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Semi-$\mathscr{I}$-submaximality

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Abstract. This paper presents the concept of semi-$\mathscr{I}$-submaximal ideal topological spaces. In particular, some characterizations of semi-$\mathscr{I}$-submaximal ideal topological spaces are investigated.

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1. Introduction

General topology has shown its fruitfulness in both pure and applied directions. The importance of general topology has appeared in many fields of applications such as computational topology for geometric design, computer-aided geometric design and engineering design. Hermann [11], Khalimsky [14] et al., Kong and Koppermann [15] applied topology in computer science and digital topology. Moore and Peters [17] investigated computational topology for geometric design. Rosen and Peters [18] used topology in computer-aided geometric design and engineering design. The concepts of maximality and submaximality of general topological spaces were introduced by Hewitt [12]. He discovered a general way of constructing maximal topologies. The existence of a maximal space that is Tychonoff is nontrivial and due to van Douwen [21]. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel'skiĭ and Collins [2]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is σ-discrete [2]. Every connected Hausdorff space which does not admit a larger connected topology is submaximal [7].

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [16] and Vaidyanathaswamy [20]. The topology τ of a space is enlarged to a topology $\tau^{\star}$ using an ideal $\mathscr{I}$ whose members are disjoint with the members of τ. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. Some early applications

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of ideal topological spaces can be found in various branches of mathematics, like a generalization of Cantor-Bendixson theorem by Freud [6], or in measure theory by Scheinberg [19]. In [13], the present authors investigated some properties of ideal topological spaces. In 2002, Hatir and Noiri [9] introduced the concepts of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets in topological spaces via ideals and used these sets to obtain certain decompositions of continuity. Hatir and Noiri [10] investigated the further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. In 2005, Açikgöz et al. [1] introduced and studied the notion of $\mathscr{I}$-submaximal ideal topological spaces. In 2010, Ekici and Noiri [4] investigated several characterizations of $\mathscr{I}$-submaximal ideal topological spaces. The purpose of the present paper is to introduce the notion semi-$\mathscr{I}$-submaximal ideal topological spaces. Moreover, several characterizations of semi-$\mathscr{I}$-submaximal ideal topological spaces are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A nonempty collection $\mathscr{I}$ of subsets of a set X is called an *ideal* on X if $\mathscr{I}$ satisfies the following two properties: (i) $A \in \mathscr{I}$ and $B \subseteq A \Rightarrow B \in \mathscr{I}$; (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$. For a topological space (X, τ) with an ideal $\mathscr{I}$ on X, a set operator $(.)^\star : \mathscr{P}(X) \to \mathscr{P}(X)$ where $\mathscr{P}(X)$ is the set of all subsets of X, called a *local function* [16] of A with respect to $\mathscr{I}$ and τ is defined as follows: for $A \subseteq X$, $A^\star(\mathscr{I}, \tau) = \{x \in X \mid G \cap A \notin \mathscr{I}$ for every $G \in \tau(x)\}$ where $\tau(x) = \{G \in \tau \mid x \in G\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\mathscr{I}, \tau)$, called the $\star$-*topology* and finer than τ, is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star$ [13]. We shall simply write $A^\star$ for $A^\star(\mathscr{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathscr{I}, \tau)$. A basis $\mathscr{B}(\mathscr{I}, \tau)$ for $\tau^\star$ can be described as follows: $\mathscr{B}(\mathscr{I}, \tau) = \{V - I' \mid V \in \tau$ and $I' \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I}, \tau)$ is not always a topology [13]. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\star$-*closed* ($\tau^\star$-*closed*) [13] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}, \tau))$ is denoted by $\mathrm{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi-$\mathscr{I}$-open* [9] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$. The complement of a semi-$\mathscr{I}$-open set is called *semi-$\mathscr{I}$-closed*. By $s\mathscr{I}O(X, \tau)$, we denote the family of all semi-$\mathscr{I}$-open sets of an ideal topological space $(X, \tau, \mathscr{I})$. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-open sets containing A is called the *semi-$\mathscr{I}$-closure* [5] of A and denoted by $s\mathrm{Cl}_{\mathscr{I}}(A)$. The *semi-$\mathscr{I}$-interior* [5], denoted by $s\mathrm{Int}_{\mathscr{I}}(A)$, is defined by the union of all semi-$\mathscr{I}$-open sets of X contained in A.

Lemma 1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sInt_{\mathscr{I}}(A)$ is semi-$\mathscr{I}$-open;

(2) $sCl_{\mathscr{I}}(A)$ is semi-$\mathscr{I}$-closed;

(3) A is semi-$\mathscr{I}$-open if and only if $A = sInt_{\mathscr{I}}(A)$;

(4) A is semi-$\mathscr{I}$-closed if and only if $A = sCl_{\mathscr{I}}(A)$;

(5) $x \in sCl_{\mathscr{I}}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every semi-$\mathscr{I}$-open set U containing x;*

(6) $X - sCl_{\mathscr{I}}(A) = sInt_{\mathscr{I}}(X - A)$;

(7) $X - sInt_{\mathscr{I}}(A) = sCl_{\mathscr{I}}(X - A)$.

Proof. (1) and (2) follows from Theorem 3.4 of [10].

(3) and (4) follows from (1) and (2).

(5) Let $x \in s\mathrm{Cl}_{\mathscr{I}}(A)$. Suppose that $U \cap A = \emptyset$ for some semi-$\mathscr{I}$-open set U containing x. Then, $A \subseteq X - U$ and $X - U$ is semi-$\mathscr{I}$-closed. Since $x \in s\mathrm{Cl}_{\mathscr{I}}(A)$, $x \in s\mathrm{Cl}_{\mathscr{I}}(X - U) = X - U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$ for every semi-$\mathscr{I}$-open set U containing x.

Conversely, assume that $U \cap A \neq \emptyset$ for every semi-$\mathscr{I}$-open set U containing x. We shall show that $x \in s\mathrm{Cl}_{\mathscr{I}}(A)$. Suppose that $x \notin s\mathrm{Cl}_{\mathscr{I}}(A)$. Then, there exists a semi-$\mathscr{I}$-closed set F such that $A \subseteq F$ and $x \notin F$. Thus, $X - F$ is a semi-$\mathscr{I}$-open set containing x such that $(X - F) \cap A = \emptyset$. This a contradiction to $U \cap A \neq \emptyset$; hence $x \in s\mathrm{Cl}_{\mathscr{I}}(A)$.

(6) Let $x \in X - s\mathrm{Cl}_{\mathscr{I}}(A)$. Then, $x \notin s\mathrm{Cl}_{\mathscr{I}}(A)$, there exists a semi-$\mathscr{I}$-open set V containing x such that $V \cap A = \emptyset$. Thus, $V \subseteq X - A$ and hence $x \in s\mathrm{Int}_{\mathscr{I}}(X - A)$. Consequently, we obtain $X - s\mathrm{Cl}_{\mathscr{I}}(A) \subseteq s\mathrm{Int}_{\mathscr{I}}(X - A)$. On the other hand, suppose that $x \in s\mathrm{Int}_{\mathscr{I}}(X - A)$. Then, there exists a semi-$\mathscr{I}$-open set V containing x such that $V \subseteq X - A$ and so $V \cap A = \emptyset$. By (5), we have $x \notin s\mathrm{Cl}_{\mathscr{I}}(A)$; hence $x \in X - s\mathrm{Cl}_{\mathscr{I}}(A)$. Thus, $s\mathrm{Int}_{\mathscr{I}}(X - A) \subseteq X - s\mathrm{Cl}_{\mathscr{I}}(A)$. This shows that $X - s\mathrm{Cl}_{\mathscr{I}}(A) = s\mathrm{Int}_{\mathscr{I}}(X - A)$.

(7) This follows from (6).

3. Semi-$\mathscr{I}$-submaximal ideal topological spaces

In this section, we introduce the notion of semi-$\mathscr{I}$-submaximal ideal topological spaces. Moreover, several characterizations of semi-$\mathscr{I}$-submaximal ideal topological spaces are discussed.

Definition 1. *A subset A of an ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be:*

(i) semi-$\mathscr{I}$-dense if $sCl_{\mathscr{I}}(A) = X$;

(ii) semi-$\mathscr{I}$-codense if $X - A$ *is semi-$\mathscr{I}$-dense.*

Definition 2. *An ideal topological space* $(X, \tau, \mathscr{I})$ *is called semi-$\mathscr{I}$-submaxiaml if each semi-$\mathscr{I}$-dense subset of X is semi-$\mathscr{I}$-open.*

Definition 3. *A subset A of an ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be:*

(i) locally semi-$\mathscr{I}$-closed if A is the intersection of a semi-$\mathscr{I}$-open set and a semi-$\mathscr{I}$-closed set;

(ii) co-locally semi-$\mathscr{I}$-closed if A is the union of a semi-$\mathscr{I}$-open set and a semi-$\mathscr{I}$-closed set.

Theorem 1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is locally semi-$\mathscr{I}$-closed;

(2) $A = U \cap sCl_{\mathscr{I}}(A)$ for some $U \in s\mathscr{I}O(X, \tau)$;

(3) $sCl_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed;

(4) $A \cup (X - sCl_{\mathscr{I}}(A))$ is semi-$\mathscr{I}$-open;

(5) $A \subseteq sInt_{\mathscr{I}}(A \cup (X - sCl_{\mathscr{I}}(A)))$.

Proof. (1) ⇒ (2): Suppose that A is locally semi-$\mathscr{I}$-closed. Then, there exist a semi-$\mathscr{I}$-open set U and a semi-$\mathscr{I}$-closed set F such that $A = U \cap F$. Since F is semi-$\mathscr{I}$-closed, $s\text{Cl}_{\mathscr{I}}(A) \subseteq s\text{Cl}_{\mathscr{I}}(F) = F$ and so $A \subseteq U \cap s\text{Cl}_{\mathscr{I}}(A) \subseteq U \cap F = A$. Thus, $A = U \cap s\text{Cl}_{\mathscr{I}}(A)$.

(2) ⇒ (3): Suppose that $A = U \cap s\text{Cl}_{\mathscr{I}}(A)$ for some $U \in s\mathscr{I}O(X, \tau)$. Since

$$\begin{aligned} s\text{Cl}_{\mathscr{I}}(A) - A &= (X - A) \cap s\text{Cl}_{\mathscr{I}}(A) \\ &= X - (U \cap s\text{Cl}_{\mathscr{I}}(A)) \cap s\text{Cl}_{\mathscr{I}}(A) \\ &= (X - U) \cap s\text{Cl}_{\mathscr{I}}(A), \end{aligned}$$

we have $s\text{Cl}_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed.

(3) ⇒ (4): Suppose that $s\text{Cl}_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed. Since $X - (s\text{Cl}_{\mathscr{I}}(A) - A) = (X - s\text{Cl}_{\mathscr{I}}(A)) \cup A$, $A \cup (X - s\text{Cl}_{\mathscr{I}}(A))$ is semi-$\mathscr{I}$-open.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (1): By (5) and Lemma 1(6),

$$\begin{aligned} X - s\text{Cl}_{\mathscr{I}}(A) &= s\text{Int}_{\mathscr{I}}(X - s\text{Cl}_{\mathscr{I}}(A)) \\ &\subseteq s\text{Int}_{\mathscr{I}}(A \cup (X - s\text{Cl}_{\mathscr{I}}(A))) \end{aligned}$$

and hence $A \cup (X - s\text{Cl}_{\mathscr{I}}(A)) \subseteq s\text{Int}_{\mathscr{I}}(A \cup (X - s\text{Cl}_{\mathscr{I}}(A)))$. Thus, $A \cup (X - s\text{Cl}_{\mathscr{I}}(A))$ is semi-$\mathscr{I}$-open. Since $A = (A \cup (X - s\text{Cl}_{\mathscr{I}}(A))) \cap s\text{Cl}_{\mathscr{I}}(A)$, we have A is locally semi-$\mathscr{I}$-closed.

Definition 4. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(i) a t-s$\mathscr{I}$-set if $sInt_{\mathscr{I}}(A) = sInt_{\mathscr{I}}(sCl_{\mathscr{I}}(A))$;

(ii) a $\mathscr{B}$-s$\mathscr{I}$-set if $A = U \cap V$, where U is a semi-$\mathscr{I}$-open set and V is a t-s$\mathscr{I}$-set.

The following theorem gives some characterizations of semi-$\mathscr{I}$-submaximal ideal topological spaces.

Theorem 2. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ *is semi-$\mathscr{I}$-submaximal;*

(2) $sCl_{\mathscr{I}}(A) - A$ *is semi-$\mathscr{I}$-closed for every subset A of X;*

(3) every subset of X is locally semi-$\mathscr{I}$-closed;

(4) every subset of X is a $\mathscr{B}$-$s\mathscr{I}$-set;

(5) every semi-$\mathscr{I}$-dense subset of X is a $\mathscr{B}$-$s\mathscr{I}$-set.

Proof. (1) ⇒ (2): Suppose that $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal. Let A be a subset of X. Since

$$\begin{aligned} X &= s\mathrm{Cl}_{\mathscr{I}}(A) \cup (X - s\mathrm{Cl}_{\mathscr{I}}(A)) \\ &\subseteq s\mathrm{Cl}_{\mathscr{I}}(A) \cup (X - s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A))) \\ &= s\mathrm{Cl}_{\mathscr{I}}(A) \cup s\mathrm{Cl}_{\mathscr{I}}(X - s\mathrm{Cl}_{\mathscr{I}}(A)) \\ &\subseteq s\mathrm{Cl}_{\mathscr{I}}(A \cup (X - s\mathrm{Cl}_{\mathscr{I}}(A))) \\ &= s\mathrm{Cl}_{\mathscr{I}}(X - (s\mathrm{Cl}_{\mathscr{I}}(A) - A)), \end{aligned}$$

we have $s\mathrm{Cl}_{\mathscr{I}}(X-(s\mathrm{Cl}_{\mathscr{I}}(A)-A)) = X$ and hence $X-(s\mathrm{Cl}_{\mathscr{I}}(A)-A)$ is semi-$\mathscr{I}$-dense. By the hypothesis, $X - (s\mathrm{Cl}_{\mathscr{I}}(A) - A)$ is semi-$\mathscr{I}$-open and so $s\mathrm{Cl}_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed.

(2) and (3) are equivalent by Theorem 1.

(3) ⇒ (4) and (4) ⇒ (5) are obvious.

(5) ⇒ (1): Let A be a semi-$\mathscr{I}$-dense subset of X. By (5), A is a $\mathscr{B}$-$s\mathscr{I}$-set and so $A = U \cap V$, where U is semi-$\mathscr{I}$-open and $s\mathrm{Int}_{\mathscr{I}}(V) = s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(V))$. Since $A \subseteq V$, $s\mathrm{Cl}_{\mathscr{I}}(A) \subseteq s\mathrm{Cl}_{\mathscr{I}}(V)$ and hence $X = s\mathrm{Cl}_{\mathscr{I}}(V)$. Thus, $X = s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(V)) = s\mathrm{Int}_{\mathscr{I}}(V)$. This implies that $V = X$. Therefore, $A = U \cap V = U \cap X = U$ and hence A is semi-$\mathscr{I}$-open. Thus, $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal.

Definition 5. *A point x of an ideal topological space $(X, \tau, \mathscr{I})$ is called semi-$\mathscr{I}$-isolated if $\{x\}$ is semi-$\mathscr{I}$-open and $(X, \tau, \mathscr{I})$ is called semi-$\mathscr{I}$-discrete if every point of X is semi-$\mathscr{I}$-isolated.*

Lemma 2. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. Then,*

$$sInt_{\mathscr{I}}(sCl_{\mathscr{I}}(A) - A) = \emptyset.$$

Proof. Let A be a subset of X. Since $s\mathrm{Int}_{\mathscr{I}}(X - A) = X - s\mathrm{Cl}_{\mathscr{I}}(A)$, we have

$$\begin{aligned} s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A) - A) &= s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A) \cap (X - A)) \\ &\subseteq s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A)) \cap s\mathrm{Int}_{\mathscr{I}}(X - A) \\ &= s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A)) \cap (X - s\mathrm{Cl}_{\mathscr{I}}(A)) \\ &\subseteq s\mathrm{Cl}_{\mathscr{I}}(A) \cap (X - s\mathrm{Cl}_{\mathscr{I}}(A)) = \emptyset. \end{aligned}$$

Theorem 3. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal;

(2) every subset of X is co-locally semi-$\mathscr{I}$-closed;

(3) every subset A of X, for which $sInt_{\mathscr{I}}(A) = \emptyset$, is semi-$\mathscr{I}$-closed;

(4) for every subset A of X, $sCl_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed;

(5) every subset of X is locally semi-$\mathscr{I}$-closed;

(6) each semi-$\mathscr{I}$-codense subset of X is semi-$\mathscr{I}$-closed.

Proof. (1) $\Rightarrow$ (2): Let A be a subset of X. Since $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, by Theorem 2, there exist a semi-$\mathscr{I}$-open set U and a semi-$\mathscr{I}$-closed set V such that $X - A = U \cap V$. Then, we have $A = (X - U) \cup (X - V)$, where $X - U$ is a semi-$\mathscr{I}$-closed set and $X - V$ is a semi-$\mathscr{I}$-open set. Thus, A is co-locally semi-$\mathscr{I}$-closed.

(2) $\Rightarrow$ (3): Let A be a subset of X and $s\mathrm{Int}_{\mathscr{I}}(A) = \emptyset$. By (2), there exist a semi-$\mathscr{I}$-open set U and a semi-$\mathscr{I}$-closed set V such that $A = U \cup V$. Then, we have

$$U = s\mathrm{Int}_{\mathscr{I}}(U) \subseteq s\mathrm{Int}_{\mathscr{I}}(A) = \emptyset$$

which yields $U = \emptyset$. Thus, $A = V$ is semi-$\mathscr{I}$-closed.

(3) $\Rightarrow$ (4): Let A be a subset of X. By Lemma 2, $s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(A) - A) = \emptyset$ and by (3), we have $s\mathrm{Cl}_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed.

(4) $\Rightarrow$ (5): It follows from Theorem 2.

(5) $\Rightarrow$ (1): Let A be a semi-$\mathscr{I}$-dense subset of X. By (5), there exist a semi-$\mathscr{I}$-open set U and a semi-$\mathscr{I}$-closed set V such that $A = U \cap V$. Since $A \subseteq V$, $s\mathrm{Cl}_{\mathscr{I}}(A) \subseteq s\mathrm{Cl}_{\mathscr{I}}(V)$ and so $X = s\mathrm{Cl}_{\mathscr{I}}(V)$. Thus, $X = s\mathrm{Int}_{\mathscr{I}}(s\mathrm{Cl}_{\mathscr{I}}(V)) = s\mathrm{Int}_{\mathscr{I}}(V)$ which yields $V = X$. Therefore, $A = U \cap V = U \cap X = U$ and hence A is semi-$\mathscr{I}$-open. This shows that $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal.

(1) $\Rightarrow$ (6): Let A be a semi-$\mathscr{I}$-codense set. Then, $X - A$ is semi-$\mathscr{I}$-dense. Since $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, we have $X - A$ is semi-$\mathscr{I}$-open and hence A is semi-$\mathscr{I}$-closed.

(6) $\Rightarrow$ (1): Let A be a semi-$\mathscr{I}$-dense subset of X. Then, $X - A$ is semi-$\mathscr{I}$-codense. By (6), $X - A$ is semi-$\mathscr{I}$-closed and so A is semi-$\mathscr{I}$-open. Thus, $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal.

Theorem 4. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal;

(2) every subset A of X, for which $sInt_{\mathscr{I}}(A) = \emptyset$, is semi-$\mathscr{I}$-closed;

(3) every subset A of X, for which $sInt_{\mathscr{I}}(A) = \emptyset$, is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete;

(4) for every subset A of X, $sCl_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete;

(5) each semi-$\mathscr{I}$-codense subset of X is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete;

(6) each semi-$\mathscr{I}$-codense subset of X is semi-$\mathscr{I}$-closed.

Proof. (1) ⇒ (2): Let A be a subset of X and $s\text{Int}_{\mathscr{I}}(A) = \emptyset$. Then, we have $s\text{Cl}_{\mathscr{I}}(X - A) = X - s\text{Int}_{\mathscr{I}}(A) = X$ and hence $X - A$ is semi-$\mathscr{I}$-dense. Since $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, $X - A$ is semi-$\mathscr{I}$-open. Thus, A is semi-$\mathscr{I}$-closed.

(2) ⇒ (3): Let A be a subset of X and $s\text{Int}_{\mathscr{I}}(A) = \emptyset$. If $B \subseteq A$, then $s\text{Int}_{\mathscr{I}}(B) \subseteq s\text{Int}_{\mathscr{I}}(A) = \emptyset$ which yields $s\text{Int}_{\mathscr{I}}(B) = \emptyset$. Thus, by (2), B is semi-$\mathscr{I}$-closed. So every subset of A is semi-$\mathscr{I}$-closed. Consequently, we obtain A is semi-$\mathscr{I}$-discrete.

(3) ⇒ (5): Let A be semi-$\mathscr{I}$-codense. Then, we have $X - A$ is semi-$\mathscr{I}$-dense and so $s\text{Cl}_{\mathscr{I}}(X - A) = X$. Therefore, $s\text{Int}_{\mathscr{I}}(A) = \emptyset$, by (3), A is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete.

(5) ⇒ (3): Let A be a subset of X and $s\text{Int}_{\mathscr{I}}(A) = \emptyset$. Then, $s\text{Cl}_{\mathscr{I}}(X - A) = X - s\text{Int}_{\mathscr{I}}(A) = X$ and hence $X - A$ is semi-$\mathscr{I}$-dense. Thus, A is semi-$\mathscr{I}$-codense, by (5), A is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete.

(3) ⇒ (4): Let A be a subset of X. By Lemma 2, $s\text{Int}_{\mathscr{I}}(s\text{Cl}_{\mathscr{I}}(A) - A) = \emptyset$ and by (3), we have $s\text{Cl}_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete.

(4) ⇒ (3): Let A be a subset of X and $s\text{Int}_{\mathscr{I}}(A) = \emptyset$. Then, $s\text{Cl}_{\mathscr{I}}(X - A) = X - s\text{Int}_{\mathscr{I}}(A) = X$ and hence $A = s\text{Cl}_{\mathscr{I}}(X - A) - (X - A)$. By (4), we have A is semi-$\mathscr{I}$-closed and semi-$\mathscr{I}$-discrete.

(5) ⇒ (6): This is obvious.

(6) ⇒ (1): Let A be a semi-$\mathscr{I}$-dense subset of X. Then, we have $X - A$ is semi-$\mathscr{I}$-codense. By (6), $X - A$ is semi-$\mathscr{I}$-closed and so A is semi-$\mathscr{I}$-open. Thus, $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal.

Theorem 5. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal;

(2) for every subset A of X, $sCl_{\mathscr{I}}(A) - A$ is semi-$\mathscr{I}$-closed;

(3) every subset of X is locally semi-$\mathscr{I}$-closed;

(4) each semi-$\mathscr{I}$-dense subset of X is locally semi-$\mathscr{I}$-closed.

Proof. (1) ⇒ (2) and (2) ⇒ (3) follows from Theorem 2.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let A be a semi-$\mathscr{I}$-dense subset of X. By (4), there exist a semi-$\mathscr{I}$-open set U and a semi-$\mathscr{I}$-closed set V such that $A = U \cap V$. Since $A \subseteq V$,

$$X = s\text{Cl}_{\mathscr{I}}(A) \subseteq s\text{Cl}_{\mathscr{I}}(V) = V$$

which yields $V = X$. Thus, $A = U \cap V = U \cap X = U$ and hence A is semi-$\mathscr{I}$-open. This shows that $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, we denote by $\tau_{|_A}$ the relative topology on A and $\mathscr{I}_{|_A} = \{A \cap I_0 \mid I_0 \in \mathscr{I}\}$ is an ideal on A.

Lemma 3. [3] *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $B \subseteq A \subseteq X$. Then, $B^\star(\tau_{|_A}, \mathscr{I}_{|_A}) = B^\star(\tau, \mathscr{I}) \cap A$.*

Lemma 4. [8] *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $B \subseteq A \subseteq X$. Then, $Cl^\star_A(B) = Cl^\star(B) \cap A$.*

Lemma 5. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq B \subseteq X$. If $(B, \tau_{|_B}, \mathscr{I}_{|_B})$ is an open subspace of $(X, \tau, \mathscr{I})$, then $sCl_{\mathscr{I}_{|_B}}(A) = sCl_{\mathscr{I}}(A) \cap B$.*

Proof. Suppose that $(B, \tau_{|_B}, \mathscr{I}_{|_B})$ is an open subspace of $(X, \tau, \mathscr{I})$ and $A \subseteq B \subseteq X$. By Lemma 13(2) of [5] and Lemma 4, we have

$$\begin{aligned} s\mathrm{Cl}_{\mathscr{I}}(A) \cap B &= (A \cup \mathrm{Cl}^\star(\mathrm{Int}(A))) \cap B \\ &= (A \cap B) \cup (\mathrm{Cl}^\star(\mathrm{Int}(A)) \cap B) \\ &= A \cup \mathrm{Cl}^\star_B(\mathrm{Int}(A)) \\ &= A \cup \mathrm{Cl}^\star_B(\mathrm{Int}(A \cap B)) \\ &= A \cup \mathrm{Cl}^\star_B(\mathrm{Int}(A) \cap B) \\ &= A \cup \mathrm{Cl}^\star_B(\mathrm{Int}_B(A)) \\ &= s\mathrm{Cl}_{\mathscr{I}_{|_B}}(A). \end{aligned}$$

Lemma 6. [10] *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $U \in \tau$ and $W \in s\mathscr{I}O(X, \tau)$, then $U \cap W \in s\mathscr{I}O(U, \tau_{|_U}, \mathscr{I}_{|_U})$.*

Theorem 6. *Let A be an open set of an ideal topological space $(X, \tau, \mathscr{I})$. If $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, then $(A, \tau_{|_A}, \mathscr{I}_{|_A})$ is semi-$\mathscr{I}_{|_A}$-submaximal.*

Proof. Suppose that $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal. Let D be a semi-$\mathscr{I}_{|_A}$-dense subset of $(A, \tau_{|_A}, \mathscr{I}_A)$. Let $U = D \cup (X - A)$. By Lemma 5, we have

$$\begin{aligned} s\mathrm{Cl}_{\mathscr{I}}(U) &= s\mathrm{Cl}_{\mathscr{I}}(D \cup (X - A)) \\ &\supseteq s\mathrm{Cl}_{\mathscr{I}}(D) \cup s\mathrm{Cl}_{\mathscr{I}}(X - A) \\ &\supseteq (s\mathrm{Cl}_{\mathscr{I}}(D) \cap A) \cup s\mathrm{Cl}_{\mathscr{I}}(X - A) \\ &= s\mathrm{Cl}_{\mathscr{I}_{|_A}}(D) \cup s\mathrm{Cl}_{\mathscr{I}}(X - A) \\ &= A \cup s\mathrm{Cl}_{\mathscr{I}}(X - A) \\ &= A \cup (X - s\mathrm{Int}_{\mathscr{I}}(A)) \\ &\supseteq A \cup (X - A) = X \end{aligned}$$

and hence $s\mathrm{Cl}_{\mathscr{I}}(U) = X$. Since $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, U is semi-$\mathscr{I}$-open. By Lemma 6, $D = A \cap U$ is semi-$\mathscr{I}_{|A}$-open in $(A, \tau_{|A}, \mathscr{I}_{|A})$. This shows that $(A, \tau_{|A}, \mathscr{I}_{|A})$ is semi-$\mathscr{I}_{|A}$-submaximal.

Next, we shall show that semi-$\mathscr{I}$-submaximal ideal topological spaces are invariant under semi-$(\mathscr{I}, \mathscr{J})$-open surjections.

Definition 6. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be semi-$(\mathscr{I}, \mathscr{J})$-open if $f(V)$ is semi-$\mathscr{J}$-open in Y for each semi-$\mathscr{I}$-open set V of X.*

Theorem 7. *Let $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ be a semi-$(\mathscr{I}, \mathscr{J})$-open surjection. If $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal, then $(Y, \sigma, \mathscr{J})$ is semi-$\mathscr{J}$-submaximal.*

Proof. Suppose that $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-submaximal. Let A be a semi-$\mathscr{J}$-dense subset of Y. Since $s\mathrm{Int}_{\mathscr{I}}(f^{-1}(Y - A)) \subseteq f^{-1}(Y - A)$, we have $f(s\mathrm{Int}_{\mathscr{I}}(f^{-1}(Y - A))) \subseteq f(f^{-1}(Y - A)) \subseteq Y - A$ and hence $s\mathrm{Int}_{\mathscr{J}}(f(s\mathrm{Int}_{\mathscr{I}}(f^{-1}(Y - A)))) \subseteq s\mathrm{Int}_{\mathscr{J}}(Y - A)$. Since f is semi-$(\mathscr{I}, \mathscr{J})$-open, $f(s\mathrm{Int}_{\mathscr{I}}(f^{-1}(Y - A))) \subseteq s\mathrm{Int}_{\mathscr{J}}(Y - A)$. Thus,

$$s\mathrm{Int}_{\mathscr{I}}(f^{-1}(Y - A)) \subseteq f^{-1}(s\mathrm{Int}_{\mathscr{J}}(Y - A)).$$

It follows that $X - s\mathrm{Cl}_{\mathscr{I}}(f^{-1}(A)) \subseteq X - f^{-1}(s\mathrm{Cl}_{\mathscr{J}}(A))$ and hence $X = f^{-1}(s\mathrm{Cl}_{\mathscr{J}}(A)) \subseteq s\mathrm{Cl}_{\mathscr{I}}(f^{-1}(A))$. This implies that $s\mathrm{Cl}_{\mathscr{I}}(f^{-1}(A)) = X$. Therefore, $f^{-1}(A)$ is semi-$\mathscr{I}$-dense and so $f^{-1}(A)$ is semi-$\mathscr{I}$-open. Since f is a semi-$(\mathscr{I}, \mathscr{J})$-open surjection, $A = f(f^{-1}(A))$ is semi-$\mathscr{J}$-open. Thus, $(Y, \sigma, \mathscr{J})$ is semi-$\mathscr{J}$-submaximal.

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