EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 261-270
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Distance k-Cost Effective Sets in the Corona and Lexicographic Product of Graphs

Julius G. Caadan[1,*], Rolando N. Paluga[2], Imelda S. Aniversario[3]

[1] *Surigao del Norte State University, 8400 Surigao City, Philippines*
[2] *Department of Mathematics, College of Mathematics and Natural Sciences, Caraga State University , 8600, Ampayon, Butuan City City, Philippines*
[3] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be a connected graph and $k \geq 1$ be an integer. The open k-neighborhood $N_G^k(v)$ of $v \in V(G)$ is the set $N_G^k(v) = \{u \in V(G) \setminus \{v\} : d_G(u,v) \leq k\}$. A set S of vertices of G is called distance k-cost effective of G if for every vertex u in S, $|N_G^k(u) \cap (V(G) \setminus S)| - |N_G^k(u) \cap S| \geq 0$. The maximum cardinality of a distance k-cost effective set of G is called the upper distance k-cost effective number of G. In this paper, we characterized the distance k-cost effective sets in the corona and lexicographic product of two graphs. Consequently, the bounds or the exact values of the upper distance k-cost effective numbers of these graphs are obtained.

2020 Mathematics Subject Classifications: 05C76, 05C12

Key Words and Phrases: Distance k-cost effective set, upper distance k-cost effective number, distance k-domonating set, very distance k-cost effective set, corona, lexicographic product

1. Introduction

Let G be a connected simple graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. The basic concepts of graph here are adapted from [2].

Let $v \in V(G)$. The **open neighborhood** $N_G(v)$ of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The **degree** $\deg_G(v)$ of a vertex $v \in V(G)$ is the cardinality of $N_G(v)$. The **minimum degree** of G is $\delta(G) = \min\{\deg_G(v) : v \in V(G)\}$ and the **maximum degree** of G is $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$. The **distancce** $d_G(u,v)$ between vertices u and v in G is the length of the shortest path from vertex u to vertex v in G. The **diameter** $diam(G)$ of G is the maximum distance between any two vertices in G.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4381

Email addresses: juliusgcaadan@gmail.com (Julius G. Caadan), rnpaluga@carsu.edu.ph (Rolando N. Paluga), imelda.aniversario@g.msuiit.edu.ph (Imelda S. Aniversario)

Let $k \geq 1$ be positive integer and $v \in V(G)$. The **open k-neighborhood** $N_G^k(v)$ of vertex v is the set of all vertices u of G such that $0 < d_G(u,v) \leq k$. That is, $N_G^k(v) = \{u \in V(G) : 0 < d_G(u,v) \leq k\}$. The **distance k-degree** of v in G , denoted by $\deg_G^k(v)$, is the cardinality of $N_G^k(v)$. The minimum distance k-degree of G, denoted by $\delta^k(G)$, is given by $\delta^k(G) = \min\{\deg_G^k(v) : v \in V(G)\}$ and the maximum distance k-degree of G, denoted by $\Delta^k(G)$, is given by $\Delta^k(G) = \max\{\deg_G^k(v); v \in V(G)\}$. Note that $\deg_G^1(v) = \deg_G(v)$, $\delta^1(G) = \delta(G)$, and $\Delta^1(G) = \Delta(G)$.

Let G be a connected graph. Haynes et al. in [7] defined a vertex $v \in S \subseteq V(G)$ as ***cost effective*** if $|N_G(v) \cap (V(G) \setminus S)| - |N_G(v) \cap S| \geq 0$. A set $S \subseteq V(G)$ is called ***cost effective*** if every vertex $v \in S$ is a cost effective. Paluga et al. in [3] applied the distance k version for this concept. Accordingly, a nonempty set $S \subseteq V(G)$ is a **distance k-cost effective** if for every $v \in V(G)$, $|N_G^k(v) \cap (V(G) \setminus S)| - |N_G^k(v) \cap S| \geq 0$. The maximum cardinality of a distance k-cost effective set in G is called ***upper distance k-cost effective number*** of G and is denoted by $\alpha_{ce}^k(G)$. A distance k-cost effective set in G of cardinality $\alpha_{ce}^k(G)$ is called an ***upper distance k-cost effective set*** and is simply called **α_{ce}^k- set** in G. For example, for any integer $n \geq 3$ and if $k = 2$, S is a distance 2-cost effective set in P_n if $|S| \leq \lfloor \frac{2n}{3} \rfloor$. Thus, $\alpha_{ce}^2(P_n) = \lfloor \frac{2n}{3} \rfloor$.

The concept of cost effective set in graph was introduced by Haynes et al. in [7]. In 2018, Chellali et al. in [4] established a generalization of this concept. However, Paluga et al. [3] considered distance concept for the cost effective set. For some investigations of the cost effective concept, we refer the readers to see [6, 9, 11]. For some practical application of distance concept, we refer the readers to [1, 4, 5, 10, 12].

In this paper, we characterized the distance k-cost effective sets in the corona and lexicographic product of two graphs. As direct consequences, we determined the bounds or the exact values of the upper distance k-cost effective numbers of these graphs.

2. Results

2.1. Preliminary Results

In this section, we present a characterization of a distance k-cost effective set in G. Some examples of the upper distance k-cost effective number of simple graphs are given. Moereover, we obtain a relationship between upper distance k-cost effective set and distance k-dominating set in G.

Theorem 1. Let G be a connected simple graph and $k \geq diam(G)$. Then S is a distance k-cost effective set in G if and only if $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Proof: Let G be a connected simple graph and $k \geq diam(G)$. Suppose S is a distance k-cost effective set in G. Then for each $u \in S$,

$$\begin{aligned}|N_G^k(u) \cap (V(G) \setminus S)| - |N_G^k(u) \cap S| &= |\big(V(G) \setminus \{u\}\big) \cap (V(G) \setminus S)| - |\big(V(G) \setminus \{u\}\big) \cap S| \\ &= |V(G)| + 1 - 2|S| \\ &\geq 0.\end{aligned}$$

Thus, $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Conversely, suppose that $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$. Then $|S| \leq \frac{|V(G)|+1}{2}$. Now,

$$\begin{aligned} |N_G^k(u) \cap (V(G) \setminus S)| - |N_G^k(u) \cap S| &= |V(G)| + 1 - 2|S| \\ &\geq |V(G)| + 1 - \big[|V(G)| + 1\big] \\ &= 0. \end{aligned}$$

Thus, S is a distance k-cost effective set in G. □

Corollary 1. Let G be a connected graph and $k \geq diam(G)$. Then $\alpha_{ce}^k(G) = \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Corollary 2. Let G be a connected graph and $k \geq 2$ be an integer. Then

i. $\alpha_{ce}^k(K_n) = \lfloor \frac{n+1}{2} \rfloor$, for positive integer n.

ii. $\alpha_{ce}^k(K_{m.n}) = \lfloor \frac{m+n+1}{2} \rfloor$, for positive integers m and n.

iii. $\alpha_{ce}^k(F_n) = \lfloor \frac{n+2}{2} \rfloor$, for integer $n \geq 3$.

iv. $\alpha_{ce}^k(W_n) = \lfloor \frac{n+2}{2} \rfloor$, for integer $n \geq 4$.

Let G be a connected graph and $k \geq 1$ be an integer. Henning et al. in [8] defined distance k-dominating set of G. Accordingly, a set $S \subseteq V(G)$ is said to be a ***distance k-dominating set*** of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) \leq k$.

Theorem 2. Every upper distance k-cost effective set in a connected graph G is a distance k-dominating set in G.

Proof: Suppose S is an upper distance k-cost effective set in G but not a distance k-dominating set in G. Then there exists $u \in V(G) \setminus S$ such that $d_G(u, s) > k$, for all $s \in S$. Let $A = S \cup \{u\}$ and $x \in A$. Suppose $x \neq u$, i.e., $x \in S$. Note that $u \notin N_G^k(x)$. Then $|N_G^k(x) \cap (V(G) \setminus A)| - |N_G^k(x) \cap A| = |N_G^k(x) \cap (V(G) \setminus S)| - |N_G^k(x) \cap S| \geq 0$. Suppose $x = u$. Then $|N_G^k(x) \cap (V(G) \setminus A)| - |N_G^k(x) \cap A| \geq 0$. Thus, A is a distance k-cost effective set in G. This is a contradiction since S is an upper distance k-cost effective set in G. Therefore, every upper distance k-cost effective set in a connected graph G is a distance k-dominating set in G. □

2.2. Corona of Graphs

This section provides a neccesary condition for a distance k-cost effective set in the corona of two graphs. Correspondingly, a lower bound for the upper distance k-cost effective number of the corona of graphs is determined.

The **corona** $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H

to the ith vertex of G. For every $v \in V(G)$, denote H_v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H_v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H_v, v \in V(G)$.

Theorem 3. Let G and H be connected graphs and $k \geq 1$ be an integer.

(i) If $k = 1, 2$ and S_x is a distance k-cost effective set in H_x, for every $x \in V(G)$, then $S = \bigcup_{x \in V(G)} S_x$ is a distance k-cost effective set in $G \circ H$.

(ii) If $k \geq 3$ and $|S_x| \leq \frac{|V(H)|\left(\delta^{k-2}(G)+1\right)+\delta^{k-1}(G)+2}{2\left(\Delta^{k-2}(G)+1\right)}$, for every $x \in V(G)$, then $S = \bigcup_{x \in V(G)} S_x$ is a distance k-cost effective set in $G \circ H$.

Proof: Suppose S_x is a distance k-cost effective set in H_x, for every $x \in V(G)$. Let $S = \bigcup_{x \in V(G)} S_x$ and $u \in S$. Then there exists $a \in V(G)$ such that $u \in S_a$. Since S_a is a distance k-cost effective set in H_a, for all $a \in V(G)$, $|N^k_{H_a}(u) \cap (V(H_a) \setminus S_a)| - |N^k_{H_a}(u) \cap S_a| \geq 0$. if $k = 1$, we have

$$|N^1_{G \circ H}(u) \cap \big(V(G \circ H) \setminus S\big)| = |N^1_{H_a}(u)\big(V(H_a) \setminus S_a\big)| + 1$$

and

$$|N^1_{G \circ H}(u) \cap S| = |N^1_{H_a}(u) \cap S_a|$$

Thus, $|N^1_{G \circ H}(u) \cap \big(V(G \circ H) \setminus S\big)| - |N^1_{G \circ H}(u) \cap S| \geq 0$. Hence, S is a distance 1-cost effective set in $G \circ H$.

If $k = 2$, then

$$|N^2_{G \circ H}(u) \cap (V(G \circ H) \setminus S)| = |N^2_{H_a}(u) \cap \big(V(H_a) \setminus S_a\big)| + \deg_G(a) + 1$$

and

$$|N^2_{G \circ H}(u) \cap S| = |N^2_{H_a}(u) \cap S_a|.$$

Thus, $|N^2_{G \circ H}(u) \cap (V(G \circ H) \setminus S)| - |N^2_{G \circ H}(u) \cap S| \geq 0$. Hence, S is a distance 2-cost effective set in $G \circ H$.

(ii) Let $k \geq 3$ be an integer and $u \in S$. Then there exists $a \in V(G)$ such that $u \in S_a$. Now,

$$\begin{aligned} |N^k_{G \circ H}(u) \cap \big(V(G \circ H) \setminus S\big)| - |N^k_{G \circ H}(u) \cap S| &= \Big[deg^{k-1}_G(a) + 1\Big] + |V(H) \setminus S_a| \\ &\quad + \sum_{x \in N^{k-2}_G(a)} \Big|\Big(V(H) \setminus S_x\Big)\Big| - \sum_{x \in N^{k-2}_G(a)} |S_x| - (|S_a| - 1) \\ &= \Big[\deg^{k-1}_G(a) + 1\Big] + |V(H)| - |S_a| + \sum_{x \in N^{k-2}_G(a)} \Big(|V(H)| - |S_x|\Big) \\ &\quad - \sum_{x \in N^{k-2}_G(a)} |S_x| - |S_a| + 1 \end{aligned}$$

$$
\begin{aligned}
&= \deg_G^{k-1}(a) + 2 + |V(H)| - 2|S_a| + \sum_{x\in N_G^{k-2}(a)} \Big(|V(H)| - 2|S_x|\Big)\\
&\geq \deg_G^{k-1}(a) + 2 + |V(H)| - 2|S_p| + |N_G^{k-2}(a)||V(H)|\\
&\qquad - 2|N_G^{k-2}(a)||S_p|, \text{where } |S_p| = \max\{|S_x| : x \in V(G)\}\\
&= \deg_G^{k-1}(a) + 2 + |V(H)| + \deg_G^{k-2}(a)|V(H)| - 2|S_p|\\
&\qquad - 2\deg_G^{k-2}(a)||S_p|\\
&= \deg_G^{k-1}(a) + 2 + |V(H)|\big(\deg_G^{k-2}(a)+1\big) - 2\big(\deg_G^{k-2}(a)+1\big)|S_p|\\
&\geq \deg_G^{k-1}(a) + 2 + |V(H)|\big(\deg_G^{k-2}(a)+1\big) \qquad -2\big(\deg_G^{k}\\
&\geq \deg_G^{k-1}(a) + 2 + +|V(H)|\big(\deg_G^{k-2}(a)+1\big)\\
&\qquad - \big(\deg_G^{k-2}(a)+1\big)\Big[\frac{|V(H)|\big(\deg_G^{k-2}(a)+1\big) + \deg_G^{k-1}(a)+2}{\big(\deg_G^{k-2}(a)+1\big)}\Big]\\
&\qquad - \Big[|V(H)|\big(\deg_G^{k-2}(a)+1\big) + \deg_G^{k-1}(a)+2\Big]\\
&= 0.
\end{aligned}
$$

Thus, S is a distance k-cost effective set in $G \circ H$. □

Corollary 3. Let G and H be connected graphs and $k \geq 1$ be an integer. Then

(i) $\alpha_{ce}^k(G \circ H) \geq |V(G)|\alpha_{ce}^k(H)$, for $k = 1, 2$.

(ii) $\alpha_{ce}^k(G \circ H) \geq |V(G)|\frac{|V(H)|\big(\delta^{k-2}(G)+1\big)+\delta^{k-1}(G)+2}{2\big(\Delta^{k-2}(G)+1\big)}$, for $k \geq 3$.

2.3. Lexicographic Product of Graphs

This section provides a neccesary condition for a distance k-cost effective set in the lexicographic product of two graphs. Consequently, a lower bound for the upper distance k-cost effective number of this graph is given.

The **lexicographic product** $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Let $(u, v) \in S$ and $k \geq 1$ be an integer. Then

$$
|N_{G[H]}^k(u,v) \cap (V(G[H]) \setminus S)| = |N_{H_u}^k(v) \cap (V(H) \setminus T_u)| + |N_G^k(u) \cap (V(G) \setminus A)||V(H)| + \sum_{x\in N_G^k(u)\cap A} |V(H) \setminus T_x| \quad (1)
$$

and

$$|N^k_{G[H]}(u,v) \cap S| = \sum_{x \in N^k_G(u) \cap A} |T_x| + |N^k_{H_u}(v) \cap T_u|. \tag{2}$$

Theorem 4. Let G and H be connected graphs and $k \geq 2$ be an integer. Let A be an α^k_{ce}-set in G. Let $S = \bigcup_{a \in A}(\{a\} \times T_a)$ such that $|\{a\} \times T_a| \leq \frac{|V(H)|+1}{2}$, for each $a \in A$. Then S is a distance k-cost effective set in $G[H]$.

Proof: Let $k \geq 2$ be an integer and A be an α^k_{ce}-set in G. Let $S = \bigcup_{a \in A}(\{a\} \times T_a)$ such that $|\{a\} \times T_a| \leq \frac{|V(H)|+1}{2}$, for each $a \in A$.

Let $(u,v) \in S$. Then using equations (1) and (2), we have

$$\begin{aligned} |N^k_{G[H]}(u,v) \cap S| &= \sum_{x \in N^k_G(u) \cap A} |T_x| + |N^k_{H_u}(v) \cap T_u| \\ &= \sum_{x \in N^k_G(u) \cap A} |T_x| + |T_u| - 1. \end{aligned}$$

and

$$\begin{aligned} |N^k_{G[H]}(u,v) \cap (V(G[H]) \setminus S)| =& |N^k_{H_u}(v) \cap (V(H) \setminus T_u)| + |N^k_G(u) \cap (V(G) \setminus A)||V(H)| \\ &+ \sum_{x \in N^k_G(u) \cap A} |V(H) \setminus T_x| \\ =& |V(H)| - |T_u| + |N^k_G(u) \cap (V(G) \setminus A)||V(H)| \\ &+ \sum_{x \in N^k_G(u) \cap A} \big[|V(H)| - |T_x|\big]. \end{aligned}$$

Hence,

$$\begin{aligned} |N^k_{G[H]}(u,v) \cap (V(G[H]) \setminus S)| - |N^k_{G[H]}(u,v) \cap S| &= |N^k_G(u) \cap (V(G) \setminus A)||V(H)| \\ &+ \sum_{x \in N^k_G(u) \cap A} \Big[|V(H)| - 2|T_x|\Big] + |V(H)| - 2|T_u| + 1 \\ &\geq |N^k_G(u) \cap (V(G) \setminus A)||V(H)| + \sum_{x \in N^k_G(u) \cap A} \Big[|V(H)| - 2|V(H)|\Big] \\ &+ |V(H)| - 2|T_u| + 1 \\ &= |N^k_G(u) \cap (V(G) \setminus A)||V(H)| - |N^k_G(u) \cap A||V(H)| \\ &+ |V(H)| - 2|T_u| + 1 \\ &= \Big[|N^k_G(u) \cap (V(G) \setminus A)| - |N^k_G(u) \cap A|\Big]|V(H)| + |V(H)| \\ &- 2|T_u| + 1 \\ &\geq \Big[|N^k_G(u) \cap (V(G) \setminus A)| - |N^k_G(u) \cap A|\Big]|V(H)| + |V(H)| \end{aligned}$$

$$- \Big[|V(H)| + 1\Big] + 1$$
$$= \Big[|N_G^k(u) \cap (V(G) \setminus A)| - |N_G^k(u) \cap A|\Big]|V(H)|.$$

Since A is an α_{ce}^k-set in G, $\Big[|N_G^k(u) \cap (V(G) \setminus A)| - |N_G^k(u) \cap A|\Big]|V(H)| \geq 0$. Thus, $|N_{G[H]}^k(u,v) \cap (V(G[H]) \setminus S)| - |N_{G[H]}^k(u,v) \cap S| \geq 0$. Therefore, S is a distance k-cost effective set in $G[H]$. □

Corollary 4. Let G and H be connected graphs and $k \geq 2$ be an integer. Then

$$\alpha_{ce}^k(G[H]) \geq \tfrac{|V(H)|+1}{2}\alpha_{ce}^k(G)$$

.

Theorem 5. Let G and H be connected graphs. Let A be an α_{ce}^1-set in G, T_a be an α_{ce}^1-set in H, for each $a \in A$, and $S = \bigcup_{a\in A}(\{a\} \times T_a)$. Then S is a distance 1-cost effective set in $G[H]$.

Proof: Let A be an α_{ce}^1-set in G, T_a be an α_{ce}^1-set in H, for each $a \in A$, and $S = \bigcup_{a\in A}(\{a\} \times T_a)$. Let $(u,v) \in S$. Then $|N_{G[H]}^1(u,v) \cap S| = |N_{H_u}^1(v) \cap T_u| + |N_G^1(u) \cap A||T_u|$ and

$$|N_{G[H]}^1(u,v) \cap (V(G[H]) \setminus S)| = |N_{H_u}^1(v) \cap (V(H) \setminus T_u)| + |N_G^1(u) \cap (V(G) \setminus A)||V(H)| + |N_G^1(u) \cap A||V(H) \setminus T_u|.$$

Hence,

$$\begin{aligned} |N_{G[H]}^1(u,v) \cap (V(G[H]) \setminus S)| - |N_{G[H]}^1(u,v) \cap S| &= |N_{H_u}^1(v) \cap (V(H) \setminus T_u)| \\ &\quad - |N_{H_u}^1(v) \cap T_u| + |N_G^1(u) \cap (V(G) \setminus A)||V(H)| \\ &\quad - |N_G^1(u) \cap A||T_u| + |N_G^1(u) \cap A||V(H) \setminus T_u| \\ &\geq |N_{H_u}^1(v) \cap (V(H) \setminus T_u)| - |N_{H_u}^1(v) \cap T_u| + |N_G^1(u) \cap (V(G) \setminus A)||V(H)| \\ &\quad - |N_G^1(u) \cap A||V(H)| \\ &= |N_H^1(v) \cap (V(H_u) \setminus T_u)| - |N_{H_u}^1(v) \cap T_u| \\ &\quad + \Big[|N_G^1(u) \cap (V(G) \setminus A)| - |N_G^1(u) \cap A|\Big]|V(H)|. \end{aligned}$$

Since T_u is an α_{ce}^1-set in $H_u, \forall\ u \in A$, $|N_{H_u}^1(v) \cap (V(H) \setminus T_u)| - |N_{H_u}^1(v) \cap T_u| \geq 0$. Also, since A is an α_{ce}^1-set in G, $|N_G^1(u) \cap (V(G) \setminus A)| - |N_G^1(u) \cap A| \geq 0$. Thus, $|N_{G[H]}^1(u,v) \cap (V(G[H]) \setminus S)| - |N_{G[H]}^1(u,v) \cap S| \geq 0$. Therefore, S is a distance 1-cost effective in $G[H]$. □

Corollary 5. Let G and H be connected graphs. Then

$$\alpha_{ce}^1(G[H]) \geq \alpha_{ce}^1(G)\alpha_{ce}^1(H).$$

Corollary 6. Let G and H be connected graphs. Let A and B be α^1_{ce}-sets in G and H, respectively. Then $A \times B$ is a distance 1-cost effective set in $G[H]$.

Definition 1. Let G be a nontrivial connected graph and $k \geq 1$ be an integer. A nonempty set $S \subseteq V(G)$ is said to be a ***very distance k-cost effective*** set in G if for every $u \in S, |N^k_G(u) \cap S^c| - |N^k_G(u) \cap S| > 0$.

Example 1. Consider the graph G as shown in Figure 1. For each $v \in S = \{v_2, v_3, v_4, v_5\}$ $|N^2_G(v) \cap (V(G) \setminus S)| - |N^2_G(v) \cap S| > 0$. Thus, S is a very distance 2-cost effective set in G.

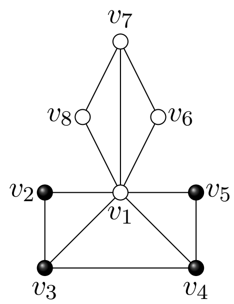

Figure 1: The Graph G with a very distance 2-cost effective set in G

Theorem 6. Let G and H be connected graphs and $k \geq 1$ be an integer. If A is a very distance k-cost effective set in G, then $A \times V(H)$ is a distance k-cost effective set in $G[H]$.

Proof: Let A be a very distance k-cost effective set in G and $k \geq 2$ be a positive integer. Let $(u, v) \in A \times V(H)$. Then

$$\begin{aligned} \left|N^k_{G[H]}(u,v) \cap (A \times V(H))\right| &= |N^k_H(v)| + |N^k_G(u) \cap A||V(H)| \\ &= |N^k_G(u) \cap A||V(H)| + |V(H)|. \end{aligned}$$

and

$$\left|N^k_{G[H]}(u,v) \cap \Big(V(G[H]) \setminus (A \times V(H))\Big)\right| = |N^k_G(u) \cap (V(G) \setminus A)||V(H)|.$$

Hence,

$$\begin{aligned} &|N^k_{G[H]}(u,v) \cap (V(G[H])\setminus(A \times V(H)))| - |N^k_{G[H]}(u,v) \cap (A \times V(H))| \\ &\quad = |N^k_G(u) \cap (V(G) \setminus A)||V(H)| - |N^k_G(u) \cap A||V(H)| - |V(H)| \\ &\quad =\Big(|N^k_G(u) \cap (V(G) \setminus A)| - |N^k_G(u) \cap A| - 1\Big)|V(H)|. \end{aligned}$$

Since A is a very distance k-cost effective set in G, $|N^k_G(u) \cap (V(G) \setminus A)| - |N^k_G(u) \cap A| > 0$. Thus, $\left|N^k_{G[H]}(u,v) \cap \Big(V(G[H])\setminus(A \times V(H))\Big)\right| - |N^k_{G[H]}(u,v) \cap (A \times V(H))| \geq 0$. Accordingly, $A \times V(H)$ is a distance k-cost effective set in $G[H]$.

Now for $k = 1$, let A be a very distance 1-cost effective set in G. Then for each $(u, v) \in A \times V(H)$, we have

$$\begin{aligned}
&|N^1_{G[H]}(u,v) \cap (V(G[H]) \backslash (A \times V(H)))| - |N^1_{G[H]}(u,v) \cap (A \times V(H))| \\
&\qquad = |N^1_G(u) \cap (V(G) \setminus A)||V(H)| - |N^1_G(u) \cap A||V(H)| - |N^1_H(v)| \\
&\qquad = \Big(|N^1_G(u) \cap (V(G) \setminus A)| - |N^1_G(u) \cap A|\Big)|V(H)| - |N^1_H(v)| \\
&\qquad \geq \Big(|N^1_G(u) \cap (V(G) \setminus A)| - |N^1_G(u) \cap A|\Big)|V(H)| - |V(H)| \\
&\qquad = \Big(|N^1_G(u) \cap (V(G) \setminus A)| - |N^1_G(u) \cap A| - 1\Big)|V(H)|.
\end{aligned}$$

Since A is a very distance 1-cost effective set in G, $|N^1_G(u) \cap (V(G) \setminus A)| - |N^1_G(u) \cap A| > 0$. Thus, $|N^1_{G[H]}(u,v) \cap \Big(V(G[H]) \setminus (A \times V(H))\Big)| - |N^1_{G[H]}(u,v) \cap (A \times V(H))| \geq 0$. Hence, $A \times V(H)$ is a distance 1-cost effective set in $G[H]$. Therefore, $A \times V(H)$ is a distance k-cost effective set in $G[H]$. □

Corollary 7. Let G and H be connected graphs and $k \geq 1$ be an integer. Then $\alpha^k_{ce}(G[H]) \geq \alpha^k_{ce}(G)|V(H)|$.

Acknowledgements

This research is funded by the Commission on Higher Education (CHED) and Mindanao State University-Iligan Institute of Technology.

References

[1] K. A. Bibi, A. Lakshmi, and R. Jothilakshmi. Applications of distance-2 dominating sets of graph in networks. *Advances in Computational Sciences and Technology*, 10(9):2801–2810, 2017.

[2] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley, Redwood City, CA, 1990.

[3] J. Caadan, R. Paluga, and I. Aniversario. Upper distance k-Cost Effective Numbers in the Join of Graphs. *European Journal of Pure and Applied Mathematics*, 13(3):701–709, 2020.

[4] T.W. Haynes, M. Chellali, and S.T. Hedetniem. Client-server and cost effective sets in graphs. *AKCE International Journal of Graphs and Combinatorics*, 15:211–218, 2018.

[5] T.W. Haynes, M. Henning, and S.T. Hedetniemi. Domination in graphs applied to electrical power networks. *J. Discrete Math*, 15(4), 2000.

[6] T.W. Haynes, I. Vasylieva, and S.T. Hedetniemi. Very cost effective bipartitions in graphs. *AKCE International Journal of Graphs and Combinatorics*, 12:155–160, 2015.

[7] S.M. Hedetniemi, T.W. Haynes, S.T. Hedetniemi, T.L. McCoy, and I. Vasylieva. Cost Effective Domination in Graphs. *Congr. Numer.*, 211:197–209, 2012.

[8] M. A. Henning, O. R. Swart, and H. C Swart. Bounds on distance domination parameters. *Journal of Combinatorics, Information and System Sciences*, 16:11–18, 1991.

[9] F. Jamil and H. Nuenay-Maglanque. Cost Effective Domination in the Join, Corona and Composition of Graphs. *European Journal of Pure and Applied Mathematics*, 12(3):978–998, 2019.

[10] A. H. Karbasi and R. E. Atani. Application of dominating sets in wireless sensor networks. *Int. J. Secur. Its Appl*, 7:185–202, 2013.

[11] J. Palco, R. Paluga, and G. Malacas. On k-cost effective domination number, cost effective domination index, and maximal cost effective domination number of simple graphs . *Far East Journal of Mathematical Sciences*, 114(1):55–68, 2019.

[12] M. Saravanan, R. Sujatha, R. Sundareswaran, and M. S. BALASUBRAMANIAN. Application of domination integrity of graphs in pmu placement in electric power networks. *Turkish Journal of Electrical Engineering and Computer Sciences*, 26(4):2066–2076, 2018.