**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 15, No. 2, 2022, 672-680
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Closed extension topological spaces

Dina Abuzaid[1], Suad Al-Qarhi[1], and Lutfi Kalantan[1,*]

[1] *King Abdulaziz University, Department of Mathematics, P.O.Box 80203, Jeddah 21589, Saudi Arabia.*

**Abstract.** Let $( X , \mathcal{T} )$ be a topological space and $p \notin X$. Put $X^p = X \cup \{ p \}$. Define a topology $\mathcal{T}^\star$ on $X^p$ by $\mathcal{T}^\star = \{ \emptyset \} \cup \{ U \cup \{ p \} : U \in \mathcal{T} \}$. The space $( X^p , \mathcal{T}^\star )$ is called the *closed extension space* of $( X , \mathcal{T} )$. We present new results about the closed extension topological spaces. Mainly weaker versions of normality.

**2020 Mathematics Subject Classifications**: 54A10, 54D35, 54D15

**Key Words and Phrases**: Closed extension, $C$-normality, $L$-normality, $CC$-normality, $S$-normality, $P$-normality, normality, mild normality, almost normality, $\pi$-normality, quasi-normality, partial normality.

We present new results about the closed extension topological spaces. Most of the results are about properties weaker than normality. Some benefits of the closed extension spaces are they work as counterexamples. Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A $T_4$ space is a $T_1$ normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a $T_1$ completely regular space. We do not assume $T_2$ in the definition of compactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset $A$ of a space $X$, int$A$ and $\overline{A}$ denote the interior and the closure of $A$, respectively. If two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a set $X$ are considered, we denote the interior of $A$ in $( X , \mathcal{T} )$ by $\text{int}_{\mathcal{T}} A$ and $\text{int}_{\mathcal{T}'} A$ for the interior of $A$ in $( X , \mathcal{T}' )$. We denote the closure of $A$ in $( X , \mathcal{T}' )$ by $\overline{A}^{\mathcal{T}'}$ and, similarly, $\overline{A}^{\mathcal{T}}$ denotes the closure of $A$ in $( X , \mathcal{T} )$.

## 1. Basic definitions and properties.

**Definition 1.** *Let $( X , \mathcal{T} )$ be a topological space and let $p$ be an object not in $X$, i.e., $p \notin X$. Put $X^p = X \cup \{ p \}$. Define a topology $\mathcal{T}^\star$ on $X^p$ by $\mathcal{T}^\star = \{ \emptyset \} \cup \{ U \cup \{ p \} : U \in \mathcal{T} \}$. The space $( X^p , \mathcal{T}^\star )$ is called the closed extension space* of $( X , \mathcal{T} )$, [15, Example 12].

*Corresponding author.

DOI: https://doi.org/10.29020/nybg.ejpam.v15i2.4379

*Email addresses:* dabuzaid@kau.edu.sa, dina.abuzaid@gmail.com (D. Abuzaid), sabdullahalqarhi@stu.kau.edu.sa and saaadah90@gmail.com (S. Al-Qarhi), lnkalantan@hotmail.com, lkalantan@kau.edu.sa (L. Kalantan)

 

Consider the particular point topology $\mathcal{T}_p$ on $X^p$,[15, Example 9]. So, $\mathcal{T}_p = \{\emptyset\} \cup \{W \subseteq X^p : p \in W\}$. Since any non-empty open set in the closed extension contains $p$, then the closed extension topology $\mathcal{T}^\star$ on $X^p$ is coarser than the particular point topology $\mathcal{T}_p$ on $X^p$. If we start with the discrete topology on $X$, then the closed extension topology $\mathcal{T}^\star$ on $X^p$ and the particular point topology $\mathcal{T}_p$ on $X^p$ will be equal. Thus, from now on, when we consider the closed extension space $(X^p, \mathcal{T}^\star)$ of a given topological space $(X, \mathcal{T})$, $X$ is assumed to have more than one element and the topology $\mathcal{T}$ on $X$ is not the discrete topology. So, in our study we have $|X| \geq 2$ and hence $|X^p| \geq 3$. It is clear that the sub-topology on $X$ inherited from $\mathcal{T}^\star$ equals the original topology on $X$, i.e., $\mathcal{T}^\star_X = \mathcal{T}$. Note that $\{p\}$ is open in the closed extension space, i.e., $\{p\} \in \mathcal{T}^\star$ because $\{p\} = \emptyset \cup \{p\}$, thus $X$ is closed in its closed extension $(X^p, \mathcal{T}^\star)$. Observe that a closed set in $(X^p, \mathcal{T}^\star)$ is of the form $X^p \setminus G$ where $G \in \mathcal{T}^\star$. So, $X^p \setminus G = X^p \setminus (U \cup \{p\})$ where $U \in \mathcal{T}$. Since $p \notin U$ for all $U \in \mathcal{T}$, then the family of all closed sets in $(X, \mathcal{T})$ is equal to the family of all closed sets in its closed extension $(X^p, \mathcal{T}^\star)$ except for $X^p$ itself.

Any closed extension space $(X^p, \mathcal{T}^\star)$ of a given space $(X, \mathcal{T})$ is always separable as $\{p\}$ is dense. If $(X, \mathcal{T})$ is first countable, then so is its closed extension $(X^p, \mathcal{T}^\star)$ because $\{\{p\}\}$ is a countable local base for $X^p$ at $p$ and for any $x \in X$, pick a countable local base $\mathfrak{B}(x) = \{U_n : n \in \mathbb{N}\}$ for $(X, \mathcal{T})$ at $x$, then the countable family $\{U_n \cup \{p\} : n \in \mathbb{N}\}$ is a local base for $(X^p, \mathcal{T}^\star)$ at $x$. Now, if $(X, \mathcal{T})$ is second countable with a countable base $\mathfrak{B} = \{B_n : n \in \mathbb{N}\}$, then the countable family $\{\{p\}, B_n \cup \{p\} : n \in \mathbb{N}\}$ is a base for its closed extension $(X^p, \mathcal{T}^\star)$.

**Remark 1.** *It is clear that the closed extension space $(X^p, \mathcal{T}^\star)$ is always hyper-connected even if $(X, \mathcal{T})$ is not.*

Recall that a space is called *hyper-connected* if any two non-empty open sets intersect, [15]. Thus the closed extension space $(X^p, \mathcal{T}^\star)$ cannot be Hausdorff $(T_2)$ nor metrizable. In fact, the closed extension $(X^p, \mathcal{T}^\star)$ is not $T_1$, even if $(X, \mathcal{T})$ is, because for an element $x \in X$ we have $x \neq p$ and any open set contains $x$ must contain $p$. Thus the closed extension is not $T_i$ where $i \in \{1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 5, 6\}$. Note that the singleton $\{p\}$ in $(X^p, \mathcal{T}^\star)$ is not closed because $X$ is not open in $(X^p, \mathcal{T}^\star)$.

**Theorem 1.** *$(X, \mathcal{T})$ is $T_0$ if and only if its closed extension $(X^p, \mathcal{T}^\star)$ is $T_0$.*

*Proof.* Assume that $(X, \mathcal{T})$ is $T_0$. Let $x, y \in X^p$ be arbitrary such that $x \neq y$. If $x, y \in X$, then $x \neq p \neq y$. Since $(X, \mathcal{T})$ is $T_0$, then there exists $U \in \mathcal{T}$ such that, without loss of generality, $x \in U \not\ni y$. Thus $U \cup \{p\} \in \mathcal{T}^\star$ with $x \in (U \cup \{p\}) \not\ni y$. Now, without loss of generality, assume that $x = p \neq y$, then $\{p\} \in \mathcal{T}^\star$ with $x = p \in \{p\} \not\ni y$. The converse is clear because $T_0$ is hereditary.

The closed extension $(X^p, \mathcal{T}^\star)$ is not regular even if $(X, \mathcal{T})$ is regular because $X$ is closed in $(X^p, \mathcal{T}^\star)$ with $p \notin X$ and $X$ and $p$ cannot be separated by disjoint open sets. Thus the closed extension $(X^p, \mathcal{T}^\star)$ is not completely regular. For the normality, we need to recall the definition of ultra-connected. A space is called *ultra-connected* if any two non-empty closed sets intersect, [15]. It is clear that any ultra-connected space is normal.

**Theorem 2.** $(X, \mathcal{T})$ *is ultra-connected if and only if its closed extension* $(X^p, \mathcal{T}^\star)$ *is normal.*

*Proof.* Assume that $(X, \mathcal{T})$ is ultra-connected. Since the closed sets in the closed extension $(X^p, \mathcal{T}^\star)$ are the same as in $(X, \mathcal{T})$, except for $X^p$, we have that $(X^p, \mathcal{T}^\star)$ is ultra-connected and hence normal.

Now, assume that $(X^p, \mathcal{T}^\star)$ is normal. Suppose that $(X, \mathcal{T})$ is not ultra-connected, then there exist two non-empty disjoint closed sets $A$ and $B$ in $(X, \mathcal{T})$. Thus $A$ and $B$ are non-empty closed disjoint sets in $(X^p, \mathcal{T}^\star)$. Since any two non-empty open sets in $(X^p, \mathcal{T}^\star)$ must intersect because both have the element $p$, see Remark 1, then $A$ and $B$ cannot be separated which gives that $(X^p, \mathcal{T}^\star)$ is not normal and this is a contradiction.

**Theorem 3.** $(X, \mathcal{T})$ *is compact* (*Lindelöf, countably compact*) *if and only if its closed extension* $(X^p, \mathcal{T}^\star)$ *is compact* (*Lindelöf, countably compact*).

*Proof.* We prove the compactness statement and the others are similar.

Assume that $(X, \mathcal{T})$ is compact. Let $\mathcal{W} = \{ W_\alpha \in \mathcal{T}^\star : \alpha \in \Lambda \}$ be any open cover for $X^p$. For each $\alpha \in \Lambda$, there exists $U_\alpha \in \mathcal{T}$ such that $W_\alpha = U_\alpha \cup \{ p \}$. Then the family $\{ U_\alpha : \alpha \in \Lambda \}$ is an open cover for $X$. By the hypothesis, there are $\alpha_1, ..., \alpha_n \in \Lambda$, where $n \in \mathbb{N}$, such that $X \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. Then $\{ W_{\alpha_1}, ..., W_{\alpha_n} \}$ is a finite subcover for $X^p$ of $\mathcal{W}$.

Now, assume that $(X^p, \mathcal{T}^\star)$ is compact. Let $\mathcal{V} = \{ V_\alpha \in \mathcal{T}: \alpha \in \Lambda \}$ be any open cover for $(X, \mathcal{T})$. Then the family $\{ V_\alpha \cup \{ p \} : \alpha \in \Lambda \}$ is an open cover for $X^p$ because $V_\alpha \cup \{ p \} \in \mathcal{T}^\star$ for each $\alpha \in \Lambda$. By the hypothesis, there are $\alpha_1, ..., \alpha_m \in \Lambda$, where $m \in \mathbb{N}$, such that $X^p \subseteq \bigcup_{i=1}^n (V_{\alpha_i} \cup \{ p \})$. Then $\{ V_{\alpha_1}, ..., V_{\alpha_n} \}$ is a finite subcover for $X$ of $\mathcal{V}$.

Observe that the closed extension $(X^p, \mathcal{T}^\star)$ is not locally compact even if $(X, \mathcal{T})$ is because $(X^p, \mathcal{T}^\star)$ is not regular. Since any hyper-connected space is connected, [15], and the closed extension space $(X^p, \mathcal{T}^\star)$ is always hyper-connected, see Remark 1, we conclude that the closed extension space $(X^p, \mathcal{T}^\star)$ is always connected even if $(X, \mathcal{T})$ is disconnected.

## 2. Closed extension and weaker versions of normality.

We begin by recalling some definitions.

**Definition 2.** *A topological space* $X$ *is called* $C$*-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [1]. $X$ is called $CC$*-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$, [6]. $X$ is called $L$*-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$, [10]. $X$ is called $P$*-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$, [9]. $X$ is called $S$*-normal* if there exist a normal space

$Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each separable subspace $A \subseteq X$, [7].

Since characterizing all compact subspaces [1] (all countably compact subspaces [6], all Lindelöf subspaces [10], all paracompact subspaces [9]) is a core subject in the notion of $C$-normality ($CC$-normality, $L$-normality, $P$-normality), we will start with characterizing all compact subspaces of the closed extension space $(X^p, \mathcal{T}^\star)$ of a given space $(X, \mathcal{T})$.

**Proposition 1.** *Let $(X, \mathcal{T})$ be a topological space. Consider the closed extension space $(X^p, \mathcal{T}^\star)$ of $(X, \mathcal{T})$. Let $A \subseteq X^p$.*
*$A$ is compact in $(X^p, \mathcal{T}^\star)$ if and only if $A \setminus \{p\}$ is compact in $(X, \mathcal{T})$.*

*Proof.* Assume that $A$ is compact in $(X^p, \mathcal{T}^\star)$. To show that $A \setminus \{p\}$ is compact in $(X, \mathcal{T})$, let $\mathcal{W} = \{W_\alpha \in \mathcal{T}: \alpha \in \Lambda\}$ be any open cover for $A \setminus \{p\}$. Observe that if $p \notin A$, then $A \setminus \{p\} = A$. The family $\{W_\alpha \cup \{p\} : \alpha \in \Lambda\}$ is an open cover for $A$ in $(X^p, \mathcal{T}^\star)$ because $W_\alpha \cup \{p\} \in \mathcal{T}^\star$ for each $\alpha \in \Lambda$. By the hypothesis, there exist $\alpha_1, ..., \alpha_n \in \Lambda$, where $n \in \mathbb{N}$, such that $A \subseteq \bigcup_{i=1}^{n}(W_{\alpha_i} \cup \{p\})$. Then $\{W_{\alpha_1}, ..., W_{\alpha_n}\}$ is a finite subcover for $A \setminus \{p\}$ of $\mathcal{W}$. Thus $A \setminus \{p\}$ is compact in $(X, \mathcal{T})$.

Now, assume that $A \setminus \{p\}$ is compact in $(X, \mathcal{T})$. Let $\mathcal{V} = \{V_\alpha \in \mathcal{T}^\star : \alpha \in \Lambda\}$ be an arbitrary open cover for $A$. For each $\alpha \in \Lambda$ there exists $U_\alpha \in \mathcal{T}$ such that $V_\alpha = U_\alpha \cup \{p\}$. Then the family $\{U_\alpha : \alpha \in \Lambda\}$ is an open cover for $A \setminus \{p\}$ in $(X, \mathcal{T})$. By the hypothesis, there exist $\alpha_1, ..., \alpha_m \in \Lambda$, where $m \in \mathbb{N}$, such that $A \setminus \{p\} \subseteq \bigcup_{i=1}^{m} U_{\alpha_i}$. Then $\{V_{\alpha_1}, ..., V_{\alpha_m}\}$ is a finite subcover for $A$ of $\mathcal{V}$.

By similar proof of the proof of Proposition 1, we conclude the following two statements.

**Proposition 2.** *Let $(X, \mathcal{T})$ be a topological space. Consider the closed extension space $(X^p, \mathcal{T}^\star)$ of $(X, \mathcal{T})$. Let $A \subseteq X^p$.*
*$A$ is countably compact (Lindelöf) in $(X^p, \mathcal{T}^\star)$ if and only if $A \setminus \{p\}$ is countably compact (Lindelöf) in $(X, \mathcal{T})$.*

Recall that a topological space $(X, \mathcal{T})$ is paracompact if any open cover has a locally finite open refinement. For a subspace $A$ of $X$, $A$ is paracompact if $(A, \mathcal{T}_A)$ is paracompact, i.e., any open (open in the subspace) cover of $A$ has a locally finite open (open in the subspace) refinement. We do not assume $T_2$ in the definition of paracompactness.

**Proposition 3.** *Let $(X, \mathcal{T})$ be a topological space. Consider the closed extension space $(X^p, \mathcal{T}^\star)$ of $(X, \mathcal{T})$. Let $A \subseteq X^p$.*

*If $p \notin A$, then $A$ is a paracompact subset in $(X^p, \mathcal{T}^\star)$ if and only if $A$ is a paracompact subset in $(X, \mathcal{T})$.*

*If $p \in A$, then $A$ is a paracompact subset in $(X^p, \mathcal{T}^\star)$ if and only if $A \setminus \{p\}$ is a compact subset in $(X, \mathcal{T})$.*

*Proof.* If $p \notin A$. We show that $A$ is paracompact subset in $(X^p, \mathcal{T}^\star)$ if and only if $A$ is paracompact subset in $(X, \mathcal{T})$. That is, $(A, \mathcal{T}^\star_A)$ is paracompact if and only if $(A, \mathcal{T}_A)$ is paracompact. Assume that $(A, \mathcal{T}^\star_A)$ is paracompact. Let $\mathcal{W} = \{W_\alpha \cap A : \alpha \in \Lambda\}$ be any open cover for $A$ where $W_\alpha \in \mathcal{T}$ for each $\alpha \in \Lambda$. Note that $(W_\alpha \cup \{p\}) \cap A = W_\alpha \cap A$ because $p \notin A$. Thus $\mathcal{W}$ is an open cover for $A$ (open in $(A, \mathcal{T}^\star_A)$). By the hypothesis, there exists a locally finite open refinement $\mathcal{V} = \{V_s \in \mathcal{T}^\star_A : s \in S\}$ of $\mathcal{W}$. That is, $A \subseteq \bigcup_{s\in S} V_s$ and for each $s \in S$ there exists $\alpha_s \in \Lambda$ such that $V_s \subseteq W_{\alpha_s} \cap A$. Then the family $\{V_s \cap A \in \mathcal{T}_A : s \in S\}$ is a locally finite open refinement of $\mathcal{W}$. Therefore, $(A, \mathcal{T}_A)$ is paracompact.

Now, assume that $(A, \mathcal{T}_A)$ is paracompact. To show that $(A, \mathcal{T}^\star_A)$ is paracompact, note that any open cover in $\mathcal{T}^\star_A$ for $A$ is an open cover in $\mathcal{T}_A$ because $p \notin A$. So, a same argument as above will work.

For the case that $p \in A$. We show that $(A, \mathcal{T}^\star_A)$ is paracompact if and only if $A \setminus \{p\}$ is compact subset in $(X, \mathcal{T})$.

Assume that $(A, \mathcal{T}^\star_A)$ is paracompact. Let $\mathcal{W} = \{W_\alpha \in \mathcal{T} : \alpha \in \Lambda\}$ be any open cover for $A \setminus \{p\}$. That is, $A \setminus \{p\} \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$. Thus $A \subseteq \bigcup_{\alpha\in\Lambda}((W_\alpha \cup \{p\}) \cap A)$ where $(W_\alpha \cup \{p\}) \cap A \in \mathcal{T}^\star_A$ for each $\alpha \in \Lambda$. Since $(A, \mathcal{T}^\star_A)$ is paracompact, then there exists a locally finite family $\mathcal{V} = \{V_s \subseteq A : s \in S\}$ which refines the family $\{(W_\alpha \cup \{p\}) \cap A : \alpha \in \Lambda\}$. i.e., $V_s \in \mathcal{T}^\star_A$ for each $s \in S$, $A \subseteq \bigcup_{s\in S} V_s$, and for each $s \in S$, there exists $\alpha_s \in \Lambda$ such that $V_s \subseteq (W_\alpha \cup \{p\}) \cap A$. Suppose that $S$ is infinite. For each $s \in S$ we have that $V_s$ is of the form $V_s = (U_s \cup \{p\}) \cap A$ where $U_s \in \mathcal{T}$. This means that $p \in V_s$ for each $s \in S$ (do not forget that $p \in A$). Since $\{p\}$ is the smallest open neighborhood of $p$ in the closed extension $(X^p, \mathcal{T}^\star)$, then any open neighborhood of $p$ in $(A, \mathcal{T}^\star_A)$ meets each $V_s$. This means that $\mathcal{V}$ is not locally finite which is a contradiction. Thus $S$ has to be finite, thus $\{U_s : s \in S\}$ is a finite refinement of $\mathcal{W}$. Thus $A \setminus \{p\}$ is compact in $(X, \mathcal{T})$.

Now, assume that $A\setminus\{p\}$ is compact in $(X, \mathcal{T})$. To show that $(A, \mathcal{T}^\star_A)$ is paracompact, let $\mathcal{W} = \{(W_\alpha \cup \{p\}) \cap A : W_\alpha \in \mathcal{T} \text{ for each } \alpha \in \Lambda\}$ be any open cover (open in $\mathcal{T}^\star_A$) for $A$. Then $\{W_\alpha : \alpha \in \Lambda\}$ is an open (open in $\mathcal{T}$) cover for $A \setminus \{p\}$. By the hypothesis, there exist a finite subcover $\{W_{\alpha_1}, ..., W_{\alpha_n}\}$ of $\{W_\alpha : \alpha \in \Lambda\}$ which covers $A \setminus \{p\}$. Now, the family $\{(W_{\alpha_i} \cup \{p\}) \cap A : i \in \{1, ..., n\}\}$ is a finite subcover of $\mathcal{W}$ which covers $A$. Since any subcover is a refinement and any finite family is locally finite, result follows. Therefore, $(A, \mathcal{T}^\star_A)$ is paracompact.

Since any normal space is $C$-normal, $CC$-normal, $L$-normal, $S$-normal, and $P$-normal, just by taking in Definition 2, $Y = X$ and $f$ to be the identity function, then by Theorem 2, we get the following theorem.

**Theorem 4.** *If $(X, \mathcal{T})$ is ultra-connected, then its closed extension $(X^p, \mathcal{T}^\star)$ is $C$-normal ($CC$-normal, $L$-normal, $S$-normal, $P$-normal ).*

Observe that a space $X$ is not ultra-connected if it has two non-empty closed disjoint subsets.

**Theorem 5.** *The closed extension space* $(X^p, \mathcal{T}^\star)$ *is not* $C$*-normal if* $(X, \mathcal{T})$ *is not ultra-connected.*

*Proof.* suppose that $(X^p, \mathcal{T}^\star)$ is $C$-normal. Let $Y$ be a normal space and $f : X^p \longrightarrow Y$ be a bijection such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A$ of $(X^p, \mathcal{T}^\star)$. For the space $Y$, we have only two cases:

**Case 1:** $Y$ is $T_1$. Take $A = \{x, p\}$, where $x \in X$. Then $A$ is a compact subspace of $(X^p, \mathcal{T}^\star)$. By assumption $f_{|_A} : A \longrightarrow f(A) = \{f(x), f(p)\}$ is a homeomorphism. Since $f(A)$ is a finite subspace of $Y$ and $Y$ is $T_1$, then $f(A)$ is a discrete subspace of $Y$. Thus, we obtain that $f_{|_A}$ is not continuous which is a contradiction as $f_{|_A}$ is a homeomorphism.

**Case 2:** $Y$ is not $T_1$. We claim that the topology on $Y$ is coarser than the particular point topology on $Y$ with $f(p)$ as its particular point. To prove this claim, we suppose not. Then there exists a non-empty open set $U \subset Y$ such that $f(p) \notin U$. Pick $y \in U$ and let $x \in X$ be the unique element such that $f(x) = y$. Consider $\{x, p\}$. Note that $x \neq p$ because $f(x) = y \in U$, $f(p) \notin U$, and $f$ is one-to-one. Consider $f_{|_{\{x,p\}}} : \{x, p\} \longrightarrow \{y, f(p)\}$. Now, $\{y\}$ is open in the subspace $\{y, f(p)\}$ of $Y$ because $\{y\} = U \cap \{y, f(p)\}$, but $f^{-1}(\{y\}) = \{x\}$ and $\{x\}$ is not open in the subspace $\{x, p\}$ of $(X^p, \mathcal{T}^\star)$, which means $f_{|_{\{x,p\}}}$ is not continuous. This is a contradiction, and our claim is proved. But any topology coarser than the particular point topology has no disjoint nonempty open sets and therefore cannot be normal , so we get a contradiction as $Y$ is assumed to be normal. Therefore, $(X^p, \mathcal{T}^\star)$ is not $C$-normal.

In [6], it was proved that $CC$-normality implies $C$-normality. In [10], it was proved that $L$-normality implies $C$-normality. In [9], it was proved that $P$-normality implies $C$-normality. So, by Theorem 5, we get the following theorem.

**Theorem 6.** *If* $(X, \mathcal{T})$ *is not ultra-connected, then the closed extension space* $(X^p, \mathcal{T}^\star)$ *is neither* $CC$*-normal,* $L$*-normal, nor* $P$*-normal .*

Now, let us study $S$-normality of the closed extension. We start with characterizing all separable subspaces in the closed extension. Note that $\{p\}$ is a countable dense subset in $(X^p, \mathcal{T}^\star)$, thus any subset of $X^p$ will be separable if it contains $p$. Since a subspace of a subspace is a subspace, we conclude the following characterizing.

**Proposition 4.** *Let* $(X^p, \mathcal{T}^\star)$ *be the closed extension of a topological space* $(X, \mathcal{T})$*. Let* $A \subseteq X^p$*.* $A$ *is separable in* $(X^p, \mathcal{T}^\star)$ *if and only if either* $p \in A$ *or* $A$ *is a separable subspace of* $(X, \mathcal{T})$*.*

**Theorem 7.** *The closed extension* $(X^p, \mathcal{T}^\star)$ *is* $S$*-normal if and only if* $(X, \mathcal{T})$ *is ultra-connected.*

*Proof.* Assume that the closed extension $(X^p, \mathcal{T}^\star)$ is $S$-normal. Pick a normal space $Y$ and a bijection function $f : X^p \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each separable subspace $A \subseteq X^p$. Since $(X^p, \mathcal{T}^\star)$ itself is separable, as $\{p\}$ is a countable dense subset, then $f$ is a homeomorphism. Thus $(X^p, \mathcal{T}^\star)$ is normal and by Theorem 2 we get that $(X, \mathcal{T})$ is ultra-connected.

Now, assume that $(X, \mathcal{T})$ is ultra-connected. By Theorem 2, we have that $(X^p, \mathcal{T}^\star)$ is normal. In Definition 2, put $Y = X^p$ and $f$ is the identity function on $X^p$ to get that the closed extension $(X^p, \mathcal{T}^\star)$ is $S$-normal.

**Definition 3.** *Two disjoint subsets $E$ and $F$ of a space $X$ are called separated* if there exist two disjoint open sets $U$ and $V$ such that $E \subseteq U$ and $F \subseteq V$. A subset $A$ of a space $X$ is called *closed domain* [2, 1.1.C], called also *regularly closed, $\kappa$-closed,* if $A = \overline{\text{int}A}$. A subset $A$ of a space $X$ is called *open domain* [2, 1.1.C], called also *regularly open, $\kappa$-open,* if $A = \text{int}(\overline{A})$. A space $X$ is called *mildly normal* [14], called also *$\kappa$-normal* [16], if any two disjoint closed domains $A$ and $B$ of $X$ are separated. In [16], Ščepin required regularity in his definition of $\kappa$-normality, see also [4, 11]. A space $X$ is called *almost normal* [13] if for two disjoint closed subsets $A$ and $B$ of $X$ one of which is closed domain are separated, see also [8]. A subset $A$ of a space $X$ is called *$\pi$-closed* [17] if $A$ is a finite intersection of closed domains. The complement of a $\pi$-closed set is called *$\pi$-open* [17]. A space $X$ is called *$\pi$-normal* [5] if any two disjoint closed subsets $A$ and $B$ of $X$ one of which is $\pi$-closed are separated. A space $X$ is called *quasi-normal* [17] if any two disjoint $\pi$-closed subsets $A$ and $B$ of $X$ are separated. In [17], Zaitsev required regularity in the definition of quasi-normal. A space $X$ is called *partially normal* if any two disjoint subsets $A$ and $B$ of $X$, where $A$ is closed domain and $B$ is $\pi$-closed, are separated [3].

Since any closed domain is $\pi$-closed and any $\pi$-closed is closed, then it is clear from the definitions that

normal $\Longrightarrow$ $\pi$-normal $\Longrightarrow$ almost normal $\Longrightarrow$ partially normal $\Longrightarrow$ mildly normal.

normal $\Longrightarrow$ $\pi$-normal $\Longrightarrow$ quasi-normal $\Longrightarrow$ partially normal $\Longrightarrow$ mildly normal.

None of the above implications is reversible. By Theorem 2, we conclude the following.

**Theorem 8.** *If $(X, \mathcal{T})$ is ultra-connected, then its closed extension $(X^p, \mathcal{T}^\star)$ is $\pi$-normal, hence quasi-normal, almost normal, partially normal, and hence mildly normal.*

In fact, we will show that any closed extension is $\pi$-normal, hence satisfies all other properties. First, we will study the closed domains in a closed extension space. Let $A = \overline{\text{int}_{\mathcal{T}^\star}(A)}^{\mathcal{T}^\star}$ be any closed domain in a closed extension space $(X^p, \mathcal{T}^\star)$ of a space $(X, \mathcal{T})$. For the subset $\text{int}_{\mathcal{T}^\star}(A)$, we have only two cases, either $\text{int}_{\mathcal{T}^\star}(A) = \emptyset$ or $\text{int}_{\mathcal{T}^\star}(A) \neq \emptyset$. If $\text{int}_{\mathcal{T}^\star}(A) = \emptyset$, then $A = \emptyset$. If $\text{int}_{\mathcal{T}^\star}(A) \neq \emptyset$, then $p \in \text{int}_{\mathcal{T}^\star}(A)$, hence $A = X^p$ because any subset of $X^p$ containing $p$ is dense in $(X^p, \mathcal{T}^\star)$. This means that there are only two closed domains in any closed extension space $(X^p, \mathcal{T}^\star)$ of a space $(X, \mathcal{T})$ and they are $\emptyset$ and $X^p$. Now, since a $\pi$-closed set is a finite intersection of closed domains, then there are only two $\pi$-closed sets in any closed extension space $(X^p, \mathcal{T}^\star)$ of a space $(X, \mathcal{T})$ and they are $\emptyset$ and $X^p$. So, if $A$ and $B$ are closed disjoint subsets in a closed extension space $(X^p, \mathcal{T}^\star)$ of a space $(X, \mathcal{T})$ such that, without loss of generality, $A$ is $\pi$-closed, then either $A = \emptyset$ or $B = \emptyset$, thus $A$ and $B$ are separated. We conclude the following theorem.

**Theorem 9.** *Any closed extension $(X^p, \mathcal{T}^\star)$ space of a given space $(X, \mathcal{T})$ is $\pi$-normal.*

**Corollary 1.** *Any closed extension $(X^p, \mathcal{T}^\star)$ space of a given space $(X, \mathcal{T})$ is quasi-normal, almost normal, partially normal, and mildly normal.*

Recall that a space $X$ is *scattered* if any non-empty subset of $X$ has an isolated point [2], i.e., if $\emptyset \neq A \subseteq X$, then there exists an element $a \in A$ and there exists an open set $U$ such that $a \in U$ and $U \cap A = \{a\}$.

**Theorem 10.** *A space* $(X, \mathcal{T})$ *is scattered if and only if its closed extension* $(X^p, \mathcal{T}^\star)$ *is scattered.*

*Proof.* Assume that $(X, \mathcal{T})$ is scattered. Let $\emptyset \neq A \subseteq X^p$ be arbitrary. There are only two cases. If $p \in A$, then $\{p\} \in \mathcal{T}^\star$ with $\{p\} \cap A = \{p\}$. If $p \notin A$, then $\emptyset \neq A \subseteq X$. Since $(X, \mathcal{T})$ is scattered, then there exists an element $a \in A$ and there exists $U \in \mathcal{T}$ such that $a \in U$ and $U \cap A = \{a\}$. Thus $U \cup \{p\} \in \mathcal{T}^\star$ with $a \in U \cup \{p\}$ and $(U \cup \{p\}) \cap A = \{a\}$. Therefore, $(X^p, \mathcal{T}^\star)$ is scattered. The other direction is true because scattered is hereditary.

Recall that a space $X$ is said to satisfy property $wD$, [12], if for every infinite closed discrete subspace $C$ of $X$, there exists a countably infinite discrete family $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X$ such that each $U_n$ intersects $C$ in exactly one point.

**Proposition 5.** *Let* $(X^p, \mathcal{T}^\star)$ *be the closed extension of a topological space* $(X, \mathcal{T})$*. If* $C \subseteq X$ *is closed and discrete in* $(X, \mathcal{T})$*, then* $C$ *is closed and discrete in* $(X^p, \mathcal{T}^\star)$*. If* $C \subset X^p$ *is closed and discrete in* $(X^p, \mathcal{T}^\star)$*, then* $p \notin C$ *and* $C$ *is closed and discrete in* $(X, \mathcal{T})$*.*

*Proof.* Let $C \subseteq X$ be closed and discrete in $(X, \mathcal{T})$. Since $(X^p, \mathcal{T}^\star)$ and $(X, \mathcal{T})$ have the same closed sets, except for $X^p$, then $C$ is closed in $(X^p, \mathcal{T}^\star)$. Let $c \in C$ be arbitrary, then there exists $U \in \mathcal{T}$ with $c \in U$ and $U \cap C = \{c\}$. Then $U \cup \{p\} \in \mathcal{T}^\star$ with $(U \cup \{p\}) \cap C = \{c\}$. Thus $C$ is discrete in $(X^p, \mathcal{T}^\star)$.

Now, let $C \subset X^p$ be closed and discrete in $(X^p, \mathcal{T}^\star)$. Suppose that $p \in C$, then there are only two cases. If $C = \{p\}$, then $\{p\}$ is not closed in $(X^p, \mathcal{T}^\star)$ because $X \notin \mathcal{T}^\star$. If there exists an element $c \in X$ with $c \in C$, then $C$ will not be discrete in $(X^p, \mathcal{T}^\star)$ because any $W \in \mathcal{T}^\star$ with $c \in W$ is of the form $W = U \cup \{p\}$ for some $U \in \mathcal{T}$ with $c \in U$, thus $W \cap C \neq \{c\}$ because $c \neq p \in W \cap C$. Therefore, $p \notin C$. Now, since $p \notin C$ and $C$ is closed in $(X^p, \mathcal{T}^\star)$, then $C$ is closed in $(X, \mathcal{T})$. Let $c \in C$ be arbitrary. Since $C$ is discrete in $(X^p, \mathcal{T}^\star)$, then there exists $V \in \mathcal{T}^\star$ with $c \in V$ and $V \cap C = \{c\}$. But $V$ is of the form $V = U \cup \{p\}$ for some $U \in \mathcal{T}$ with $c \in U$. Then $U \cap C = \{c\}$ because $p \notin C$.

**Theorem 11.** *Any closed extension* $(X^p, \mathcal{T}^\star)$ *space of a given space* $(X, \mathcal{T})$ *does not satisfy property* $wD$ *even if* $(X, \mathcal{T})$ *does.*

*Proof.* Let $C$ be any infinite closed discrete subspace of $X^p$. By Proposition 5, $p \notin C$. Pick a countably infinite subset $\{c_n : n \in \mathbb{N}\} \subseteq C$. Now, any countably infinite family $\{U_n \in \mathcal{T}^\star : n \in \mathbb{N}\}$ with $c_n \in U_n$ for each $n \in \mathbb{N}$ cannot be discrete because $p \in U_n$ for each $n \in \mathbb{N}$. Therefore, $(X^p, \mathcal{T}^\star)$ does not satisfy property $wD$.

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