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Some applications of (Λ, sp)-open sets in topological spaces

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Abstract. Our main purpose is to introduce some weak separation axioms by utilizing the concepts of (Λ, sp)-open sets and the (Λ, sp)-closure operator. In particular, some characterizations of (Λ, sp)-R_0 and (Λ, sp)-R_1 topological spaces are investigated.

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1. Introduction

The concept of R_0 topological spaces was first introduced by Shanin [18] In 1961, Davis [7] introduced the concept of a separation axiom called R_1. Dube [9] and Naimpally [15] further investigated characterizations of R_0 topological spaces and several interesting results have been obtained in various contexts. Murdeshwar and Naimpally [14] and Dube [10] studied some of the fundamental properties of R_1 topological spaces. As natural generalizations of the separation axioms R_0 and R_1, the concepts of semi-R_0 and semi-R_1 were introduced and investigated by Maheshwari and Prasad [13] and Dorsett [8]. In [4], the concepts of the (Λ, θ)-closure and (Λ, θ)-open sets were introduced by using the θ-closure operator and θ-open sets due to Velčko [19]. Caldas et al. [5] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 by using the notions of (Λ, θ)-open sets and the (Λ, θ)-closure operator. In 2005, Cammaroto and Noiri [6] introduce a weak separation axiom m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [12]. In 2006, Noiri [16] introduced the notion of m-R_1 spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. Abd El-Monsef et al. [11] introduced a weak form of open sets called β-open sets. This notion was also called semi-preopen sets in the sense of Andrijević [1]. Noiri and Hatir [17] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [2], the author introduced the concepts

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of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. In this paper, introduce some weak separation axioms by utilizing the concepts of (Λ, sp)-open sets and the (Λ, sp)-closure operator. Furthermore, several characterizations of (Λ, sp)-R_0 and (Λ, sp)-R_1 topological spaces are discussed.

2. Preliminaries

We begin with some definitions and known results which will be used throughout this paper. In the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), $\text{Cl}(A)$ and $\text{Int}(A)$ represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [11] if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [17] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset B of a topological space (X, τ) is called a *Λ_{sp}-set* [17] if $B = \Lambda_{sp}(B)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [2] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. The family of all (Λ, sp)-open (resp. (Λ, sp)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$ (resp. $\Lambda_{sp}C(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [2] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [2] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 1. [2] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A^{(\Lambda,sp)} = A$.

Lemma 2. [2] *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ and $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ is (Λ, sp)-open.

(4) A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

3. Characterizations of (Λ, sp)-R_0 topological spaces

In this section, we introduce the notion of (Λ, sp)-R_0 topological spaces. Moreover, several characterizations of (Λ, sp)-R_0 topological spaces are discussed.

Definition 1. *A topological space (X, τ) is called (Λ, sp)-R_0 if, for each (Λ, sp)-open set U and each $x \in U$, $\{x\}^{(\Lambda,sp)} \subseteq U$.*

Theorem 1. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* (Λ, sp)-R_0.

(2) For each (Λ, sp)-closed set F and each $x \in X - F$, there exists $U \in \Lambda_{sp}O(X, \tau)$ such that $F \subseteq U$ and $x \notin U$.

(3) For each (Λ, sp)-closed set F and each $x \in X - F$, $F \cap \{x\}^{(\Lambda,sp)} = \emptyset$.

(4) For any distinct points x, y in X, $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$ or $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$.

Proof. (1) ⇒ (2): Let F be a (Λ, sp)-closed set and let $x \in X - F$. Since (X, τ) is (Λ, sp)-R_0, we have $\{x\}^{(\Lambda,sp)} \subseteq X - F$. Put $U = X - \{x\}^{(\Lambda,sp)}$. Thus, by Lemma 1, $U \in \Lambda_{sp}O(X, \tau)$, $F \subseteq U$ and $x \notin U$.

(2) ⇒ (3): Let F be a (Λ, sp)-closed set and let $x \in X - F$. By (2), there exists $U \in \Lambda_{sp}O(X, \tau)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in \Lambda_{sp}O(X, \tau)$, $U \cap \{x\}^{(\Lambda,sp)} = \emptyset$ and hence $F \cap \{x\}^{(\Lambda,sp)} = \emptyset$.

(3) ⇒ (4): Let x and y be distinct points of X. Suppose that $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} \neq \emptyset$. By (3), $x \in \{y\}^{(\Lambda,sp)}$ and $y \in \{x\}^{(\Lambda,sp)}$. By Lemma 1, $\{x\}^{(\Lambda,sp)} \subseteq \{y\}^{(\Lambda,sp)} \subseteq \{x\}^{(\Lambda,sp)}$ and hence $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$.

(4) ⇒ (1): Let $V \in \Lambda_{sp}O(X, \tau)$ and let $x \in V$. For each $y \notin V$, $V \cap \{y\}^{(\Lambda,sp)} = \emptyset$ and hence $x \notin \{y\}^{(\Lambda,sp)}$. Thus, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. By (4), for each $y \notin V$,

$$\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset.$$

Since $X - V$ is (Λ, sp)-closed, $y \in \{y\}^{(\Lambda,sp)} \subseteq X - V$ and $\cup_{y \in X-V}\{y\}^{(\Lambda,sp)} = X - V$. Thus, $\{x\}^{(\Lambda,sp)} \cap (X - V) = \{x\}^{(\Lambda,sp)} \cap [\cup_{y\in X-V}\{y\}^{(\Lambda,sp)}] = \cup_{y\in X-V}[\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)}] = \emptyset$ and hence $\{x\}^{(\Lambda,sp)} \subseteq V$. This shows that (X, τ) is (Λ, sp)-R_0.

Corollary 1. *A topological space (X, τ) is (Λ, sp)-R_0 if and only if, for any points x and y in X, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$ implies $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$.*

Proof. This is obvious by Theorem 1.

Conversely, let $U \in \Lambda_{sp}O(X, \tau)$ and let $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,sp)} = \emptyset$. Thus, $x \notin \{y\}^{(\Lambda,sp)}$ and $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. By the hypothesis, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$ and hence $y \notin \{x\}^{(\Lambda,sp)}$. This shows that $\{x\}^{(\Lambda,sp)} \subseteq U$. Thus, (X, τ) is (Λ, sp)-R_0.

Definition 2. [3] *Let A be a subset of a topological space (X,τ). A subset $\Lambda_{(\Lambda,sp)}$ is defined as follows:* $\Lambda_{(\Lambda,sp)}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda_{sp}O(X,\tau)\}$.

Lemma 3. [3] *For subsets A, B of a topological space (X,τ), the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,sp)}(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_{(\Lambda,sp)}(A) \subseteq \Lambda_{(\Lambda,sp)}(B)$.

(3) $\Lambda_{(\Lambda,sp)}[\Lambda_{(\Lambda,sp)}(A)] = \Lambda_{(\Lambda,sp)}(A)$.

(4) If A is (Λ, sp)-open, $\Lambda_{(\Lambda,sp)}(A) = A$.

Lemma 4. [3] *Let (X,τ) be a topological space and $x, y \in X$. Then, the following properties hold:*

(1) $y \in \Lambda_{(\Lambda,sp)}(\{x\})$ *if and only if* $x \in \{y\}^{(\Lambda,sp)}$.

(2) $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\})$ *if and only if* $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$.

Theorem 2. *A topological space (X,τ) is (Λ, sp)-R_0 if and only if, for each points x and y in X,* $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$ *implies* $\Lambda_{(\Lambda,sp)}(\{x\}) \cap \Lambda_{(\Lambda,sp)}(\{y\}) = \emptyset$.

Proof. Let (X,τ) be (Λ, sp)-R_0. Suppose that $\Lambda_{(\Lambda,sp)}(\{x\}) \cap \Lambda_{(\Lambda,sp)}(\{y\}) \neq \emptyset$. Let $z \in \Lambda_{(\Lambda,sp)}(\{x\}) \cap \Lambda_{(\Lambda,sp)}(\{y\})$. Then, $z \in \Lambda_{(\Lambda,sp)}(\{x\})$ and by Lemma 4, $x \in \{z\}^{(\Lambda,sp)}$. Thus, $x \in \{z\}^{(\Lambda,sp)} \cap \{x\}^{(\Lambda,sp)}$ and by Corollary 1, $\{z\}^{(\Lambda,sp)} = \{x\}^{(\Lambda,sp)}$. Similarly, we have $\{z\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$ and hence $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$, by Lemma 4,

$$\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\}).$$

Conversely, we show the sufficiency by using Corollary 1. Suppose that

$$\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}.$$

By Lemma 4, $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$ and hence $\Lambda_{(\Lambda,sp)}(\{x\}) \cap \Lambda_{(\Lambda,sp)}(\{y\}) = \emptyset$. Thus, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$. In fact, assume that $z \in \{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)}$. Then, $z \in \{x\}^{(\Lambda,sp)}$ implies $x \in \Lambda_{(\Lambda,sp)}(\{z\})$ and hence $x \in \Lambda_{(\Lambda,sp)}(\{z\}) \cap \Lambda_{(\Lambda,sp)}(\{x\})$. By the hypothesis, $\Lambda_{(\Lambda,sp)}(\{z\}) = \Lambda_{(\Lambda,sp)}(\{x\})$ and by Lemma 4, $\{z\}^{(\Lambda,sp)} = \{x\}^{(\Lambda,sp)}$. Similarly, we have $\{z\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$ and hence $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$. This contradicts that

$$\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}.$$

Thus, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$. This shows that (X,τ) is (Λ, sp)-R_0.

Theorem 3. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) *is* (Λ, sp)-R_0.

(2) $x \in \{y\}^{(\Lambda,sp)}$ *if and only if* $y \in \{x\}^{(\Lambda,sp)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $x \in \{y\}^{(\Lambda,sp)}$. By Lemma 4, $y \in \Lambda_{(\Lambda,sp)}(\{x\})$ and hence $\Lambda_{(\Lambda,sp)}(\{x\}) \cap \Lambda_{(\Lambda,sp)}(\{y\}) \neq \emptyset$. By Theorem 2, $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\})$ and hence $x \in \Lambda_{(\Lambda,sp)}(\{y\})$. Thus, by Lemma 4, $y \in \{x\}^{(\Lambda,sp)}$. The converse is similarly shown.

(2) $\Rightarrow$ (1): Let $U \in \Lambda_{sp}O(X,\tau)$ and let $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,sp)} = \emptyset$. Thus, $x \notin \{y\}^{(\Lambda,sp)}$ and $y \notin \{x\}^{(\Lambda,sp)}$. This implies that $\{x\}^{(\Lambda,sp)} \subseteq U$. Therefore, (X,τ) is (Λ,sp)-R_0.

Theorem 4. *For a topological space* (X,τ)*, the following properties are equivalent:*

(1) (X,τ) *is* (Λ,sp)-R_0.

(2) For each nonempty subset A *of* X *and each* $U \in \Lambda_{sp}O(X,\tau)$ *such that* $A \cap U \neq \emptyset$*, there exists a* (Λ,sp)*-closed set* F *such that* $A \cap F \neq \emptyset$ *and* $F \subseteq U$.

(3) $F = \Lambda_{(\Lambda,sp)}(F)$ *for each* (Λ,sp)*-closed set* F.

(4) $\{x\}^{(\Lambda,sp)} = \Lambda_{(\Lambda,sp)}(\{x\})$ *for each* $x \in X$.

(5) $\{x\}^{(\Lambda,sp)} \subseteq \Lambda_{(\Lambda,sp)}(\{x\})$ *for each* $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let A be a nonempty subset of X and let $U \in \Lambda_{sp}O(X,\tau)$ such that $A \cap U \neq \emptyset$. Then, there exists $x \in A \cap U$ and hence $\{x\}^{(\Lambda,sp)} \subseteq U$. Put $F = \{x\}^{(\Lambda,sp)}$, by Lemma 1, F is (Λ,sp)-closed, $A \cap F \neq \emptyset$ and $F \subseteq U$.

(2) $\Rightarrow$ (3): Let F be any (Λ,sp)-closed set of X. By Lemma 3, we have $F \subseteq \Lambda_{(\Lambda,sp)}(F)$. Next, we show $F \supseteq \Lambda_{(\Lambda,sp)}(F)$. Let $x \notin F$. Then, $x \in X - F \in \Lambda_{sp}O(X,\tau)$ and by (2), there exists a (Λ,sp)-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, $F \subseteq U \in \Lambda_{sp}O(X,\tau)$ and $x \notin U$. Thus, $x \notin \Lambda_{(\Lambda,sp)}(F)$. This shows that $F \supseteq \Lambda_{(\Lambda,sp)}(F)$.

(3) $\Rightarrow$ (4): Let $x \in X$ and let $y \notin \Lambda_{(\Lambda,sp)}(\{x\})$. Then, there exists $U \in \Lambda_{sp}O(X,\tau)$ such that $x \in U$ and $y \notin U$. Thus, $U \cap \{y\}^{(\Lambda,sp)} = \emptyset$. By (3), $U \cap \Lambda_{(\Lambda,sp)}(\{y\}^{(\Lambda,sp)}) = \emptyset$. Since $x \notin \Lambda_{(\Lambda,sp)}(\{y\}^{(\Lambda,sp)})$, there exists $V \in \Lambda_{sp}O(X,\tau)$ such that $\{y\}^{(\Lambda,sp)} \subseteq V$ and $x \notin V$. Thus, $V \cap \{x\}^{(\Lambda,sp)} = \emptyset$. Since $y \in V$, $y \notin \{x\}^{(\Lambda,sp)}$ and hence $\{x\}^{(\Lambda,sp)} \subseteq \Lambda_{(\Lambda,sp)}(\{x\})$. Moreover, $\{x\}^{(\Lambda,sp)} \subseteq \Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \Lambda_{(\Lambda,sp)}(\{x\}^{(\Lambda,sp)}) = \{x\}^{(\Lambda,sp)}$. This shows that $\{x\}^{(\Lambda,sp)} = \Lambda_{(\Lambda,sp)}(\{x\})$.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (1): Let $U \in \Lambda_{sp}O(X,\tau)$ and let $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,sp)} = \emptyset$ and $x \notin \{y\}^{(\Lambda,sp)}$. By Lemma 4, $y \notin \Lambda_{(\Lambda,sp)}(\{x\})$ and by (5), $y \notin \{x\}^{(\Lambda,sp)}$. Thus, $\{x\}^{(\Lambda,sp)} \subseteq U$ and hence (X,τ) is (Λ,sp)-R_0.

Corollary 2. *A topological space* (X,τ) *is* (Λ,sp)-R_0 *if and only if* $\Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \{x\}^{(\Lambda,sp)}$ *for each* $x \in X$.

Proof. This is obvious by Theorem 4.

Conversely, let $x \in \{y\}^{(\Lambda,sp)}$. Thus, by Lemma 4, we have $y \in \Lambda_{(\Lambda,sp)}(\{x\})$ and hence $y \in \{x\}^{(\Lambda,sp)}$. Similarly, if $y \in \{x\}^{(\Lambda,sp)}$, then $x \in \{y\}^{(\Lambda,sp)}$. It follows from Theorem 3 that (X,τ) is (Λ,sp)-R_0.

Definition 3. [3] *Let (X,τ) be a topological space and $x \in X$. A subset $\langle x\rangle_{sp}$ is defined as follows: $\langle x\rangle_{sp} = \Lambda_{(\Lambda,sp)}(\{x\}) \cap \{x\}^{(\Lambda,sp)}$.*

Corollary 3. *A topological space (X,τ) is (Λ,sp)-R_0 if and only if $\langle x\rangle_{sp} = \{x\}^{(\Lambda,sp)}$ for each $x \in X$.*

Proof. Let $x \in X$. By Theorem 4, $\Lambda_{(\Lambda,sp)}(\{x\}) = \{x\}^{(\Lambda,sp)}$. Thus,

$$\langle x\rangle_{sp} = \Lambda_{(\Lambda,sp)}(\{x\}) \cap \{x\}^{(\Lambda,sp)} = \{x\}^{(\Lambda,sp)}.$$

Conversely, let $x \in X$. By the hypothesis,

$$\{x\}^{(\Lambda,sp)} = \langle x\rangle_{sp} = \Lambda_{(\Lambda,sp)}(\{x\}) \cap \{x\}^{(\Lambda,sp)} \subseteq \Lambda_{(\Lambda,sp)}(\{x\}).$$

It follows from Theorem 4 that (X,τ) is (Λ,sp)-R_0.

4. Characterizations of (Λ,sp)-R_1 topological spaces

We begin this section by introducing the notion of (Λ,sp)-R_1 topological spaces.

Definition 4. *A topological space (X,τ) is said to be (Λ,sp)-R_1 if, for each points x,y in X with $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$, there exist disjoint (Λ,sp)-open sets U and V such that $\{x\}^{(\Lambda,sp)} \subseteq U$ and $\{y\}^{(\Lambda,sp)} \subseteq V$.*

Theorem 5. *A topological space (X,τ) is (Λ,sp)-R_1 if and only if, for any points x,y in X with $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$, there exist (Λ,sp)-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.*

Proof. Let x and y be any points in X with $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Then, there exist disjoint $U,V \in \Lambda_{sp}O(X,\tau)$ such that $\{x\}^{(\Lambda,sp)} \subseteq U$ and $\{y\}^{(\Lambda,sp)} \subseteq V$. Now, put $F = X - V$ and $K = X - U$. Then, F and K are (Λ,sp)-closed sets of X such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.

Conversely, let x and y be any points in X such that $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Then, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$. In fact, if $z \in \{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)}$, then $\{z\}^{(\Lambda,sp)} \neq \{x\}^{(\Lambda,sp)}$ or $\{z\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. In case $\{z\}^{(\Lambda,sp)} \neq \{x\}^{(\Lambda,sp)}$, by the hypothesis, there exists a (Λ,sp)-closed set F such that $x \in F$ and $z \notin F$. Then, $z \in \{x\}^{(\Lambda,sp)} \subseteq F$. This contradicts that $z \notin F$. In case $\{z\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$, similarly, this leads to the contradiction. Thus, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$, by Corollary 1, (X,τ) is (Λ,sp)-R_0. By the hypothesis, there exist (Λ,sp)-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$. Put $U = X - K$ and $V = X - F$. Then, $x \in U \in \Lambda_{sp}O(X,\tau)$ and $y \in V \in \Lambda_{sp}O(X,\tau)$. Since (X,τ) is (Λ,sp)-R_0, we have $\{x\}^{(\Lambda,sp)} \subseteq U$, $\{y\}^{(\Lambda,sp)} \subseteq V$ and also $U \cap V = \emptyset$. This shows that (X,τ) is (Λ,sp)-R_1.

Definition 5. [2] *Let A be a subset of a topological space (X,τ). The $\theta(\Lambda,sp)$-closure of A, $A^{\theta(\Lambda,sp)}$, is defined as follows:*

$$A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap\ U^{(\Lambda,sp)} \neq \emptyset \text{ for each } U \in \Lambda_{sp}O(X,\tau) \text{ containing } x\}.$$

Lemma 5. *If a topological space (X,τ) is (Λ,sp)-R_1, then (X,τ) is (Λ,sp)-R_0.*

Proof. Let $U \in \Lambda_{sp}O(X,\tau)$ and let $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,sp)} = \emptyset$ and $x \notin \{y\}^{(\Lambda,sp)}$. Therefore, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Since (X,τ) is (Λ,sp)-R_1, there exists $V \in \Lambda_{sp}O(X,\tau)$ such that $\{y\}^{(\Lambda,sp)} \subseteq V$ and $x \notin V$. Thus, $V \cap \{x\}^{(\Lambda,sp)} = \emptyset$ and hence $y \notin \{x\}^{(\Lambda,sp)}$. Therefore, $\{x\}^{(\Lambda,sp)} \subseteq U$. This shows that (X,τ) is (Λ,sp)-R_0.

Theorem 6. *A topological space (X,τ) is (Λ,sp)-R_1 if and only if $\langle x\rangle_{sp} = \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$.*

Proof. Let (X,τ) be (Λ,sp)-R_1. By Lemma 5, (X,τ) is (Λ,sp)-R_0 and by Corollary 3, $\langle x\rangle_{sp} = \{x\}^{(\Lambda,sp)} \subseteq \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$. Thus, $\langle x\rangle_{sp} \subseteq \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$. In order to show the opposite inclusion, suppose that $y \notin \langle x\rangle_{sp}$. Then, $\langle x\rangle_{sp} \neq \langle y\rangle_{sp}$. Since (X,τ) is (Λ,sp)-R_0, by Corollary 3, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Since (X,τ) is (Λ,sp)-R_1, there exist disjoint (Λ,sp)-open sets U and V of X such that $\{x\}^{(\Lambda,sp)} \subseteq U$ and $\{y\}^{(\Lambda,sp)} \subseteq V$. Since $\{x\} \cap V^{(\Lambda,sp)} \subseteq U \cap V^{(\Lambda,sp)} = \emptyset$, $y \notin \{x\}^{\theta(\Lambda,sp)}$. Thus, $\{x\}^{\theta(\Lambda,sp)} \subseteq \langle x\rangle_{sp}$ and hence $\{x\}^{\theta(\Lambda,sp)} = \langle x\rangle_{sp}$.

Conversely, suppose that $\{x\}^{\theta(\Lambda,sp)} = \langle x\rangle_{sp}$ for each $x \in X$. Then,

$$\langle x\rangle_{sp} = \{x\}^{\theta(\Lambda,sp)} \supseteq \{x\}^{(\Lambda,sp)} \supseteq \langle x\rangle_{sp}$$

and $\langle x\rangle_{sp} = \{x\}^{(\Lambda,sp)}$ for each $x \in X$. By Corollary 3, (X,τ) is (Λ,sp)-R_0. Suppose that $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Thus, by Corollary 1, $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$. By Corollary 3, $\langle x\rangle_{sp} \cap \langle y\rangle_{sp} = \emptyset$ and hence $\{x\}^{\theta(\Lambda,sp)} \cap \{y\}^{\theta(\Lambda,sp)} = \emptyset$. Since $y \notin \{x\}^{\theta(\Lambda,sp)}$, there exists a (Λ,sp)-open set U of X such that $y \in U \subseteq U^{(\Lambda,sp)} \subseteq X - \{x\}$. Let $V = X - U^{(\Lambda,sp)}$, then $x \in V \in \Lambda_{sp}O(X,\tau)$. Since (X,τ) is (Λ,sp)-R_0, $\{y\}^{(\Lambda,sp)} \subseteq U$, $\{x\}^{(\Lambda,sp)} \subseteq V$ and $U \cap V = \emptyset$. This shows that (X,τ) is (Λ,sp)-R_1.

Corollary 4. *A topological space (X,τ) is (Λ,sp)-R_1 if and only if $\{x\}^{(\Lambda,sp)} = \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$.*

Proof. Let (X,τ) be (Λ,sp)-R_1. By Theorem 6, we have

$$\{x\}^{(\Lambda,sp)} \supseteq \langle x\rangle_{sp} = \{x\}^{\theta(\Lambda,sp)} \supseteq \{x\}^{(\Lambda,sp)}$$

and hence $\{x\}^{(\Lambda,sp)} = \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$.

Conversely, suppose that $\{x\}^{(\Lambda,sp)} = \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$. First, we show that (X,τ) is (Λ,sp)-R_0. Let $U \in \Lambda_{sp}O(X,\tau)$ and $x \in U$. Let $y \notin U$. Then,

$$U \cap \{y\}^{(\Lambda,sp)} = U \cap \{y\}^{\theta(\Lambda,sp)} = \emptyset.$$

Thus, $x \notin \{y\}^{\theta(\Lambda,sp)}$. There exists $V \in \Lambda_{sp}O(X,\tau)$ such that $x \in V$ and $y \notin V^{(\Lambda,sp)}$. Since $\{x\}^{(\Lambda,sp)} \subseteq V^{(\Lambda,sp)}$, $y \notin \{x\}^{(\Lambda,sp)}$. This shows that $\{x\}^{(\Lambda,sp)} \subseteq U$ and hence (X,τ) is (Λ,sp)-R_0. By Corollary 3, $\langle x\rangle_{sp} = \{x\}^{(\Lambda,sp)} = \{x\}^{\theta(\Lambda,sp)}$ for each $x \in X$. Thus, by Theorem 6, (X,τ) is (Λ,sp)-R_1.

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