P_2[S]_H = V(H)\}$.
Case 1. Suppose that H is a connected noncomplete graph and $G = H + K_p$, then $\gamma_{wcg}(G) \leq \omega$.
Case 2. Suppose that $S \in \mathcal{W}(G)$. Let $S \subseteq V(G)$ be a γ_{wcg}-*set* of G. Then by Theorem 16, $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H. Hence $\gamma_{wcg}(G) =|S|\geq \omega$.
Consequently, by combining these two inequalities the conclusion follows.

□

Corollary 7. *Let H be a connected noncomplete graph and let $G = H+K_p$. If $diam(H) = 2$, then $\gamma_{wcg}(G) = \gamma_{wcg}(H)$.*

Proof. Suppose that $G = H + K_p$ where H is a noncomplete graph with $diam(H) = 2$.
Case 1. Let $S \subseteq V(H)$ such that $S \in \mathcal{W}(G)$. Then by Corollary 6, $\gamma_{wcg}(G) =|S|\geq \gamma_{wcg}(H)$.
Case 2. Let S be a $\gamma_{wcg}(G)$-*set* of G. Then by Theorem 16 , $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H. Thus, by Theorem 15, $S \in \mathcal{W}(G)$. Hence, by Corollary 6, $\gamma_{wcg}(G) =|S|\leq \gamma_{wcg}(H)$.
Consequently, combining these two inequalities the conclusion follows.

□

Theorem 17. *Let $G = H + K$ where H and K are connected noncomplete graphs. If S is a γ_{wcg}-set of G, then either*

(i.) $S \subseteq V(H)$, *where S is a 2-path closure absorbing set in H, or*

(ii.) $S \subseteq V(K)$, *where S is a 2-path closure absorbing set in K.*

Proof. Let $G = H + K$ where H and K are connected noncomplete graphs. Suppose $\gamma_{wcg}(G) = k$ and let $S = \{y_1, y_2, ..., y_k\} \in \mathcal{W}(G)$. If $\langle S\rangle$ is a complete subgraph of G and $I_G[S] = V(G)$, then $N_G[S] = V(G)$ and $E_w(S)$ is connected which implies that $\langle S\rangle_w$ is connected. Hence, $V(\langle S\rangle_w) = V(G)$, a contradiction. Thus, there exist integers i, j,

$1 \leq i < j \leq k$ such that $d_G(y_i, y_j) = 2$. Either $y_i, y_j \in V(H)$ or $y_i, y_j \in V(K)$. Suppose $y_i, y_j \in V(H)$. We claim that $S \cap V(K) = \varnothing$. Clearly, $V(K) \subseteq I_G[y_i, y_j]$.

Suppose that $S \cap V(K) = \{z\}$ and let $z = y_l$. Then $l < j$. We consider the set $S^* = \{x_i, x_2, ..., x_{k-1}\}$ where

$$x_n = \begin{cases} y_n, \text{if } 1 \leq n \leq l-1 \\ y_{n+1}, \text{if } l \leq n \leq k-1 \end{cases}.$$

Since $d_G(y_l, y_n) = 1$ for all $n = 1, 2, ..., l-1, l+1, ..., k$, $I_G[S^*] = V(G)$. This implies that $S^* \in C^*(G)$. Since G is connected, for every $x_i, x_j \in S^*$, $d_G(x_i, x_j) = 2$. Then there exists $z \in V(G) \setminus S$ such that z lies in x_i-x_j geodesic. Thus, for $N_G[S^*] = V(G)$ and x_i, $x_j \in E_w(S)$ for all $z \in V(G) \setminus S$. This implies that $S^* \in \mathcal{W}(G)$, contrary to the assumption that $\gamma_{wcg}(G) = k$.

Suppose that $|S \cap V(K)| \geq 2$. In here, we consider two subcases,
Subcase 1. When $d_G(x, y) = 1$ for all x, $y \in S \cap V(K)$; and
Subcase 2. When for some x, $y \in S \bigcap V(K)$, $d_G(x, y) = 2$.
Suppose that $d_G(x, y) = 1$ for all x, $y \in S \cap V(K) = \{y_{r_1}, y_{r_2}, ..., y_{r_l}\}$. Then $r_n < j$ for all $n = 1, 2, ..., l$. We consider the set $S^* = S \cap V(H)$. Write $S^* = \{x_1, x_2, ..., x_{k-l}\}$ such that if $x_n = y_p$ and $x_m = y_q$, then $n < m$ if and only if $p < q$. Since y_i, $y_j \in S^*$, we have for every $n = 1, 2, ..., l$, $I_G[x, y_{r_n}] = \{x, y_{r_n}\} \subseteq I_G[S^*]$ for all $x \in S$. Thus, $I_G[S^*] = I_G[S] = V(G)$. Hence, there exists $z \in V(G) \setminus S^*$ such that z lies in x-y_{r_n} geodesic. Thus, $N_G[S^*] = N_G[S] = V(G)$ and xz, $yz \in E(\langle S \rangle_w)$ for all $z \in V(G) \setminus S^*$. This means that $S^* \in \mathcal{W}(G)$. The fact that k-$l < k$, a contradiction. Lastly, suppose that $d_G(y_m, y_n) = 2$ for some y_m, $y_n \in S \cap V(K)$ with $m < n$. Again, we must have $n < j$. But, if $d_G(y_m, y_n) = 2$, then $V(H) \subseteq I_G[y_m, y_n]$, and in particular, $y_j \in I_G[y_m, y_n]$. But by definition of S, $y_j \notin I_G[S_n]$. It follows that $y_j \notin N_G[S_n]$. Hence, $N_G[S_n] \neq V(G)$. Thus, $S_n \notin \mathcal{W}(G)$, a contradiction.

Now, we are left to show that S is a 2-path closure absorbing in H. Suppose that $S \subseteq V(H)$. By Theorem 16 and Lemma 3, $P_2[S]_G = V(G)$. Let $z \in V(H) \setminus S$. Then $z \in V(G) \setminus S$, and there exist x, $y \in S$ such that $z \in I_G[x, y]$ and $d_G(x, y) = 2$. This implies that $[x, z, y]$ is a x-y geodesic in H. Thus, $z \in I_H[x, y]$ and $d_H(x, y) = 2$. This means that $P_2[S]_H = V(H)$, and so S is a 2-path closure absorbing in H.

Similarly, if y_i, $y_j \in V(K)$, then $S \subseteq V(K)$. Moreover, if $S \subseteq V(K)$, then S is a 2-path closure absorbing in K.

□

Theorem 18. *Let $G = H + K$, where H and K are connected noncomplete graphs. If either*

(i.) $S \subseteq V(H)$, where S is a 2-path closure absorbing set in H and $S \in \mathcal{W}(H)$ or

(ii.) $S \subseteq V(K)$, where S is a 2-path closure absorbing set in K and $S \in \mathcal{W}(K)$,

then $S \in \mathcal{W}(G)$.

Theorem 19. *Let $G = H + K$, where H and K are connected noncomplete graphs. Then $\gamma_{wcg}(G) = \min\{\Gamma(H), \Gamma(K)\}$, where $\Gamma(H) = \min\{|S|: S \subseteq V(H), S \in \mathcal{W}(G)$ and $P_2[S]_H = V(H)\}$ and $\Gamma(K) = \min\{|S|: S \subseteq V(K), S \in \mathcal{W}(G)$ and $P_2[S]_K = V(K)\}$.*

Proof. Let G be a connected graph and let $G = H + K$ where H and K are connected noncomplete graphs. Assume that $S \subseteq V(G)$ is a γ_{wcg}-set of G. Then by Theorem 17, we have $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H or $S \subseteq V(K)$ and S is a 2-path closure absorbing set in K. Hence, $\gamma_{wcg}(G) \geq \min\{\Gamma(H), \Gamma(K)\}$, where $\Gamma(H) = \min\{|S|: S \subseteq V(H), S \in \mathcal{W}(G)$ and $P_2[S]_H = V(H)\}$ and $\Gamma(K) = \min\{|S|: S \subseteq V(K), S \in \mathcal{W}(G)$ and $P_2[S]_K = V(K)\}$. By Theorem 18, $\gamma_{wcg}(G) \leq \min\{\Gamma(H), \Gamma(K)\}$. Consequently, $\gamma_{wcg}(G) = \min\{\Gamma(H), \Gamma(K)\}$.

□

Corollary 8. *The weakly connected closed geodetic domination number of the join graph of the Path graph P_n, cycle graph C_n, and complete bipartite graph $K_{m,n}$ are given as follows.*

(i.) $\gamma_{wcg}(P_m + P_n) = \min\{\lceil \frac{m+1}{2} \rceil, \lceil \frac{n+1}{2} \rceil\}$, *for* $m, n > 2$.

(ii.) $\gamma_{wcg}(C_m + C_n) = \min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$, *for* $m, n > 3$.

(iii.) $\gamma_{wcg}(K_{m,n} + K_p) = \min\{m, n\}$, *for* $m, n > 2$.

(iv.) $\gamma_{wcg}(K_{m,n} + K_{p,q}) = \min\{m, n, p, q\}$, *for* $m, n, p, q \geq 2$.

Corollary 9. *Let H and K are connected noncomplete graphs and $G = H + K$. If $diam(H) = diam(K) = 2$, then $\gamma_{wcg}(G) = \min\{\gamma_{wcg}(H), \gamma_{wcg}(K)\}$.*

The corona of graphs G and H, $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v$, $v \in V(G)$, Harary [2].

Theorem 20. Jamil, et.al [11] Let $G = H \circ K$, where H is a nontrivial connected graph and K a noncomplete graph, and let $S \subseteq V(G)$. Then $S \in C^*(G)$ if and only if $S = (\bigcup_{v \in V(H)} S_v) \cup S_0$, where $S_v \subseteq V(K^v)$ and $S_v \in C^*(v + K^v)$, and S_0 is a closed geodetic subset of $V(H)$.

Lemma 4. *Let $G = H \circ K$, where H is a nontrivial connected graph, and K a noncomplete graph. If $S \in \mathcal{W}(G)$, then $S \cap V(K^v) \in \mathcal{W}(v + K^v)$ for all $v \in V(H)$.*

Lemma 5. *Let $G = H \circ K$, where H is a nontrivial connected graph of order m and K a noncomplete graph. Let $S_v \subseteq V(K^v)$ for all $v \in V(H)$. If $S_v \in \mathcal{W}(v + K^v)$ for each $v \in V(H)$, then $S = \bigcup_{v \in V(H)} S_v \in \mathcal{W}(G)$.*

Theorem 21. *Let $G = H \circ K$, where H is a nontrivial connected graph and K a noncomplete graph, and let $S_v \subseteq V(G)$. Then $S \in \mathcal{W}(G)$ if and only if $S = (\bigcup_{v \in V(H)} S_v) \cup S_0$, where $S_v \subseteq V(K^v)$ and $S_v \in \mathcal{W}(v + K^v)$, and S_0 is a weakly connected closed geodetic subset of $V(H)$.*

Proof. Suppose that $S \in \mathcal{W}(G)$. Then $S \in C^*(G)$. By Theorem 20, $S = (\bigcup_{v \in V(H)} S_v) \cup S_0$. where $S_v \subseteq V(K)$ and $S_v \in C^*(v + K^v)$ and S_0 is a closed geodetic subset. By Lemma 4, $S \cap V(K^v) \in \mathcal{W}(v + K^v)$ for all $v \in V(H)$. Thus, $S_0 = S \setminus \bigcup_{v \in V(H)} S_v$ is a closed geodetic subset is also a weakly connected closed geodetic subset of V(H). It remeains to show that $S_v \in \mathcal{W}(v + K^v)$. That is, S_v is a weakly connected closed geodetic dominating set of $v + K^v$.

Now for any $x, y \in S_v$ there exists $z \in S_v$ such that $xz, yz \in E(v+K^v)$. Thus, $E_w(S_V)$ will induce a connected subgraph since $N[S^v] = V(v + K^v)$, we have $S_v \in \mathcal{W}(v + K^v)$.

Conversely, suppose that $S = (\bigcup_{v \in V(H)} S_v) \cup S_0$. where $S_v \subseteq V(K)$ and $S_v \in \mathcal{W}(v+K^v)$ and S_0 is a weakly connected closed geodetic subset of $V(H)$. By Lemma 5, $\bigcup_{v \in V(H)} S_v \in \mathcal{W}(G)$. If $S_0 = \varnothing$, then we are done. Suppose that $S_0 \neq \varnothing$. By Theorem 20 and and Lemma 3, $S = (\bigcup_{v \in V(H)} S_v) \cup S_0$ gives $S_0 = S \setminus \bigcup_{v \in V(H)}$, where $S \in C^*(G)$ and $\bigcup_{v \in V(H)} \in C^*(G)$ and S_0 is a weakly connected closed geodetic subset of V(H). Thus, for any $x, y \in V(G) \setminus S$ there exists $s \in S$ such that $xs, ys \in E(G)$. Hence, $E_w(S)$ will induce a weakly connected subgraph of G. Therefore, $S = (\bigcup_{v \in V(H)} S_v) \cup S_0 \in \mathcal{W}(G)$.

□

Corollary 10. *Let $G = H \circ K$, where H is a connected graph and K a noncomplete graph. Then S is γ_{wcg}-set of G if and only if $S = \bigcup_{v \in V(H)} S_v$, where each $S_v \subseteq V(v + K^v)$ is γ_{wcg}-set of $v + K^v$.*

Corollary 11. *Let $G = H \circ K$, where H is a connected graph of order m and K a noncomplete graph. Then $\gamma_{wcg}(G) = m \cdot \gamma_{wcg}(K_1 \circ K)$.*

Theorem 22. *Let $G = H \circ K$, where H is a connected graph of order m. If $n \geq 3$, then $\gamma_{wcg}(H \circ C_n) = m \cdot \lceil \frac{n}{2} \rceil$.*

Proof. Let $G = H \circ K$, where H is a connected graph of order m and $K = C_n$ be a noncomplete graph. Then, we have

$$\gamma_{wcg}(H \circ C_n) = m \cdot \gamma_{wcg}(K_1 \circ C_n), \text{ by Corollary 11}$$

$$= m \cdot \gamma_{wcg}(W_{n+1})$$
$$= m \cdot \left\lceil \frac{n+1-1}{2} \right\rceil, \text{ by Corolary 1 (vi)}$$
$$= m \cdot \left\lceil \frac{n}{2} \right\rceil$$

□

Corollary 12. *If $G = P_m \circ C_n$. Then $\gamma_{wcg}(G) = m \cdot \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 3$.*

Theorem 23. *Let $G = H \circ K$, where H is a connected graph of order m. If $n \geq 3$, then $\gamma_{wcg}(H \circ P_n) = m \cdot \lfloor \frac{n+2}{2} \rfloor$.*

Proof. Let $G = H \circ K$, where H is a connected graph of order m and $K = P_n$ be a noncomplete graph. Then, we have

$$\gamma_{wcg}(H \circ P_n) = m \cdot \gamma_{wcg}(K_1 \circ P_n), \text{ by Corollary 11}$$
$$= m \cdot \gamma_{wcg}(F_{n+1})$$
$$= m \cdot \left\lceil \frac{n+1}{2} \right\rceil, \text{ by Corolary 1 (v)}$$

□

Corollary 13. *If $G = C_m \circ P_n$. Then $\gamma_{wcg}(G) = m \cdot \left\lceil \frac{n+1}{2} \right\rceil$, $n \geq 3$.*

Theorem 24. *Let H be a nontrivial connected graph of order n and $K = K_n$. Then $S = \bigcup_{v \in V(H)} V(v + K^v)$ is a γ_{wcg}-set of $H \circ K_n$.*

Proof. Let H be a nontrivial connected graph of order n and $K = K_n$. Suppose that $S = \bigcup_{v \in V(H)} S_v$, $S_v = V(v + K^v)$. By Lemma 5, $S = \bigcup_{v \in V(H)} S_v \in \mathcal{W}(H \circ K_n)$. Then, by Corollary 10, $S = \bigcup_{v \in V(H)} V(v + K^v)$ is a γ_{wcg}-set of $H \circ K_n$.

□

Corollary 14. *Let $G = H \circ K$, where H is a nontrivial connected graph of order m and $K = K_n$ with $n \geq 4$. Then $\gamma_{wcg}(G) = m \cdot (n+1)$.*

The Cartesian product of two graphs G and H, denoted by $G \square H$ is the graph with $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H)$ satisfying the following conditions:

$$(u_1, v_1)(u_2, v_2) \in E(G \square H)$$

if and only if either $u_1 u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$, Harary [2].

Lemma 6. Chellathurai, et.al [6] Let $G = (V, E)$ be the Cartesian product $H\Box K$ of connected graphs $H = (V_1, E_1)$ and $K = (V_2, E_2)$. If $S \subseteq V$, then $I_G[S] \subseteq I_G[S_1]\Box I_G[S_2]$.

Lemma 7. Chellathurai, et.al [6] Let $G = (V, E)$ be the Cartesian product $H\Box K$ of connected graphs $H = (V_1, E_1)$ and $K = (V_2, E_2)$. If $S \subseteq V$,then $N_G[S] \subseteq N_G[S_1]\Box N_G[S_2]$

Lemma 8. *Let H and J be graphs of order m and n respectively, and let $G = H\Box J$ be the Cartesian product of graphs H and J.*

(i.) If $S \subseteq V(H)$ (or $S \subseteq V(J)$), then $V[S \times \{v_i\}] \subseteq V(H_i)(orV(J_i))$ for $v \in J_i$ (or H_i).

(ii.) If $S \subseteq V(H)$ (or $S \subseteq V(J)$) is a γ_{wcg}-set of a graph H (or J), then $V[S \times \{v_i\}]$ is a γ_{wcg}-set of graph H_i (or J_i). But, $V[S \times \{v_i\}]$ is not a γ_{wcg}-set of G.

Remark 4. *Let H and J be graphs of order m and n respectively, and let $H\Box J$ be the Cartesian product of graphs H and J. If $S \subseteq V(H)$ (or $S \subseteq V(J)$) is a γ_{wcg}-set of graphs H (or J), then $S \times \{v_i\}$ is a γ_{wcg}-set of graph H_i (or J_i).*

Theorem 25. *Let H and J be connected graphs. Then*

$$\gamma_{wcg}(H\Box J) \geq \max\{\gamma_{wcg}(H), \gamma_{wcg}(J)\}.$$

Equality holds if H and J are complete graphs.

Proof. Let $S \subseteq V(H\Box J)$ be a γ_{wcg}-set of $H\Box J$. Then by Lemma 6 and 7, $V(H\Box J) = I_G[S] \subseteq I_G[S_1]\Box I_G[S_2]$ and $V(H\Box J) = N_G[S] \subseteq N_G[S_1]\Box N_G[S_2]$. Since G is connected and $N_G[S] = V(H\Box J)$, there exists xv, $vy \in E(H\Box J)$ such that $x \in S$ or $y \in S$ for some $v \in V(H\Box J)$. Hence, $\langle S\rangle_w \subseteq \langle S_1\rangle_w\Box\langle S_2\rangle_w$ is also connected. Thus, S_1 and S_2 are γ_{wcg}-sets of H and J respectively, with $\gamma_{wcg}(H) \leq |S_1|$ and $\gamma_{wcg}(J) \leq |S_2|$. Therefore, $\gamma_{wcg}(H\Box J) = |S| \geq \max\{|S_1|\,|S_2|\} \geq \max\{\gamma_{wcg}(H), \gamma_{wcg}(K)\}$. So, equality holds.

□

Corollary 15. *For every nontrivial connected graph H,*

$$\gamma_{wcg}(H) \leq \gamma_{wcg}(H\Box K_n).$$

Theorem 26. *Let H be a connected graph of order at least 3 and diameter at most 2. Then H has γ_{wcg}-set S with a vertex x such that every vertex of H lies on some u-v geodesic in H for some $w \in S$ and $\langle S\rangle_w = \langle N_H[S], E_w\rangle$ is connected if and only if $\gamma_{wcg}(H) = \gamma_{wcg}(H\Box K_2)$.*

Proof. Let $H\Box K_2$ be formed from two copies H_1 and H_2 of H and S be a minimum weakly connected closed geodetic dominating set of H_1 such that S contains a vertex v with the property that every vertex of H_1 lies on some $u - v$ geodesic in H_1 for some $v \in S$. Let D consists of vertex x together with those vertices of H_2 corresponding to those vertices in $S - \{u\}$. Hence, $|D| = |S|$. We show that D is weakly conneceted closed geodetic dominating set of $H\Box K_2$. Let $x \notin D$ be a vertex of $H\Box K_2$. First, suppose that $x \in V(H_1)$. Since, $I_H[S] = V(H_1)$ and $diam(H_1) \leq 2$, it follows that, $x \in I_H[u, v] = I_H[S]$ and $v \neq x$. Since

$v^{'}$ is the corresponding vertex of $v \in S$, $v^{'} \in D$ and $x \in N[D]$ where $v \neq x$. Also, since $N_H[S] = V(H_1)$ and $E_w = \{uv^{'} \in E(H_1) : u \in S \text{ or } v^{'} \in S\}$ which implies that $\langle S \rangle_w$ is connected, and $diam(H_1) \leq 2$, $x \in N_H[D]$ where $v \neq x$. Therefore, D is a weakly connected closed geodetic dominating set of $H \Box K_2$. Next, suppose that $x \in I_H[u^{'}, v^{'}]$, where $u^{'}$ is the vertex in $V(H_2)$ corresponding to v and $v^{'} \in D$. Since $diam(H_2) \leq 2$, $x \in I_H[u, v^{'}] \subseteq I_H[D]$ and $x \in N_H[v^{'}] \subseteq N_H[D]$, and $N_H[S] = V(H_1)$ and $E_w = \{uv^{'} \in E(H_1) : u \in S \text{ or } v^{'} \in S\}$ which implies that $\langle S \rangle_w$ is connected. Therefore D is a weakly connected closed geodetic dominating set of $H \Box K_2$. Now, $\gamma_{wcg}(H \Box K_2) \leq |D| = |S| = \gamma_{wcg}(H)$. Consequently, by Corollary 15, $\gamma_{wcg}(H) = \gamma_{wcg}(H \Box K_2)$.

Conversely, suppose that $\gamma_{wcg}(H) = \gamma_{wcg}(H \Box K_2)$ where $H \Box K_2$ is formed from two copies of H_1 and H_2 of H. Let D be a minimum weakly connected closed geodetic dominaing set of $H \Box K_2$. Clearly, $D \cap V(H_i) \neq \varnothing$, $i = 1, 2$. Let $x \in D \cap V(H_1)$ and let S consist of vertices of $D \cap V(H_1)$ together with those vertices in $D \cap V(H_2)$. Clearly, S is a weakly connected closed geodetic dominating set of H_1 and $|S| = |D|$. Since, D is a minimum weakly connected closed geodetic dominating set of H_1. We show that every vertex of H_1 lies on some $u - v$ geodesic for some $v \in S$ and $\langle S \rangle_w = \langle N_H[S], E_w \rangle$ is connected. Suppose that there exists a vertex $x \in V(H_1)$ such that $x \in I_H[u, v]$ for all $v \in S$. Then $x \notin N_H[u]$ and $d(u, x) = d(u, v) + d(u, x) > 2$, a contradiction, Consequently, $diam(H_1) \leq 2$.

□

Conlusion: The paper has introduced the concept of weakly connected closed geodetic dominating sets of some graphs and the join, corona, and Cartesian product of two graphs are characterized. The weakly connected closed geodetic domination numbers of these graphs are determined. Also, some relationships between weakly connected closed geodetic dominating set, weakly connected closed geodetic set, geodetic dominating set, and geodetic connected dominating set are established. A worthwhile direction for further investigate is to establish other variations of the concept of the weakly connected closed geodetic dominating sets, the weakly connected closed geodetic sets, geodetic dominating sets, and geodetic connected dominating sets, characterize the weakly connected closed geodetic dominating sets in the lexicographic and composition of two graphs and determine the exact values of the weakly connected closed geodetic domination numbers of graphs associated with the lexicographic and composition of two graphs.

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References

[1] I. Aniversario, F. Jamil, Jr. S. Canoy, The closed Geodetic Numbers of Graph, Utilitas Mathematica, 74 (2007), pp. 3 - 18.

[2] F. Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., USA, 1969.

[3] G. Cagaanan., On Geodetic Covexity in Graphs. Disertation. MSU-Iligan Institute of Technology, March 2004.

[4] G. Chartrand, F. Harary, P. Zhang, Geodetics Sets in Graphs, Discussiones Mathematicae Graph Theory 20 (2000) 129-138

[5] Jr. S. Canoy, G. Cagaanan, S. Gervacio, Covexity, Geodetic, and Hull Numbers of the Join of Graphs. Utilitas Mathematica 71 (2006), To appear.

[6] R. Chellathurai, S. Padma Vijaya, , The Geodetic Domination Number For The Product Of Graphs,Transactions on Combinatorics, ISSN (print): 2251-8657, ISSN (on-line): 2251-8665 Vol. 3 No. 4 (2014), pp. 19-30.

[7] J. Dunbar, J. Grossman, J. Hattingh, S. Hedetniemi, A. McRae, On Weakly Connected Domination in Graphs, Discrete Mathematics 167/168 (1997) 261 -269

[8] W. Duckworth, B. Mans, Connected domination of regular graphs, Discrete Mathematics 309 (2009) 2305–2322

[9] H. Escuadro, R. Gera, A. Hansberg,N. Jafari Rad, and L. Volkmann, Geodetic Domination in Graphs. J. Combin. Math. Combin Comput 77 (2011), 89-101

[10] A. Hansberg, L. Volkmann, On the geodetic and geodetic domination numbers of a graph, Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany, Discrete Mathematics 310 (2010) 2140–2146

[11] F. Jamil,I. Aniversario,Jr. S. Canoy, The Closed Geodetic Numbers of the Corona and Composition of Graphs, MSU - Iligan Institute of Technology 9200 Iligan City, Utilitas Mathematica 74(2007), pp.3-18

[12] R. Patangan, I. Aniversario, and Jr. A. Rosalio, Weakly Connected Closed Geodetic Numbers of Graphs, International Journal of Mathematical Analysis Vol. 10, 2016, no. 6, 257 - 270

[13] E. Sandueta, Jr. S. Canoy, Weakly Connected Domination in Graphs Resulting from Some Graph Operations, International Mathematical Forum, Vol. 6, 2011, no. 21, 1031 - 1035

[14] J. Tarr, S. Suen, Domination in Graphs, Graduate Theses and Dissertations,University of South Florida, Scholar Commons, Graduate Theses and Dissertations, 5-19-2010

[15] K.M. Tejaswini, Venkanagouda M. Goudar, Venkatesha, Geodetic Connected Domination Number of a Graph, Journal of Advances in Mathematics, Vol. 9, No. 7, 2014, 2812-2816