



Contra- (Λ, sp) -continuity and $\delta(\Lambda, sp)$ -closed sets

Chawalit Boonpok¹, Chokchai Viriyapong^{1,*}

¹ *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

Abstract. This paper deals with the concepts of upper and lower contra- (Λ, sp) -continuous multifunctions. Moreover, some characterizations of upper and lower contra- (Λ, sp) -continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: $\delta(\Lambda, sp)$ -closed set, upper contra- (Λ, sp) -continuous multifunction, lower contra- (Λ, sp) -continuous multifunction

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. The concept of contra-continuity was introduced and studied by Dontchev [7]. In 1999, Dontchev and Noiri [9] considered a slightly weaker form of contra-continuity called contra-semicontinuity and investigated the class of strongly S -closed spaces. In 2001, Caldas and Jafari [6] introduced and investigated the concept of contra- β -continuous functions. In 2002, Jafari and Noiri [14] introduced and studied a new form of functions called contra-precontinuous functions. In 2004, Ekici [10] presented and studied almost contra-precontinuity as a new generalization of regular set-connectedness [8], contra-precontinuity [14], contra-continuity [7], almost s -continuity [17] and perfect continuity [16]. In 2005, Nasef [15] defined a new class of functions called contra- γ -continuous functions which lies between classes of contra-semicontinuous functions and contra- β -continuous functions. The first initiation of the concept of contra-continuous multifunctions has been done by Ekici et al. [10]. In 2009, Ekici et al. [11] introduced and studied a new generalization of contra-continuous multifunctions called almost contra-continuous multifunctions. Noiri and Popa [19] introduced and investigated the notion of weakly precontinuous multifunctions. In 2010, Ekici et al. [12] introduced and studied two new concepts namely contra-precontinuous multifunctions and almost contra-precontinuous multifunctions which are containing the

*Corresponding author.

DOI: <https://doi.org/10.29020/nybg.ejpam.v15i4.4370>

Email addresses: chawalit.b@msu.ac.th (C. Boonpok), chokchai.v@msu.ac.th (C. Viriyapong)

class of contra-continuous multifunctions and contained in the class of weakly precontinuous multifunctions. The concept of β -open sets due to Abd El-Monsef et al. [13] or semi-preopen sets in the sense of Andrijević [1] plays a significant role in general topology. Noiri and Hatir [18] introduced the concept of Λ_{sp} -sets in terms of the concept of β -open sets and investigated the notion of Λ_{sp} -closed sets by using Λ_{sp} -sets. In [3], the author introduced the concepts of (Λ, sp) -open sets and (Λ, sp) -closed sets which are defined by utilizing the notions of Λ_{sp} -sets and β -closed sets. Moreover, some characterizations of Λ_{sp} -extremally disconnected spaces are investigated in [3]. The purpose of the present paper is to introduce the notions of upper and lower contra- (Λ, sp) -continuous multifunctions. In particular, several characterizations of upper and lower contra- (Λ, sp) -continuous multifunctions are discussed.

2. Preliminaries

Let A be a subset of a topological space (X, τ) . The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be β -open [13] if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$. The complement of a β -open set is called β -closed. The family of all β -open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [18] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a Λ_{sp} -set [18] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called (Λ, sp) -closed [3] if $A = T \cap C$, where T is a Λ_{sp} -set and C is a β -closed set. The complement of a (Λ, sp) -closed set is called (Λ, sp) -open. The family of all (Λ, sp) -open sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$. Let A be a subset of a topological space (X, τ) . A point $x \in X$ is called a (Λ, sp) -cluster point [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp) -open set U of X containing x . The set of all (Λ, sp) -cluster points of A is called the (Λ, sp) -closure [3] of A and is denoted by $A^{(\Lambda, sp)}$. The union of all (Λ, sp) -open sets contained in A is called the (Λ, sp) -interior [3] of A and is denoted by $A_{(\Lambda, sp)}$.

Lemma 1. [3] *Let A and B be subsets of a topological space (X, τ) . For the (Λ, sp) -closure, the following properties hold:*

- (1) $A \subseteq A^{(\Lambda, sp)}$ and $[A^{(\Lambda, sp)}]_{(\Lambda, sp)} = A^{(\Lambda, sp)}$.
- (2) If $A \subseteq B$, then $A^{(\Lambda, sp)} \subseteq B^{(\Lambda, sp)}$.
- (3) $A^{(\Lambda, sp)} = \cap\{F \mid A \subseteq F \text{ and } F \text{ is } (\Lambda, sp)\text{-closed}\}$.
- (4) $A^{(\Lambda, sp)}$ is (Λ, sp) -closed.
- (5) A is (Λ, sp) -closed if and only if $A = A^{(\Lambda, sp)}$.

Lemma 2. [3] *Let A and B be subsets of a topological space (X, τ) . For the (Λ, sp) -interior, the following properties hold:*

- (1) $A_{(\Lambda, sp)} \subseteq A$ and $[A_{(\Lambda, sp)}]_{(\Lambda, sp)} = A_{(\Lambda, sp)}$.

- (2) If $A \subseteq B$, then $A_{(\Lambda, sp)} \subseteq B_{(\Lambda, sp)}$.
- (3) $A_{(\Lambda, sp)}$ is (Λ, sp) -open.
- (4) A is (Λ, sp) -open if and only if $A_{(\Lambda, sp)} = A$.
- (5) $[X - A]^{(\Lambda, sp)} = X - A_{(\Lambda, sp)}$.
- (6) $[X - A]_{(\Lambda, sp)} = X - A^{(\Lambda, sp)}$.

A subset A of a topological space (X, τ) is called $r(\Lambda, sp)$ -open [3] if $A = [A^{(\Lambda, sp)}]_{(\Lambda, sp)}$. The complement of a $r(\Lambda, sp)$ -open set is said to be $r(\Lambda, sp)$ -closed. The family of all $r(\Lambda, sp)$ -open sets in a topological space (X, τ) is denoted by $r\Lambda_{sp}O(X, \tau)$.

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces and $F : X \rightarrow Y$ (resp. $f : X \rightarrow Y$) presents a multivalued (resp. single valued) function. By a multifunction $F : X \rightarrow Y$, we mean a point-to-set correspondence from X into Y , and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \rightarrow Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then, F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Moreover, $F : X \rightarrow Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [20].

3. Upper and lower contra- (Λ, sp) -continuous multifunctions

In this section, we introduce the notions of upper and lower contra- (Λ, sp) -continuous multifunctions. Moreover, several characterizations of upper and lower contra- (Λ, sp) -continuous multifunctions are discussed.

Definition 1. A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (1) *upper contra- (Λ, sp) -continuous at $x \in X$ if, for each (Λ, sp) -closed set K of Y such that $x \in F^+(K)$, there exists a (Λ, sp) -open set U of X containing x such that $U \subseteq F^+(K)$;*
- (2) *lower contra- (Λ, sp) -continuous at $x \in X$ if, for each (Λ, sp) -closed set K of Y such that $x \in F^-(K)$, there exists a (Λ, sp) -open set U of X containing x such that $U \subseteq F^-(K)$;*
- (3) *upper (resp. lower) contra- (Λ, sp) -continuous if F has this property at each point of X .*

Theorem 1. For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is upper contra- (Λ, sp) -continuous;
- (2) $F^+(K)$ is (Λ, sp) -open in X for every (Λ, sp) -closed set K of Y ;
- (3) $F^-(V)$ is (Λ, sp) -closed in X for every (Λ, sp) -open set V of Y ;
- (4) for each $x \in X$ and each (Λ, sp) -closed set K of Y containing $F(x)$, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that if $y \in U$, then $F(y) \subseteq K$.

Proof. (1) \Leftrightarrow (2): Let K be any (Λ, sp) -closed set of Y and let $x \in F^+(K)$. Since F is upper contra- (Λ, sp) -continuous, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $F(U) \subseteq K$. Thus, $x \in U \subseteq F^+(K)$ and hence $F^+(K)$ is (Λ, sp) -open in X . The converse is similar.

(2) \Leftrightarrow (3): It follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y .

(1) \Leftrightarrow (4): This is obvious.

Theorem 2. For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is lower contra- (Λ, sp) -continuous;
- (2) $F^-(K)$ is (Λ, sp) -open in X for every (Λ, sp) -closed set K of Y ;
- (3) $F^+(V)$ is (Λ, sp) -closed in X for every (Λ, sp) -open set V of Y ;
- (4) for each $x \in X$ and each (Λ, sp) -closed set K of Y such that $F(x) \cap K \neq \emptyset$, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that if $y \in U$, then $F(y) \cap K \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1.

Definition 2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be contra- (Λ, sp) -continuous if, for each $x \in X$ and each (Λ, sp) -closed set K of Y containing $f(x)$, there exists a (Λ, sp) -open set U of X containing x such that $f(U) \subseteq K$.

Corollary 1. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) f is contra- (Λ, sp) -continuous;
- (2) $f^{-1}(K)$ is (Λ, sp) -open in X for every (Λ, sp) -closed set K of Y ;
- (3) $f^{-1}(V)$ is (Λ, sp) -closed in X for every (Λ, sp) -open set V of Y .

Let A be a subset of a topological space (X, τ) . A point $x \in X$ is called a $\delta(\Lambda, sp)$ -cluster point [21] of A if $A \cap [U^{(\Lambda, sp)}]_{(\Lambda, sp)} \neq \emptyset$ for every (Λ, sp) -open set U of X containing x . The set of all $\delta(\Lambda, sp)$ -cluster points of A is called the $\delta(\Lambda, sp)$ -closure [21] of A and is denoted by $A^{\delta(\Lambda, sp)}$. If $A = A^{\delta(\Lambda, sp)}$, then A is said to be $\delta(\Lambda, sp)$ -closed [21]. The complement of a $\delta(\Lambda, sp)$ -closed set is said to be $\delta(\Lambda, sp)$ -open [21]. The union of all $\delta(\Lambda, sp)$ -open sets contained in A is called the $\delta(\Lambda, sp)$ -interior [21] of A and is denoted by $A_{\delta(\Lambda, sp)}$.

Definition 3. A topological space (X, τ) is said to be semi- (Λ, sp) -regular if, for each (Λ, sp) -open set U of X and each $x \in U$, there exists a $r(\Lambda, sp)$ -open set V such that $x \in V \subseteq U$.

Lemma 3. Let (X, τ) be a semi- (Λ, sp) -regular space. Then, the following properties hold:

- (1) $A^{(\Lambda, sp)} = A^{\delta(\Lambda, sp)}$ for every subset A of X .
- (2) Every (Λ, sp) -open set is $\delta(\Lambda, sp)$ -open.

Theorem 3. For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, where (Y, σ) is a semi- (Λ, sp) -regular space, the following properties are equivalent:

- (1) F is upper contra- (Λ, sp) -continuous;
- (2) $F^+(B^{\delta(\Lambda, sp)})$ is (Λ, sp) -open in X for every subset B of Y ;
- (3) $F^+(K)$ is (Λ, sp) -open in X for every $\delta(\Lambda, sp)$ -closed set K of Y ;
- (4) $F^-(V)$ is (Λ, sp) -closed in X for every $\delta(\Lambda, sp)$ -open set V of Y .

Proof. (1) \Rightarrow (2): Let B be any subset of Y . Thus, by Lemma 3, $B^{\delta(\Lambda, sp)}$ is a (Λ, sp) -closed set of Y and by Theorem 1, $F^+(B^{\delta(\Lambda, sp)})$ is (Λ, sp) -open in X .

(2) \Rightarrow (3): Let K be any $\delta(\Lambda, sp)$ -closed set of Y . Then, $K^{\delta(\Lambda, sp)} = K$. By (2), $F^+(K)$ is (Λ, sp) -open in X .

(3) \Rightarrow (4): Let V be any $\delta(\Lambda, sp)$ -open set of Y . Then, $Y - V$ is a $\delta(\Lambda, sp)$ -closed set of Y . By (3), we have $X - F^-(V) = F^+(Y - V)$ is (Λ, sp) -open in X and hence $F^-(V)$ is (Λ, sp) -closed.

(4) \Rightarrow (1): Let V be any (Λ, sp) -open set of Y . Since (Y, σ) is semi- (Λ, sp) -regular, by Lemma 3, V is a $\delta(\Lambda, sp)$ -open set of Y . By (4), $F^-(V)$ is (Λ, sp) -closed in X and by Theorem 1, F is upper contra- (Λ, sp) -continuous.

Theorem 4. For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, where (Y, σ) is a semi- (Λ, sp) -regular space, the following properties are equivalent:

- (1) F is lower contra- (Λ, sp) -continuous;
- (2) $F^-(B^{\delta(\Lambda, sp)})$ is (Λ, sp) -open in X for every subset B of Y ;
- (3) $F^-(K)$ is (Λ, sp) -open in X for every $\delta(\Lambda, sp)$ -closed set K of Y ;

(4) $F^+(V)$ is (Λ, sp) -closed in X for every $\delta(\Lambda, sp)$ -open set V of Y .

Proof. The proof is similar to that of Theorem 3.

Corollary 2. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, where (Y, σ) is a semi- (Λ, sp) -regular space, the following properties are equivalent:

- (1) f is contra- (Λ, sp) -continuous;
- (2) $f^{-1}(B^{\delta(\Lambda, sp)})$ is (Λ, sp) -open in X for every subset B of Y ;
- (3) $f^{-1}(K)$ is (Λ, sp) -open in X for every $\delta(\Lambda, sp)$ -closed set K of Y ;
- (4) $f^{-1}(V)$ is (Λ, sp) -closed in X for every $\delta(\Lambda, sp)$ -open set V of Y .

Definition 4. A subset K of a topological space (X, τ) is called strongly $S\Lambda_{sp}$ -closed (resp. Λ_{sp} -compact) relative to X if every cover of K by (Λ, sp) -closed (resp. (Λ, sp) -open) sets of X has a finite subcover. A topological space (X, τ) is called strongly $S\Lambda_{sp}$ -closed (resp. Λ_{sp} -compact [22]) if X is strongly $S\Lambda_{sp}$ -closed (resp. Λ_{sp} -compact) relative to X .

Theorem 5. Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be an upper contra- (Λ, sp) -continuous surjective multifunction such that $F(x)$ is strongly $S\Lambda_{sp}$ -closed relative to Y for each $x \in X$. If A is Λ_{sp} -compact relative to X , then $F(A)$ is strongly $S\Lambda_{sp}$ -closed relative to Y .

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any cover of $F(A)$ by (Λ, sp) -closed sets of Y . For each $x \in A$, $F(x)$ is strongly $S\Lambda_{sp}$ -closed relative to Y and there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subseteq \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Put $V(x) = \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Then, $F(x) \subseteq V(x)$. Since F is upper contra- (Λ, sp) -continuous, there exists a (Λ, sp) -open $U(x)$ of X containing x such that $F(U(x)) \subseteq V(x)$. Since $\{U(x) \mid x \in A\}$ is a cover of A by (Λ, sp) -open sets of X , there exists a finite number of points of A , say, x_1, x_2, \dots, x_n such that $A \subseteq \cup\{U(x_i) \mid 1 \leq i \leq n\}$. Thus, $F(A) \subseteq F(\cup_{i=1}^n U(x_i)) \subseteq \cup_{i=1}^n F(U(x_i)) \subseteq \cup_{i=1}^n V(x_i) \subseteq \cup_{i=1}^n [\cup_{\alpha \in \nabla(x_i)} V_\alpha]$ and hence $F(A)$ is strongly $S\Lambda_{sp}$ -closed relative to Y .

Corollary 3. Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be an upper contra- (Λ, sp) -continuous surjective multifunction such that $F(x)$ is strongly $S\Lambda_{sp}$ -closed relative to Y for each $x \in X$. If X is Λ_{sp} -compact, then Y is strongly $S\Lambda_{sp}$ -closed.

Corollary 4. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a contra- (Λ, sp) -continuous surjective function and A is Λ_{sp} -compact relative to X , then $f(A)$ is strongly $S\Lambda_{sp}$ -closed relative to Y .

Definition 5. [4] Let A be a subset of a topological space (X, τ) . The (Λ, sp) -frontier of A , denoted by (Λ, sp) -Fr(A), is defined by

$$(\Lambda, sp)\text{-Fr}(A) = A^{(\Lambda, sp)} \cap [X - A]^{(\Lambda, sp)} = A^{(\Lambda, sp)} - A_{(\Lambda, sp)}.$$

Theorem 6. *The set of all points x of X at which a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is not upper contra- (Λ, sp) -continuous is identical with the union of the (Λ, sp) -frontiers of the upper inverse images of (Λ, sp) -closed sets of Y containing $F(x)$.*

Proof. Let $x \in X$ at which F is not upper contra- (Λ, sp) -continuous. Then, there exists a (Λ, sp) -closed set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in \Lambda_{sp}O(X, \tau)$ containing x . Thus, $x \in [X - F^+(V)]^{(\Lambda, sp)}$. On the other hand, we have $x \in F^+(V) \subseteq [F^+(V)]^{(\Lambda, sp)}$ and hence $x \in (\Lambda, sp)\text{-Fr}(F^+(V))$.

Conversely, let V be any (Λ, sp) -closed set of Y containing $F(x)$ such that

$$x \in (\Lambda, sp)\text{-Fr}(F^+(V)).$$

If F is upper contra- (Λ, sp) -continuous at x , then there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $U \subseteq F^+(V)$; hence $x \in [F^+(V)]_{(\Lambda, sp)}$. This is a contradiction and hence F is not upper contra- (Λ, sp) -continuous at x .

Theorem 7. *The set of all points x of X at which a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is not lower contra- (Λ, sp) -continuous is identical with the union of the (Λ, sp) -frontiers of the lower inverse images of (Λ, sp) -closed sets of Y meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 6.

Corollary 5. *The set of all points x of X at which a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is not contra- (Λ, sp) -continuous is identical with the union of the (Λ, sp) -frontiers of the inverse images of (Λ, sp) -closed sets of Y containing $f(x)$.*

Definition 6. [5] *Let A be a subset of a topological space (X, τ) . A subset $\Lambda_{(\Lambda, sp)}(A)$ is defined as follows: $\Lambda_{(\Lambda, sp)}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda_{sp}O(X, \tau)\}$.*

Lemma 4. [5] *For subsets A, B of a topological space (X, τ) , the following properties hold:*

- (1) $A \subseteq \Lambda_{(\Lambda, sp)}(A)$.
- (2) If $A \subseteq B$, then $\Lambda_{(\Lambda, sp)}(A) \subseteq \Lambda_{(\Lambda, sp)}(B)$.
- (3) $\Lambda_{(\Lambda, sp)}[\Lambda_{(\Lambda, sp)}(A)] = \Lambda_{(\Lambda, sp)}(A)$.
- (4) If A is (Λ, sp) -open, $\Lambda_{(\Lambda, sp)}(A) = A$.

Theorem 8. *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction. If $[F^-(B)]^{(\Lambda, sp)} \subseteq F^-(\Lambda_{(\Lambda, sp)}(B))$ for every subset B of Y , then F is upper contra- (Λ, sp) -continuous.*

Proof. Let V be any (Λ, sp) -open set of Y . By Lemma 4,

$$[F^-(V)]^{(\Lambda, sp)} \subseteq F^-(\Lambda_{(\Lambda, sp)}(V)) = F^-(V)$$

and hence $[F^-(V)]^{(\Lambda, sp)} = F^-(V)$. Thus, $F^-(V)$ is (Λ, sp) -closed in X , by Theorem 1, F is upper contra- (Λ, sp) -continuous.

Corollary 6. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. If $[f^{-1}(B)]^{(\Lambda, sp)} \subseteq f^{-1}(\Lambda_{(\Lambda, sp)}(B))$ for every subset B of Y , then f is contra- (Λ, sp) -continuous.

Theorem 9. Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction. If $F(B^{(\Lambda, sp)}) \subseteq \Lambda_{(\Lambda, sp)}(F(B))$ for every subset B of Y , then F is lower contra- (Λ, sp) -continuous.

Proof. Let V be any (Λ, sp) -open set of Y . Then, $F([F^+(V)]^{(\Lambda, sp)}) \subseteq \Lambda_{(\Lambda, sp)}(V)$ and $[F^+(V)]^{(\Lambda, sp)} \subseteq F^+(\Lambda_{(\Lambda, sp)}(V))$. By Lemma 4, $[F^+(V)]^{(\Lambda, sp)} \subseteq F^+(\Lambda_{(\Lambda, sp)}(V)) = F^+(V)$. Thus, $[F^+(V)]^{(\Lambda, sp)} = F^+(V)$ and hence $F^+(V)$ is (Λ, sp) -closed in X , by Theorem 2, F is lower contra- (Λ, sp) -continuous.

Corollary 7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. If $f(B^{(\Lambda, sp)}) \subseteq \Lambda_{(\Lambda, sp)}(f(B))$ for every subset B of Y , then f is contra- (Λ, sp) -continuous.

Definition 7. [3] A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (i) upper (Λ, sp) -continuous if, for each $x \in X$ and each (Λ, sp) -open set V of Y such that $F(x) \subseteq V$, there exists a (Λ, sp) -open set U of X containing x such that $F(U) \subseteq V$;
- (ii) lower (Λ, sp) -continuous if, for each $x \in X$ and each (Λ, sp) -open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp) -open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

Lemma 5. [3] For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is upper (Λ, sp) -continuous;
- (2) $F^+(V)$ is (Λ, sp) -open in X for every (Λ, sp) -open set V of Y ;
- (3) $F^-(K)$ is (Λ, sp) -closed in X for every (Λ, sp) -closed set K of Y ;
- (4) $[F^-(B)]^{(\Lambda, sp)} \subseteq F^-(B^{(\Lambda, sp)})$ for every subset B of Y ;
- (5) $F^+(B_{(\Lambda, sp)}) \subseteq [F^+(B)]_{(\Lambda, sp)}$ for every subset B of Y .

Theorem 10. If $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper (Λ, sp) -continuous multifunction and $G : (Y, \sigma) \rightarrow (Z, \rho)$ is an upper contra- (Λ, sp) -continuous multifunction, then

$$G \circ F : (X, \tau) \rightarrow (Z, \rho)$$

is upper contra- (Λ, sp) -continuous.

Proof. Let V be any (Λ, sp) -closed set of Z . From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$. Since G is upper contra- (Λ, sp) -continuous, by Theorem 1, $G^+(V)$ is (Λ, sp) -open in Y . Since F is upper (Λ, sp) -continuous, by Lemma 5, $F^+(G^+(V))$ is (Λ, sp) -open in X . Thus, $(G \circ F)^+(V)$ is (Λ, sp) -open in X and hence $G \circ F$ is upper contra- (Λ, sp) -continuous.

Lemma 6. [3] For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is lower (Λ, sp) -continuous;
- (2) $F^-(V)$ is (Λ, sp) -open in X for every (Λ, sp) -open set V of Y ;
- (3) $F^+(K)$ is (Λ, sp) -closed in X for every (Λ, sp) -closed set K of Y ;
- (4) $[F^+(B)]^{(\Lambda, sp)} \subseteq F^+(B^{(\Lambda, sp)})$ for every subset B of Y ;
- (5) $F(A^{(\Lambda, sp)}) \subseteq [F(A)]^{(\Lambda, sp)}$ for every subset A of X ;
- (6) $F^-(B_{(\Lambda, sp)}) \subseteq [F^-(B)]_{(\Lambda, sp)}$ for every subset B of Y .

Theorem 11. If $F : (X, \tau) \rightarrow (Y, \sigma)$ is a lower (Λ, sp) -continuous multifunction and

$$G : (Y, \sigma) \rightarrow (Z, \rho)$$

is a lower contra- (Λ, sp) -continuous multifunction, then $G \circ F : (X, \tau) \rightarrow (Z, \rho)$ is lower contra- (Λ, sp) -continuous.

Proof. Let V be any (Λ, sp) -closed set of Z . From the definition of $G \circ F$, we have $(G \circ F)^-(V) = F^-(G^-(V))$. Since G is lower contra- (Λ, sp) -continuous, by Theorem 2, $G^-(V)$ is (Λ, sp) -open in Y . Since F is lower (Λ, sp) -continuous, by Lemma 6, $F^-(G^-(V))$ is (Λ, sp) -open in X . Thus, $(G \circ F)^-(V)$ is (Λ, sp) -open in X and hence $G \circ F$ is lower contra- (Λ, sp) -continuous.

Definition 8. [22] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be (Λ, sp) -continuous at a point $x \in X$ if, for each (Λ, sp) -open set V of Y containing $f(x)$, there exists a (Λ, sp) -open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be (Λ, sp) -continuous if f has this property at each point of X .

Lemma 7. [22] For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) f is (Λ, sp) -continuous;
- (2) $f^{-1}(V)$ is (Λ, sp) -open in X for every (Λ, sp) -open set V of Y ;
- (3) $f(A^{(\Lambda, sp)}) \subseteq [f(A)]^{(\Lambda, sp)}$ for every subset A of X ;
- (4) $[f^{-1}(B)]^{(\Lambda, sp)} \subseteq f^{-1}(B^{(\Lambda, sp)})$ for every subset B of Y ;
- (5) $f^{-1}(B_{(\Lambda, sp)}) \subseteq [f^{-1}(B)]_{(\Lambda, sp)}$ for every subset B of Y ;
- (6) $f^{-1}(F)$ is (Λ, sp) -closed in X for every (Λ, sp) -closed set F of Y .

Corollary 8. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a (Λ, sp) -continuous function and*

$$h : (Y, \sigma) \rightarrow (Z, \rho)$$

is a contra- (Λ, sp) -continuous function, then $h \circ f : (X, \tau) \rightarrow (Z, \rho)$ is contra- (Λ, sp) -continuous.

Proof. Let V be any (Λ, sp) -closed set of Z . We have $(h \circ f)^{-1}(V) = f^{-1}(h^{-1}(V))$. Since h is contra- (Λ, sp) -continuous, by Corollary 1, $h^{-1}(V)$ is (Λ, sp) -open in Y . Since f is (Λ, sp) -continuous, by Lemma 7, $f^{-1}(h^{-1}(V))$ is (Λ, sp) -open in X . Thus, $(h \circ f)^{-1}(V)$ is (Λ, sp) -open in X and hence $h \circ f$ is contra- (Λ, sp) -continuous.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

- [1] D. Andrijević. On b -open sets. *Matematički Vesnik*, 48:59–64, 1996.
- [2] C. Berge. *Espaces topologiques fonctions multivoques*. Dunod, Paris, 1959.
- [3] C. Boonpok. (Λ, sp) -closed sets and related topics in topological spaces. *WSEAS Transactions on Mathematics*, 19:321–322, 2020.
- [4] C. Boonpok and J. Khampakdee. Slight (Λ, sp) -continuity and Λ_{sp} -extremally disconnectedness. *European Journal of Pure and Applied Mathematics*, 15(3):1180–1188, 2022.
- [5] C. Boonpok and C. Viriyapong. On generalized (Λ, sp) -closed sets. (to appear).
- [6] M. Caldas and J. Jafari. Some properties of contra β -continuous functions. *Memoirs of the Faculty of Science, Kochi University. Series A Mathematics*, 22:19–28, 2001.
- [7] J. Dontchev. Contra-continuous functions and strongly S -closed spaces. *International Journal of Mathematics and Mathematical Sciences*, 19(2):303–310, 1996.
- [8] J. Dontchev, M. Ganster, and I. Reilly. More on almost s -continuity. *Indian Journal of Mathematics*, 41:139–146, 1999.
- [9] J. Dontchev and T. Noiri. Contra-semicontinuous functions. *Mathematica Pannonica*, 10(2):159–168, 1999.
- [10] E. Ekici, S. Jafari, and T. Noiri. On upper and lower contra-continuous multifunctions. *Annals of the "Alexandru Ioan Cuza" University of Iași (New Series). Mathematics*, 54(1):75–85, 2008.

- [11] E. Ekici, S. Jafari, and V. Popa. On almost contra-continuous multifunctions. *Lobachevskii Journal of Mathematics*, 30(2):124–131, 2009.
- [12] E. Ekici, S. Jafari, and V. Popa. On contra-precontinuous and almost contra-precontinuous. *Journal of Advanced Research in Pure Mathematics*, 2(1):11–25, 2010.
- [13] M. E. Abd El-Monsef, S. N. El-Deeb, and R. A. Mahmoud. β -open sets and β -continuous mappings. *Bulletin of the Faculty of Science. Assiut University.*, 12:77–90, 1983.
- [14] S. Jafari and T. Noiri. On contra-precontinuous functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 25:115–128, 2002.
- [15] A. A. Nasef. Some properties of contra- γ -continuous functions. *Chaos, Solitons & Fractals*, 24:471–477, 2005.
- [16] T. Noiri. Super-continuity and some strong forms of continuity. *Indian Journal of Pure and Applied Mathematics*, 15:241–250, 1984.
- [17] T. Noiri, B. Ahmad, and M. Khan. Almost s -continuous functions. *Kyungpook Mathematical Journal*, 35:311–322, 1995.
- [18] T. Noiri and E. Hatir. Λ_{sp} -sets and some weak separation axioms. *Acta Mathematica Hungarica*, 103(3):225–232, 2004.
- [19] T. Noiri and V. Popa. Almost weakly continuous multifunctions. *Demonstratio Mathematica*, 26:363–380, 1993.
- [20] V. I. Ponomarev. Properties of topological spaces preserved under multivalued continuous mappings on compacta. *American Mathematical Society Translations*, 38(2):119–140, 1964.
- [21] C. Viriyapong and C. Boonpok. (Λ, sp) -continuity and $\delta(\Lambda, sp)$ -closed sets. *International Journal of Mathematics and Computer Science*, 17(4):1685–1689, 2022.
- [22] C. Viriyapong and C. Boonpok. (Λ, sp) -continuous functions. *WSEAS Transactions on Mathematics*, 21:380–385, 2022.