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Toeplitz matrix and Nyström method for solving linear fractional integro-differential equation

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Abstract. In this paper, the Volterra-Fredholm integral equation is derived from a linear integro-differential equation with a fractional order $0 < \alpha < 1$ using Riemann–Liouville fractional integral. The existence and uniqueness of the solution are proved using the Picard method. Popular numerical methods; the Toeplitz matrix, and the product Nyström are used in the solution. These methods will prove their effective in solving this type of equation. Two examples are solved using the mentioned methods and the estimation error is calculated. Finally, a comparison between the numerical results is made.

2020 Mathematics Subject Classifications: 45F15, 45K05, 45E10, 74H15

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1. Introduction

The wide range of applications for fractional equations has led to increased interest in recent years. The list of applications has expanded and become more diverse in a short period of time. One example of such uses is electromagnetic fields. [17] Constructed fractional integro-differential equations **(FI-DEs)** from electromagnetic waves in a dielectric material. The existence, uniqueness and the convergence of the solution of **FI-DEs** by using the Picard method were discussed in [9]. The existence and uniqueness of the solution of the linear fractional Volterra **I-DEs** with initial conditions relied on applying the Picard iteration method to obtain uniformly convergent series for the exact solution were discussed by [13]. The existence and uniqueness of mild solutions for **FI-DEs** were investigated by using Holder's inequality, p-mean continuity, and Schauder's fixed point theorem in Banach spaces in [3]. Several authors are interested in solving **FI-DEs** by analytical methods. Furthermore, numerical methods are used to approximate the solutions. In [12] introduced the generalized hat functions and operational matrix of the fractional integration to solve the **FI-DEs** of Bratu-type numerically. The decomposition

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method for approximating the solution of **FI-DEs** systems was implemented by [11]. In [15] applied the Legendre wavelets method to approximate the solution of **FI-DEs**. In [5], Sinc-Collocation method was introduced to solve Volterra-Fredholm **FI-DEs**. Numerical solutions of linear Fredholm-Volterra **FI-DEs** using Laguerre polynomials were investigated by [6]. Chebyshev polynomials were used to solve the Fredholm and Volterra integro-differential equations in [7]. Using coupled mathematical solutions, the generalized monotone iterative technique was developed to solve the Caputo **FI-DEs** of order q[8].

2. The linear fractional integro-differential equation

Consider the linear fractional integro-differential equation **(LFI-DE)**

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = f(x,t) + \lambda \int_a^b k(x,y)u(y,t)dy \ , \quad (0 < \alpha < 1) \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x).$$

In (1) the unknown function appears on one side of the equation under the fractional order derivative and appears on the other side under integration.

Definition 1. *[16] For all $t \in [a,b]$ the left Riemann-Liouville fractional integral of order $\alpha > 0$, of the function $\phi : (0,\infty) \to R$ is defined as*

$${}_aI_t^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}\phi(\tau)d\tau, \quad t > a. \tag{2}$$

Applying relation (2) to equation (1), we obtain

$$u(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_a^b (t-s)^{\alpha-1} k(x,y)u(y,s)dyds, \tag{3}$$

where

$$\eta(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s)ds, \tag{4}$$

equation (3) is a singular Volterra-Fredholm integral equation **(V-FIE)** of Abel's type. It is clear that equation (3) is equivalent to equation (1).

3. Picard method for singularV-FIE [1]

In this section, the Picard method is used to prove the existence of a unique solution to singular **V-FIE** (3) where

$$F(t,s) = (t-s)^{\alpha-1}.$$

For this goal, consider the following conditions:

(i) Volterra integral's kernel $F(t,s)$ satisfies the discontinuity condition

$$\Big[\int_0^T\int_0^T |F(t,s)|^2 dsdt\Big]^{\frac{1}{2}} = Q, \qquad \text{(Q is a constant)}.$$

(ii) The kernel of Fredholm $k(x,y)$ belongs to the class $C[a,b]$, and it is bounded, i.e.

$$|k(x,y)| \le B, \qquad \forall x,y \in [a,b], \qquad \text{(B is a constant)}.$$

(iii) The function $f(x,t)$, and its partial derivatives with respect to x and t, are continuous in the Banach space $L_2[0,T]\times C[a,b], T<1$. Its norm is defined as

$$||f(x,t)|| = \max_{a\le x\le b}\Big|\int_a^x \Big[\int_0^t |f(x,s)|^2 ds\Big]^{\frac{1}{2}} dy\Big| = H, \qquad \text{(H is a constant)}.$$

(iv) The function $u_0(x)$ belongs to the space $C[a,b]$ and has the norm

$$||u_0(x)|| = \max_{a\le x\le b}|u_0(x)| = A, \qquad \text{(A is a constant)}.$$

Theorem 1. *Equation* (3) *has a unique solution in the Banach space* $L_2[0,T]\times C[a,b]$, *under the condition*

$$\lambda(b-a)BQ < \Gamma(\alpha).$$

Proof.

The existence of a unique solution of equation (3) can be proved by using the method of successive approximations which is also called "Picard's method ". This method consists of the following simple iteration.

$$u_n(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_a^b F(t,s)k(x,y)u_{n-1}(y,s)dyds, \qquad (n\ge 1), \tag{5}$$

with

$$u_0(x,t) = \eta(x,t).$$

For ease manipulation it is convenient to introduce

$$\xi_n = u_n(x,t) - u_{n-1}(x,t) \tag{6}$$

where

$$u_n(x,t) = \sum_{i=0}^{n}\xi_i(x,t), \qquad \xi_0(x,t) = \eta(x,t), \tag{7}$$

subtracting from equation (5) a similar equation with n replaced by $n-1$, we get

$$u_n(x,t) - u_{n-1}(x,t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_a^b F(t,s)k(x,y)[u_{n-1}(y,s) - u_{n-2}(y,s)]dyds, \tag{8}$$

inserting (6) in (8), we have

$$|\xi_n(x,t)| = \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t \int_a^b |F(t,s)||k(x,y)||\xi_{n-1}(y,s)|dyds, \tag{9}$$

using condition (ii), then applying Cauchy-Schwarz inequality, we obtain

$$||\xi_n(x,t)|| \leq \frac{|\lambda|B}{\Gamma(\alpha)}\Big(\int_0^T \int_0^T |F(t,s)|^2 dsdt\Big)^{\frac{1}{2}} \max_{a\leq x\leq b}\Big|\int_a^x \Big(\int_0^t |\xi_{n-1}(y,s)|^2 ds\Big)^{\frac{1}{2}} dy\Big| \int_a^b dy,$$

in view of condition (i), we get

$$||\xi_n(x,t)|| \leq \frac{|\lambda|BQ(b-a)}{\Gamma(\alpha)}||\xi_{n-1}(x,t)||. \tag{10}$$

Inequality (10) for $n=1$, yields

$$||\xi_1(x,t)|| \leq \sigma||\xi_0(x,t)|| \quad ; \quad \big(\sigma = \frac{|\lambda|BQ(b-a)}{\Gamma(\alpha)}\big), \tag{11}$$

from (4) and (7), we have

$$|\xi_0(x,t)| \leq |u_0(x)| + \frac{1}{\Gamma(\alpha)}\int_0^t |F(t,s)||f(x,s)|ds,$$

after applying Cauchy-Schwarz inequality, we get

$$||\xi_0(x,t)|| \leq \max_{a\leq x\leq b}|u_0(x)| + \frac{1}{\Gamma(\alpha)}\Big(\int_0^T \int_0^T |F(t,s)|^2 dsdt\Big)^{\frac{1}{2}} \max_{a\leq x\leq b}\Big|\int_a^x \Big(\int_0^t |f(x,s)|^2 ds\Big)^{\frac{1}{2}} dy\Big|,$$

using conditions (i), (iii) and (iv), the above inequality takes the form

$$||\xi_0(x,t)|| \leq G \quad ; \qquad \Big(G = A + \frac{QH}{\Gamma(\alpha)} \quad \text{is a constant}\Big). \tag{12}$$

Introducing (11) in (12), we have

$$||\xi_1(x,t)|| \leq \sigma G,$$

by induction, we can prove that

$$||\xi_n(x,t)|| \leq \sigma^n G \quad ; \quad n = 0,1,2,.... \tag{13}$$

Since (13) is obviously true for $n = 0,1$, then it holds for all n. This bound makes the sequence $\{\xi_n(x,t)\}$ converges under the condition $\sigma < 1$, and therefore the sequence $\{u_n(x,t)\}$ in (7) converges. Hence we can write

$$u(x,t) = \sum_{i=0}^{\infty} \xi_i(x,t), \tag{14}$$

the series (14) is uniformly convergent since the terms $\xi_i(x,t)$ are limited by σ^i.
To prove that $u(x,t)$ defined by (14) satisfies equation (3), set

$$u(x,t) = u_n(x,t) + \Delta_n(x,t), \quad (\Delta_n(x,t) \to 0 \quad \text{as} \quad n \to \infty), \tag{15}$$

from equation (3), we get

$$u(x,t) - \Delta_n(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_a^b F(t,s)k(x,y)[u(y,s) - \Delta_{n-1}(y,s)]dyds,$$

therefore, we have

$$\Big| u(x,t) - \eta(x,t) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_a^b F(t,s)k(x,y)u(y,s)dyds \Big|$$
$$\leq \big|\Delta_n(y,s)\big| + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t \int_a^b |F(t,s)||k(x,y)||\Delta_{n-1}(y,s)|dyds,$$

using condition (ii), then applying Cauchy-Schwarz inequality to the integral term in the right-hand side and in view of condition (i), we get

$$||u(x,t) - \eta(x,t) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_a^b F(t,s)k(x,y)u(y,s)dyds|| \leq ||\Delta_n(x,t)|| + \sigma||\Delta_{n-1}(x,t)||, \tag{16}$$

by taking n large enough, the right-hand side of (16) can be made as small as desired. Consequently, the function $u(x,t)$ defined by (15) satisfies

$$u(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_a^b F(t,s)k(x,y)u(y,s)dyds,$$

and is therefore the solution to equation (3).
To show that $u(x,t)$ is the only solution of equation (3), we assume the existence of another solution $\tilde{u}(x,t)$, then

$$|u(x,t) - \tilde{u}(x,t)| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t \int_a^b |F(t,s)||k(x,y)||u(y,s) - \tilde{u}(y,s)|dyds,$$

using condition (ii), then applying Cauchy-Schwarz inequality and finally in view of condition (i), we deduce that

$$||u(x,t) - \tilde{u}(x,t)|| \leq \sigma||u(x,t) - \tilde{u}(x,t)||, \tag{17}$$

because of $\sigma < 1$, this can be true if $u(x,t) = \tilde{u}(x,t)$; that is, the solution of equation (3) is unique.

4. A system of Volterra integral equations [1, 2]

In this section, the **V-FIE** will be converted to a system of Volterra integral equation by dividing the interval $[a, b]$, into M subintervals, such that $a = x_0 < x_1 < x_2 < ...... < x_m < ...... < x_M = b$ where $x = x_m, m = 0, 1, 2..., M$. Equation (3) becomes

$$u_m(t) = \eta_m(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_a^b k(x_m, y) u(y, s) dy ds \tag{18}$$

The Fredholm integral part of equation (18), after using the quadrature formula, takes the form

$$\int_a^b k(x_m, y) u(y, s) dy = \sum_{l=0}^{M} \overline{w}_l k(x_m, x_l) u(x_l, s), \qquad l = 0, 1, ...M \tag{19}$$

Using (19) in (18), we have the following Volterra integral equations system

$$u_m(t) = \eta_m(t) + \frac{\lambda}{\Gamma(\alpha)} \sum_{l=0}^{M} \overline{w}_l k_{m,l} \int_0^t (t-s)^{\alpha-1} u_l(s) ds. \tag{20}$$

where $u_m(t) = u(x_m, t)$, $\eta_m(t) = \eta(x_m, t)$, $k_{m,l} = k(x_m, x_l)$, $\overline{w}_0 = \overline{w}_m = \frac{1}{2} h_0$, $\overline{w}_r = h_r, (r \neq 0, m)$.

5. Numerical methods to solve singular Volterra integral equation

5.1. The Toeplitz matrix method [2, 4]

Consider the Volterra integral equation:

$$u^*(t) - \lambda \int_0^t (t-s)^{(\alpha-1)} u^*(s) ds = f^*(t), \tag{21}$$

write the integral term in the form

$$\int_0^t (t-s)^{(\alpha-1)} u^*(s) ds = \sum_{n=0}^{N-1} \int_{nh}^{nh+h} (t-s)^{(\alpha-1)} u^*(s) ds, \qquad (h = \frac{T}{N}), \tag{22}$$

approximate the integral in the right hand side of equation (22) by

$$\int_{nh}^{nh+h} (t-s)^{(\alpha-1)} u^*(s) ds = A_n(t) u^*(nh) + B_n(t) u^*(nh+h), \tag{23}$$

where

$$A_n(t) = \frac{1}{h}[(nh+h) I(t) - J(t)], \qquad B_n(t) = \frac{1}{h}[J(t) - nh I(t)], \tag{24}$$

$$I(x)=\int_{nh}^{nh+h}(t-s)^{(\alpha-1)}ds, \qquad J(x)=\int_{nh}^{nh+h}s(t-s)^{(\alpha-1)}ds. \tag{25}$$

If we set $t=ph$, the integral equation (21) becomes

$$u^*(ph)-\lambda D_{n,p}u^*(nh)=f^*(ph), \tag{26}$$

or by the matrix expression

$$[I-\lambda D]U^*=F^* \tag{27}$$

The function F^* is a vector of $N+1$ elements, but D is a matrix whose elements are given by

$$D_{n,p}{=}a_{n,p}+a'_{n,p}$$

The matrix $a_{n,p}=A_n(ph)+B_{n-1}(ph)$ is the Toeplitz matrix of order $N+1$, $0\le n,p\le N$. Whereas, the elements of the second matrix $a'_{n,p}$ are zeros except for $a_{0,p}=B_1(ph)$ and $a_{p-1,p}=B_p(ph)$. Therefore, the solution of the formula (27) will take the form

$$U^*=[I-\lambda D]^{-1}F^*, \qquad |I-\lambda D|\neq 0. \qquad \text{(I is the identity matrix.)} \tag{28}$$

5.2. The product Nyström method [1, 10, 14]

Assume the Volterra integral equation can be written in the form

$$u^*(t)-\lambda\int_0^t p(x,y)\overline{k}(x,y)u^*(s)ds=f^*(t), \tag{29}$$

where $p(x,y)=(t-s)^{\alpha-1}$ and $\overline{k}(x,y)$ are 'badly behaved' and 'well behaved' of their arguments, respectively. Equation (29) can be written in the form

$$u^*(t_i)-\lambda\sum_{j=0}^{i}w_{ij}(t_i-s)^{\alpha-1}u^*(s_i)=f^*(t_i), \tag{30}$$

here $x_i=y_i=a+ih$, $i=0,1,..,N$ with $h=\dfrac{T}{N}$, N must be even, and w_{ij} are the weights to be determined. If we approximate the nonsingular part by the second degree of Lagrange interpolation polynomial, therefore,

$$\begin{aligned}\omega_{i,0}&=\beta_1(y_i), &\qquad \omega_{i,2j+1}&=2\gamma_{j+1}(y_i),\\ \omega_{i,2j}&=\zeta_j(y_i)+\beta_{j+1}(y_i), &\qquad \omega_{i,N}(y_i)&=\zeta_{\frac{N}{2}}(y_i)\end{aligned} \tag{31}$$

Assume $v=i-2j+2$, then

$$\begin{aligned}\zeta_j(y_i)&=\frac{h}{2}\int_0^2\rho(\rho-1)(vh-\rho h)^{(\alpha-1)}d\rho,\\ \beta_j(y_i)&=\frac{h}{2}\int_0^2(\rho-1)(\rho-2)(vh-\rho h)^{(\alpha-1)}d\rho,\\ \gamma_j(y_i)&=\frac{h}{2}\int_0^2\rho(2-\rho)(vh-\rho h)^{(\alpha-1)}d\rho.\end{aligned} \tag{32}$$

Now,

$$\psi_i = \int_0^2 \rho^i (vh - \rho h)^{(\alpha-1)} d\rho,$$

equation (31) becomes

$$\begin{aligned}
\omega_{i,0} &= \frac{h}{2}[2\psi_0(v) - 3\psi_1(v) + \psi_2(v)], \qquad v = i \\
\omega_{i,2j} &= \frac{h}{2}[\psi_2(v) - \psi_1(v) + 2\psi_0(v-2) - 3\psi_1(v-2) + \psi_2(v-2)], \qquad v = i - 2j + 2 \\
\omega_{i,2j+1} &= h[2\psi_1(v) - \psi_2(v)], \qquad v = i - 2j \\
\omega_{i,N} &= \frac{h}{2}[\psi_2(v) - \psi_1(v)], \qquad v = i - N + 2
\end{aligned} \tag{33}$$

Therefore, the integral equation (29) is reduced to a system of linear algebraic equations

$$(I - \lambda W)U^* = \overline{F}, \qquad \overline{F} = f_i^* = f^*(t_i) \quad ; i = 0, 1, 2, ..., N \tag{34}$$

which has the solution

$$U^* = [I - \lambda W]^{-1}\overline{F}, \qquad |I - \lambda W| \neq 0 \qquad \text{(I is the identity matrix.)} \tag{35}$$

6. Numerical results

In this section, The fractional integro-differential equation will be solved using the Toeplitz matrix and the product Nyström methods. The results are calculated at $T = 0.009, 0.05, 0.2$, and $\alpha = 0.6, 0.75, 0.9$. Maple 18 software will be used in programming.

Example 1. *Consider the* ***FI-DE***

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = x\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \lambda x(t - t^2 e + t^2) + \lambda \int_0^1 x e^y u(y,t)dy, \quad u_0(x) = 0, \tag{36}$$

after using the Riemann-Liouville fractional integral to the equation (36)*, we obtain*

$$u(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \int_0^1 (t-s)^{\alpha-1} x e^y u(y,s)dyds, \tag{37}$$

where

$$\eta(x,t) = xt - t^2 - \lambda x\left[\frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}(1-e)\right]. \tag{38}$$

Applying the Toeplitz matrix and the product Nyström methods to equation (37)*, the numerical results are shown in the following tables.*

Table 1: The numerical solution at $\alpha = 0.6$ and $T = 0.009$.

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.0018	1.7968×10^{-3}	1.7969×10^{-3}	1.0585×10^{-7}	1.7961×10^{-3}	1.9918×10^{-7}
0.0036	3.5870×10^{-3}	3.5880×10^{-3}	9.6681×10^{-7}	3.5882×10^{-3}	1.1977×10^{-6}
0.0054	5.3708×10^{-3}	5.3731×10^{-3}	3.1263×10^{-6}	5.3743×10^{-3}	3.4511×10^{-6}
0.0072	7.1482×10^{-3}	7.1553×10^{-3}	7.1190×10^{-6}	7.1555×10^{-3}	7.3083×10^{-6}
0.0090	8.9190×10^{-3}	8.9346×10^{-3}	1.5609×10^{-5}	8.9321×10^{-3}	1.3067×10^{-5}

Table 2: The numerical solution at $\alpha = 0.9$ and $T = 0.009$.

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.0018	1.7968×10^{-3}	1.7968×10^{-3}	2.5004×10^{-8}	1.7968×10^{-3}	2.8887×10^{-8}
0.0036	3.5870×10^{-3}	3.5872×10^{-3}	1.8726×10^{-7}	3.5872×10^{-3}	1.9739×10^{-7}
0.0054	5.3708×10^{-3}	5.3715×10^{-3}	6.1311×10^{-7}	5.3715×10^{-3}	6.2659×10^{-7}
0.0072	7.1482×10^{-3}	7.1496×10^{-3}	1.4257×10^{-6}	7.1496×10^{-3}	1.4294×10^{-6}
0.0090	8.9190×10^{-3}	8.9218×10^{-3}	2.7961×10^{-6}	8.9217×10^{-3}	2.7133×10^{-6}

Table 3: The numerical solution at $\alpha = 0.6$ and $T = 0.2$.

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.04	3.84×10^{-2}	3.8413×10^{-2}	1.3325×10^{-5}	3.8426×10^{-2}	2.6277×10^{-5}
0.08	7.36×10^{-2}	7.3717×10^{-2}	1.1738×10^{-4}	7.3748×10^{-2}	1.4837×10^{-4}
0.12	1.056×10^{-1}	1.0596×10^{-1}	3.5658×10^{-4}	1.0591×10^{-1}	3.9846×10^{-4}
0.16	1.344×10^{-1}	1.3516×10^{-1}	7.5928×10^{-4}	1.3518×10^{-1}	7.8131×10^{-4}
0.20	1.60×10^{-1}	1.6162×10^{-1}	1.6062×10^{-3}	1.6128×10^{-1}	1.2842×10^{-3}

Table 4: The numerical solution at $\alpha = 0.9$ and $T = 0.2$.

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.04	3.84×10^{-2}	3.8408×10^{-2}	8.2129×10^{-6}	3.8401×10^{-2}	9.5921×10^{-6}
0.08	7.36×10^{-2}	7.3658×10^{-2}	5.7747×10^{-5}	7.3661×10^{-2}	6.1222×10^{-5}
0.12	1.056×10^{-1}	1.0578×10^{-1}	1.7594×10^{-4}	1.0578×10^{-1}	1.8036×10^{-4}
0.16	1.344×10^{-1}	1.3478×10^{-1}	3.7816×10^{-4}	1.3478×10^{-1}	3.7916×10^{-4}
0.20	1.60×10^{-1}	1.6068×10^{-1}	6.8461×10^{-4}	1.6066×10^{-1}	6.5791×10^{-4}

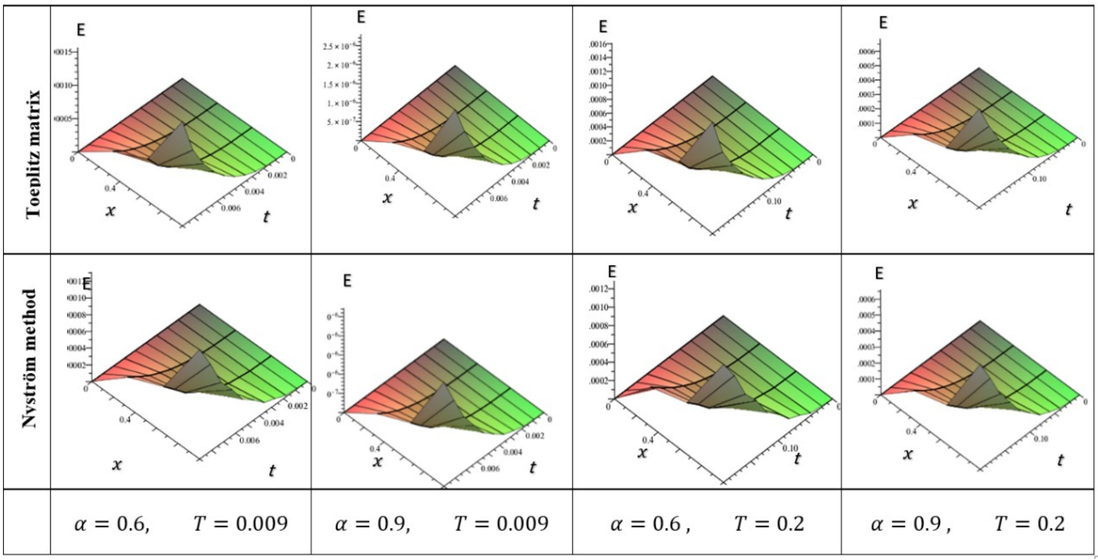

Figure 1: The error by Toeplitz matrix and Nyström product methods at various values for $\alpha = 0.6, 0.75, 0.9$ and at $T = 0.009, 0.05, 0.2$.

Tables and the figures show that the maximum error by Toeplitz matrix is 1.6062×10^{-3} and by Nyström method is 1.2842×10^{-3} both at $\alpha = 0.6$ and $T = 0.2$

Example 2. *Consider the **FI-DE***

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = x^2\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \lambda t^2(\frac{x}{4}+\frac{x^2}{5}) + \lambda\int_0^1 (xy+x^2y^2)u(y,t)dy, \quad u_0(x) = 0 \tag{39}$$

Applying the Riemann-Liouville fractional integral to equation (39)*, we get*

$$u(x,t) = \eta(x,t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_0^1 (t-s)^{\alpha-1}(xy+x^2y^2)u(y,s)dyds, \tag{40}$$

and

$$\eta(x,t) = x^2t^2 - \lambda x^2\frac{2t^{2+\alpha}}{5\Gamma(3+\alpha)} - \lambda x\frac{2t^{2+\alpha}}{4\Gamma(3+\alpha)}. \tag{41}$$

The numerical results after using the Toeplitz matrix and the product Nyström methods to equation (40) *are demonstrated in the following tables.*

Table 5: The numerical results at $\alpha = 0.6$ and $T = 0.009$

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.0018	3.24×10^{-6}	3.2400×10^{-6}	3.0434×10^{-11}	3.2401×10^{-6}	1.2663×10^{-10}
0.0036	1.296×10^{-5}	1.2961×10^{-5}	7.1041×10^{-10}	1.2961×10^{-5}	1.2172×10^{-9}
0.0054	2.916×10^{-5}	2.9164×10^{-5}	3.7224×10^{-9}	2.915×10^{-5}	4.8778×10^{-9}
0.0072	5.184×10^{-5}	5.1852×10^{-5}	1.1978×10^{-8}	5.1853×10^{-5}	1.3282×10^{-8}
0.0090	8.1×10^{-5}	8.1039×10^{-5}	3.9302×10^{-8}	8.1029×10^{-5}	2.9079×10^{-8}

Table 6: The numerical results at $\alpha = 0.9$ and $T = 0.009$

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.0018	3.24×10^{-6}	3.2400×10^{-6}	1.5934×10^{-11}	3.2400×10^{-6}	1.8594×10^{-11}
0.0036	1.296×10^{-5}	1.2962×10^{-5}	1.8965×10^{-10}	1.2960×10^{-5}	2.0850×10^{-10}
0.0054	2.916×10^{-5}	2.9161×10^{-5}	8.7897×10^{-10}	2.9161×10^{-5}	9.2193×10^{-10}
0.0072	5.184×10^{-5}	5.1843×10^{-5}	2.6682×10^{-9}	5.1843×10^{-5}	2.7020×10^{-9}
0.0090	8.1×10^{-5}	8.1007×10^{-5}	6.6027×10^{-9}	8.1006×10^{-5}	6.2751×10^{-9}

Table 7: The numerical results at $\alpha = 0.6$ and $T = 0.2$

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.04	1.6×10^{-3}	1.6001×10^{-3}	9.6612×10^{-8}	1.6004×10^{-3}	4.0211×10^{-7}
0.08	6.4×10^{-3}	6.4023×10^{-3}	2.2562×10^{-6}	6.4039×10^{-3}	3.8664×10^{-6}
0.12	1.44×10^{-2}	1.4412×10^{-2}	1.1827×10^{-5}	1.4416×10^{-2}	1.5501×10^{-5}
0.16	2.56×10^{-2}	2.5638×10^{-2}	3.8076×10^{-5}	2.5642×10^{-2}	4.2227×10^{-5}
0.20	4×10^{-2}	4.0125×10^{-2}	1.2499×10^{-4}	4.0093×10^{-2}	9.2503×10^{-5}

Table 8: The numerical results at $\alpha = 0.9$ and $T = 0.2$

t	Exact solution	Toeplitz solution	Error	Nyström solution	Error
0	0	0	0	0	0
0.04	1.6×10^{-3}	1.6001×10^{-3}	1.2824×10^{-7}	1.6001×10^{-3}	1.4965×10^{-7}
0.08	6.4×10^{-3}	6.4015×10^{-3}	1.5266×10^{-6}	6.4017×10^{-3}	1.6784×10^{-6}
0.12	1.44×10^{-2}	1.4407×10^{-2}	7.0765×10^{-6}	1.4407×10^{-2}	7.4226×10^{-6}
0.16	2.56×10^{-2}	2.5621×10^{-2}	2.1487×10^{-5}	2.5622×10^{-2}	2.1759×10^{-5}
0.20	4×10^{-2}	4.0053×10^{-2}	5.3186×10^{-5}	4.0051×10^{-2}	5.0548×10^{-5}

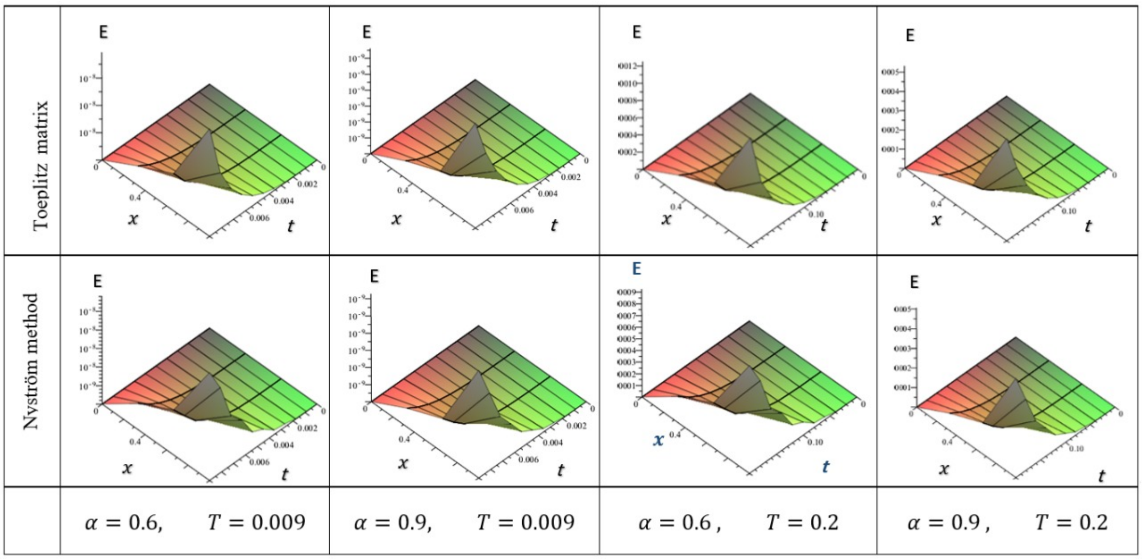

Figure 2: The error by Toeplitz matrix and Nyström product methods at various values for $\alpha = 0.6, 0.75, 0.9$ and at $T = 0.009, 0.05, 0.2$.

It is clear the largest error by Toeplitz matrix and Nyström product methods is at $\alpha = 0.6$ *and* $T = 0.2$ *. The largest error by Toeplitz matrix is* 1.2499×10^{-4}*, where as it by Nyström method is* 9.2503×10^{-5}

Notes

Through the results given in the previous tables, we note that both the Toeplitz matrix and the Nyström product methods are effective in solving our problem. The tables and the figures show that the numerical solutions were obtained using Toeplitz matrix and the product Nyström are very close to the exact solution. However, the results using Nyström are better than corvesponding results by Toeplitz matrix . Also, for any choice of the time value T, there is a negative proportionality between α and the error, i.e. the higher the value of α, the smaller the error. We also note that with increasing time values, the error for any values of α increases.

7. Conclusion

We have assumed a **LFI-DE** with a fractional-order $0 < \alpha < 1$. Then, the Riemann-Liouville fractional integral of order $\alpha > 0$ was applied to the **FI-DE** to convert it to a **V-FIE** with Abel's kernel. After that, the uniqueness of the solution of the **V-FIE**, which is equivalent to the **LFI-DE**, has been proved by using the Picard method. Finally, two numerical methods; the Toeplitz matrix and product Nyström methods were used to find the numerical solution. The results showed the efficiency and accuracy of the two methods through the multi-values of α and T.

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