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Convergence and Stability of Optimal two-step fourth-order and its expanding to sixth order for solving nonlinear equations

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Abstract. In this paper, we provided a new fourth-order optimal method. This method demands three functional evaluations, and according to Kung-Traub it is considered as one of the optimal methods with efficiency indicator I that reaches 1.587. Furthermore, we can extend its convergence to obtain a new sixth-order method where its efficiency indicator is 1.565. In this paper, we also discuss the convergence analysis of our new methods as it was established that the new methods have convergence orders four and six. Moreover, we will illustrate our study of the stability criterion of the new methods, and we will present the stability theorems along with some examples which prove that our methods are stable. Finally, we have discussed attraction basins for those suggested methods and compared them with methods that have the same order, and we have applied them for numerical examples to clarify the performance and efficiency of the proposed methods.

2020 Mathematics Subject Classifications: 41A25, 65H05, 65K05.

Key Words and Phrases: Nonlinear equations, basins of attraction, efficiency index, iterative methods, complex dynamics.

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1. Introduction

It seems that a non-linear equation $f(x) = 0$ is one of the most important problems in numerical analysis in which you can find the root for these equations using iterative methods. One of these methods is Newton's classic method where it converges to the second order. In recent years, researchers suggested a large number of iterative methods that solve the non-linear equations with different orders in which they constructed iterative methods that consist of two steps with the third, fourth and fifth convergence order in [1, 5, 7], as well as iterative methods constructed of three steps with the sixth, seventh and eighth convergence order in [10, 16, 17] and references therein. Efficiency indicator

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EI is measured for iterative methods by p^m form, where p is the convergence order and m is the number of evaluation functions [6].

In this paper, our main objective is to theoretically prove that the error equations are of fourth-order from Jarrett type for solving non-linear equations where the efficiency indicator E1 for this method is 1.578 and this method is optimal according to Kung-Traub conjecture [6]. In addition to expanding the fourth-order method to the sixth-order method by adding a third step which is mentioned in [14]; this method is not optimal because the efficiency indicator EI will be 1.565. And the final objective is to study theories for the stability of fixed point in map f for the new methods.

The rest of the paper will be organized in sections, as in Section 2, the iterative method of the fourth-order and the equation error is given theoretically to show that the proposed method has fourth-order convergence. Section 3 will expand the fourth-order method to the sixth-order method. Section 4 will include the study of iterative methods stability and will introduce stability theories analysis for the proposed methods. In Section 5, the results of our methods for some examples in the real domain and their comparison with different methods of the same order will be shown. In Section 6, the attraction basins for the fourth-order and sixth-order methods and comparing them with ones from the same order will be fully discussed.

2. Construction of the new fourth order method

In this section, we will introduce a new fourth-order optimal method. We have constructed the following iterative scheme via two steps; the first one is by the Jarratt method step. In the second one we have involved weight function (W) depending on k. The iterative expression is

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - W(k_n)\frac{f(x_n)}{f'(x_n)+f'(y_n)} \end{aligned} \tag{1}$$

Where the weight function is $W(k) = \dfrac{Ak+Bk^2}{C+k}$, $k = \frac{f'(y)}{f'(x)}$. We can notice that the probability functions in the iterative scheme (1) are three. Thus, the efficiency indicator will be $I = 4^{\frac{1}{3}} = 1.587$.

2.1. Convergence Analysis

The convergence analysis will be discussed for the iterative method in the following theorem, where we used mathematica program 11 to prove that the convergence order for this method is four.

Theorem 1. *Let$f : I \subset R \to R$ be a real sufficiently differentiable function in an open interval I and let $a \in I$ be a simple root of $f(x) = 0$. If x_0 is close enough to a when*

$W(k) = \dfrac{Ak + Bk^2}{C + k}$ *satisfies* $A = \dfrac{17}{24}$ $, B = -\dfrac{41}{120}$ *and* $C = -\dfrac{9}{20}$, *then iterative family* (1) *converges to a with order of convergence four, and its error equation is:*

$$e_{n+1} = \left(\frac{43}{99}c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + O\left(e_n^5\right) \tag{2}$$

Proof. Using Taylor expansion, $f\left(x_n\right)$ *and* $f'\left(x_n\right)$ can be obtained as

$$f\left(x_n\right) = f'(a)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_4^4\right] + O\left(e_n^5\right) \tag{3}$$

and

$$f'\left(x_n\right) = f'(a)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_4^3\right] + O\left(e_n^4\right) \tag{4}$$

then from (3) and (4), we have

$$\frac{f\left(x_n\right)}{f'\left(x_n\right)} = e_n - c_2e_n^2 + \left(2c_2^2 - 2c_3\right)e_n^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right)e_n^4 + O(e_n^5) \tag{5}$$

Now from (1) and (5) , we get

$$\begin{aligned} y_n - a &= e_n - \frac{2}{3}\frac{f\left(x_n\right)}{f'\left(x_n\right)} \\ &= \frac{1}{3}e_n + \frac{2}{3}c_2e_n^2 - \left(\frac{4}{3}c_2^2 - \frac{4}{3}c_3\right)e_n^3 + \left(\frac{8}{3}c_2^3 - \frac{14}{3}c_2c_3 + 2c_4\right)e_n^4 + O(e_n^5) \end{aligned} \tag{6}$$

From (6), we calculate $f'\left(y_n\right)$ as:

$$\begin{aligned} f'\left(y_n\right) = &f'\left(a\right)\left[1 + \frac{2}{3}c_2e_n + \left(\frac{4}{3}c_2^2 - \frac{1}{3}c_3\right)e_n^2 + \left(-\frac{8}{3}c_2^3 + 4c_2c_3 + \frac{4}{27}c_4\right)e_n^3\right. \\ &\left. + \left(\frac{16}{3}c_2^4 - \frac{32}{3}c_2^2c_3 + \frac{8}{3}c_3^2 + \frac{44}{9}c_2c_4 + \frac{5}{81}c_5\right)e_n^4 + O(e_n^5)\right] \end{aligned} \tag{7}$$

The expansion of the weight function varible k is

$$\begin{aligned} k = &\frac{f'(y)}{f'(x)} = 1 - \frac{4}{3}c_2e_n + \left(4c_2^2 - \frac{8}{3}c_3\right)e_n^2 + \left(\frac{-32}{3}c_2^3 + \frac{40}{3}c_2c_3 - \frac{104}{27}c_4\right)e_n^3 \\ &+ \left(\frac{80}{3}c_2^4 - \frac{148}{3}c_2^2c_3 + \frac{32}{3}c_3^2 + \frac{484}{27}c_2c_4 - \frac{400}{81}c_5\right)e_n^4 + O(e_n^5) \end{aligned}$$

and, therefore, weight $W\left(k\right)$ around one results in

$$W(k) = \frac{Ak + Bk^2}{C + k}$$

$$= \frac{A+B}{1+C} - \left(\frac{4\left(B+AC+2BC\right)}{3\left(1+C\right)^2}\right) c_2 e_n + \frac{4}{9\left(1+C\right)^3}\Big[\big(AC\left(5+9C\right)+B\big(9+27C + 22C^2\big)\big)c_2^2 - 6\left(AC\left(1+C\right)+B\left(1+3C+2C^2\right)\right)c_3\Big]e_n^2 - \frac{8}{27\left(1+C\right)^4}\big[\big(4AC\big(2+9C + 9C^2\big) + 4B\left(9+36C+52C^2+27C^3\right)\big)c_2^3 - 3\left(1+C\right)\left(AC\left(7+15C\right)+B\big(15+C\big(45 + 38C\big)\big)\right)c_2c_3 + 13\left(1+C\right)^2\left(B+AC+2BC\right)c_4\big]e_n^3 + \frac{4}{81\left(1+C\right)^5}\big[AC\big(4\left(2+3C\right)\left(1+15C\left(2+3C\right)\right)c_2^4 - 9\left(1+C\right)\left(15+C\left(94+111C\right)\right)c_2^2c_3 + \left(1+C\right)^2\left(155+363C\right)c_2c_4 + 4\left(1+C\right)^2\left(18\left(1+3C\right)c_3^2 - 25\left(1+C\right)c_5\right)\big) + B\big(4\big(135+C\big(675+C\big(1348 + 99C\left(13+5C\right)\big)\big)\big)c_2^4 - 9\left(1+C\right)\left(111+C\left(444+7C\left(93+50C\right)\right)\right)c_2^2c_3 + \left(1+C\right)^2\left(363+C\left(1089+934C\right)\right)c_2c_4 + 4\left(1+C\right)^2\left(18\left(3+C\left(9+8C\right)\right)c_3^2 - 25\left(1+C\right)\left(1+2C\right)c_5\right)\big)\big]e_n^4 + O(e_n^5) \tag{8}$$

And so, from (4),(5) and (7) we have

$$\frac{f\left(x_n\right)}{f'\left(x_n\right)+f'\left(y_n\right)} = \frac{1}{2}e_n - \frac{1}{6}c_2e_n^2 + \frac{1}{9}\left(-c_2^2-3c_3\right)e_n^3 + \frac{1}{54}\left(50c_2^3-15c_2c_3-29c_4\right)e_n^4 + O(e_n^5) \tag{9}$$

Finally, when using (8) and (9), the error equation of any method of (1) becames:

$$e_{n+1} = y_n - a - W(k)\frac{f(x_n)}{f'(x_n)+f'(y_n)} = \left(\frac{1}{3}-\frac{A+B}{2+2C}\right)e_n + \left(\frac{2}{3}+\frac{A+5AC+B\left(5+9C\right)}{6\left(1+C\right)^2}\right)c_2e_n^2 + \frac{1}{9\left(1+C\right)^3}\Big[c_2^2\Big(-12\left(1+C\right)^3 - A\left(-1+10C+19C^2\right) - B\left(19+58C+47C^2\right)\Big) + 3\left(1+C\right)c_3\big(A+5AC+4\left(1+C\right)^2 + B\left(5+9C\right)\big)\Big]e_n^3 + \frac{1}{54\left(1+C\right)^4}\Big[c_2^3\big(144\left(1+C\right)^4 + 2A\left(-25-37C+89C^2+133C^3\right) + 2B\left(133+557C+835C^2+443C^3\right)\big) - 3\left(1+C\right)c_2c_3\big(84\left(1+C\right)^3 + A\big(-5+62C + 131C^2\big) + B\left(131+398C+331C^2\right)\big) + \left(1+C\right)^2c_4\big(108\left(1+C\right)^2 + A\left(29+133C\right) + B\left(133+237C\right)\big)\Big]e_n^4 + O(e_n^5) \tag{10}$$

From (10) the conditions on the wight function W are:

$$\left.\begin{aligned} 2\left(1+C\right)-3\left(A+B\right)&=0\\ 4\left(1+C\right)^2+A\left(1+5C\right)+B\left(5+9C\right)&=0\\ -12\left(1+C\right)^3+A\left(-1+10C+19C^2\right)-B\left(19+58C+47C^2\right)&=0 \end{aligned}\right\}$$

Now by solving the system above one gets the values A,B and C we get $A = \frac{17}{24}$, $B = -\frac{41}{120}$ and $C = -\frac{9}{20}$. Replacing these values on (10),the error equation of any method of (1) is

$$e_{n+1} = \left(\frac{43}{99}c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + O\left(e_n^5\right) \tag{11}$$

Consequently, the typical method for our proven theory can be stated in the following:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \left[\frac{85f'(y_n)f'(x_n) - 41f'(y_n)^2}{-54f'(x_n)^2 + 120f'(y_n)f'(x_n)}\right]\frac{f(x_n)}{f'(x_n) + f'(y_n)} \end{aligned} \tag{12}$$

We will call our new method (12) by AMF1, note that it requires three functional evaluations per step.

3. Extension of the fourth order method to a sixth order method

This section presents the development we have made on the new fourth-order method AMF1 into a sixth-order method by adding one step as shown in [14]. We can notice that, this method requires four probability functions for each step thus, their efficiency indication is $I = 6^{\frac{1}{4}} = 1.565$ and it is worse than $I = 1.587$. We consider another method consisting of three steps as follows:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \left[\frac{85f'(y_n)f'(x_n) - 41f'(y_n)^2}{-54f'(x_n)^2 + 120f'(y_n)f'(x_n)}\right]\frac{f(x_n)}{f'(x_n) + f'(y_n)} \\ x_{n+1} &= z_n - \left(\frac{1}{3f'(x_n)} - \frac{8}{15f'(x_n) - 27f'(y_n)}\right)\cdot f(z_n) \end{aligned} \tag{13}$$

Here, we will clarify-through the following theory- that the convergence order of this (13) method will reach the sixth order, we have proved that using Mathematica 11.

Theorem 2. *Let $f : I \subset R \to R$ be a smooth function with continuous derivatives up to fourth order. If f(x)=0 has a simple root a in the open interval I where x_0 is chosen in a sufficiently small neighborhood of a, then the iterative scheme given by* (13) *has a sixth-order convergence and the error equation is:*

$$e_{n+1} = \left(-\frac{43}{99}c_2^3c_3 - c_2c_3^2 - \frac{1}{9}c_3c_4\right)e_n^6 + O\left(e_n^7\right)$$

Proof. Using Taylor epansion $f(z_n)$ about a gives :

$$f(z_n) = f'(a)\Big[\left(\frac{43}{99}c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + \left(-\frac{148}{121}c_2^4 + \frac{152}{33}c_2^2c_3 - 2c_3^2 - \frac{20}{9}c_2c_4 + \frac{8}{27}c_5\right)e_n^5$$
$$+ \frac{2}{107811}\big(114071c_2^5 - 602613c_2^3c_3 + 394581c_2^2c_4 + 16335c_2\left(37c_3^2 - 11c_5\right)$$
$$+ 3993\left(-99c_3c_4 + 7c_6\right)\big)e_n^6\Big] + O\left(e_n^7\right) \tag{14}$$

though the use of equations (4), (7) and (14) in equation (13) we obtain:

$$e_{n+1} = \left(-\frac{43}{99}c_2^3c_3 - c_2c_3^2 - \frac{1}{9}c_3c_4\right)e_n^6 + O\left(e_n^7\right) \tag{15}$$

Thus (15) shows that method (13) has sixth-order convergence.

As a result, from the theory (2) we can verify that the iterative method (13) which will be denoted as AMF2 is a convergent method of sixth-order. It considered as an extension of the method AMF1.

4. Stability Analysis

Lately, many researchers have steadily studied iterative methods to solve nonlinear equations Najmuddin Ahmad and Vimal Pratap have studied the stability of some iterative methods mentioned in [11]; as they have set some simple-but powerful criteria- for the local stability of a fixed point and provided stability analysis theories to the methods as well as compared them with many other methods through their stability and have also clarified that their methods are more stable. In addition, in research [2], the researchers have studied suggested family stability with sixth-order where stability is studied through complex dynamics and numerical examples.

Now, we will discuss the stability of our new methods (AMF1 and AMF2). We will present some theories to prove whether the new methods are stable or not, and we will perform several numerical tests, in order to check the theoretical stability results of our new methods. Before we present the stability theorems, we will introduce the theorem of the hyperbolic fixed point.

Theorem 3 (hyperbolic fixed point [4])**.** *If f is map continuously differentiable at a and let a be a hyperbolic fixed point of f. The following statements then hold true*

- *If $|f'(a)| < 1$, then a is asymptotically stable*
- *If $|f'(a)| > 1$, then a is unstable*

Theorem 4. *(Stability Analysis of AMF1) Let $f : I \subset R \to R$ be a real sufficiently differentiable function in an open interval I and let $a \in I$ be a simple root of $f(x) = 0$. If x_0 is sufficiently close to a then the method* (12) *is stable*

Proof. Considering

$$y_N(x) = x - \frac{2}{3}\frac{f(x)}{f'(x)}$$
$$f_N(x) = y_N(x) - \left[\frac{85f'(y_N(x))\,f'(x) - 41f'(y_N(x))^2}{-54f'(x)^2 + 120f'(y_N)\,f'(x)}\right]\frac{f(x)}{f'(x) + f'(y_N(x))} \tag{16}$$

Let a be a simple zero of f and x^* is a fixed points of f. We need to make sure that the fixed point $x^* = a$ is stable.
To do this, we evaluate $f'_N(a)$ as follows:

$$y'_N(x) = \frac{1}{3} + \frac{2f(x)\,f''(x)}{3\,[f'(x)]^2}$$

We put the root a instead of x

$$y'_N(a) = \frac{1}{3} + \frac{2f(a)\,f''(a)}{3\,[f'(a)]^2}$$

Since a is simple zero of f then $f(a) = 0$ and we get

$$y'_N(a) = \frac{1}{3}$$

Now we calculate $f'_N(x)$

$$\begin{aligned}
f'_N(x) = \frac{1}{3} + \frac{2f(x)\,f''(x)}{3f'(x)^2} &- \frac{f'(x)\left(121f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right) - 61f'\left(x-\frac{2f(x)}{3f'(x)}\right)^2\right)}{\left(2f'(x)+f'\left(x-\frac{2f(x)}{3f'(x)}\right)\right)\left(114f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right)-54f'(x)^2\right)} \\
&+ \frac{f(x)\left(121f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right) - 61f'\left(x-\frac{2f(x)}{3f'(x)}\right)^2\right)\left(2f''(x)+\left(\frac{2f(x)f''(x)}{3f'(x)^2}+\frac{1}{3}\right)f''\left(x-\frac{2f(x)}{3f'(x)}\right)\right)}{\left(2f'(x)+f'\left(x-\frac{2f(x)}{3f'(x)}\right)\right)^2\left(114f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right)-54f'(x)^2\right)} \\
&+ \frac{f(x)\left(121f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right) - 61f'\left(x-\frac{2f(x)}{3f'(x)}\right)^2\right)}{\left(2f'(x)+f'\left(x-\frac{2f(x)}{3f'(x)}\right)\right)} \\
&\cdot \frac{\left(-108f'(x)f''(x)+114f'\left(x-\frac{2f(x)}{3f'(x)}\right)f''(x)+114f'(x)\left(\frac{2f(x)f''(x)}{3f'(x)^2}+\frac{1}{3}\right)f''\left(x-\frac{2f(x)}{3f'(x)}\right)\right)}{\left(114f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right)-54f'(x)^2\right)^2} \\
&- \frac{f(x)\left(121f'\left(x-\frac{2f(x)}{3f'(x)}\right)f''(x)+121f'(x)\left(\frac{2f(x)f''(x)}{3f'(x)^2}+\frac{1}{3}\right)\right)}{\left(2f'(x)+f'\left(x-\frac{2f(x)}{3f'(x)}\right)\right)}
\end{aligned}$$

$$\cdot \frac{f(x)\left(f''\left(x-\frac{2f(x)}{3f'(x)}\right)-122f'\left(x-\frac{2f(x)}{3f'(x)}\right)\left(\frac{2f(x)f''(x)}{3f'(x)^2}+\frac{1}{3}\right)f''\left(x-\frac{2f(x)}{3f'(x)}\right)\right)}{\left(114f'(x)f'\left(x-\frac{2f(x)}{3f'(x)}\right)-54f'(x)^2\right)}$$

put $x = a$ in the first step of equation (16), we get $y_N(a) = a$
Substituting the value $y_N(a)$ and $x = a$ in $f_N(x)$, we get $f'_N(a) = 0$ and $|f'_N(a)| < 1$
Then from the theorem of hyperbolic fixed point we get that a is asymptotically stable
This shows that the method AMF1 is stability for the fixed point $x = a$ of f_N.

Theorem 5. *(Stability Analysis of AMF2) Let $f : I \subset R \to R$ be a real sufficiently differentiable function in an open interval I and let $a \in I$ be a simple root of $f(x) = 0$.If x_0 is sufficiently close to a then the method* (13) *is stable*

Proof. Considering

$$\begin{aligned} y_N(x) &= x - \frac{2}{3}\frac{f(x)}{f'(x)} \\ z_N(x) &= y_N(x) - \left[\frac{85f'(y_N(x))f'(x) - 41f'(y_N(x))^2}{-54f'(x)^2 + 120f'(y_N)f'(x)}\right]\frac{f(x)}{f'(x)+f'(y_N(x))} \\ f_N(x) &= z_N(x) - \left(\frac{1}{3f'(x)} - \frac{8}{15f'(x) - 27f'(y_N(x))}\right)\cdot f(z_N(x)) \end{aligned} \tag{17}$$

Let a be a simple zero of f and x^* is a fixed points of f. We need to prove that the fixed point $x^* = a$ is stable.
To do this, we evaluate $f'_N(a)$ as:

$$y_N(a) = a - \frac{2}{3}\frac{f(a)}{f'(a)}$$

Since a is simple zero of f then $f(a) = 0$ and we get

$$y_N(a) = a \quad \text{and} \quad z_N(a) = a$$

Now $f'_N(a) = 0$, $|f'_N(a)| < 1$.
Then from the theorem of hyperbolic, fixed point we get that a is asymptotically stable
This shows that the method AMF2 is stability for the fixed point $x = a$ of f_N.

4.1. Numerical Results to study stability analysis to the new methods

We present some examples existing in research [2] to check the stability results of the new methods obtained in the previous section. These new methods are applied on four nonlinear equations, which their given expressions and correspondence roots are:

$$f_1(x) = sin(x) - x^2 + 1, \quad a = -0.6367326508$$

$$f_2(x) = cos\,(x) - xe^x + x^2, \quad a = 0.6391540963$$
$$f_3(x) = x^3 + 4x^2 - 10, \quad a = 1.3652300134$$
$$f_4(x) = \sqrt{x^2 + 2x + 5} - 2sin\,(x) - x^2 + 3, \quad a = 2.3319676559$$

We carried out a stability analysis of the new methods of orders four and six where we start the iterations with different initial estimates: close ($x_0 \approx a$), far ($x_0 \approx 10a$), and very far ($x_0 \approx 100a$) to the root $x = a$, respectively. This enables us to measure, how demanding the methods related to the initial estimation for finding a solution are. Numerical computations have been carried out in the Mathematica 11 with 128 significant digits of mantissa. We analyze the required number of iterations (iter) to converge to the solution, so that the stopping criteria $|x_{n+1} - x_n| < 10^{-15}$ or $|f\,(x_{n+1})| < 10^{-15}$ are satisfied, we note that $|x_{n+1} - x_n|$ represents the error estimation between two consecutive iterations and $|f\,(x_{n+1})|$ is the residual error of the nonlinear test function.

In order to check the theoretical order of convergence, we calculate the approximate computational order of convergence (ACOC) given by Cordero and Torregrosa in [1].

In Tables 1 and 2, we illustrate the numerical performance of the new iterative methods associated with close, far, and very far initial estimations. On one hand, we observe that the methods always converge to the solution, although the number of iterations (iter) needed differs from initial guess and nonlinear equation to another. As a result, in estimations close to the root, the methods converge to a with number iterations $3 \leq iter \leq 4$. When the initial guess is far from the root, they converge to a with a $4 \leq iter \leq 6$. When the starting guess is very far from the root, the iterative methods converge to a with a $5 \leq iter \leq 11$.

Table 1: Numerical performance of AMF1 method for some nonlinear equations

Function	x_0	$\lvert x_{n+1} - x_n\rvert$	$\lvert f(x_{n+1})\rvert$	*iter*	*ACOC*
	Close to a				
f_1	-1.6	4.8721×10^{-41}	$0. \times 10^{-127}$	4	3.99
f_2	-0.4	8.3094×10^{-20}	1.3×10^{-77}	4	4.02
f_3	0.4	3.1224×10^{-24}	3.4×10^{-95}	4	3.99
f_4	1.3	5.6497×10^{-49}	$0. \times 10^{-125}$	4	4.00
	Far from a				
f_1	-6	9.1996×10^{-48}	$0. \times 10^{-125}$	5	3.99
f_2	6	3.2676×10^{-45}	$0. \times 10^{-121}$	6	4.00
f_3	14	1.7088×10^{-17}	3.0×10^{-68}	5	3.97
f_4	23	6.4591×10^{-32}	$0. \times 10^{-122}$	5	3.98
	Very far from a				
f_1	-60	3.8177×10^{-34}	$0. \times 10^{-122}$	6	3.99
f_2	60	1.6463×10^{-42}	$0. \times 10^{-109}$	11	4.00
f_3	140	2.2842×10^{-49}	$0. \times 10^{-118}$	8	3.99

f_4	230	6.8461×10^{-29}	3.5×10^{-115}	7	3.97

Table 2: Numerical performance of AMF2 method for some nonlinear equations

Function	x_0	$\lvert x_{n+1} - x_n \rvert$	$\lvert f(x_{n+1}) \rvert$	*iter*	*ACOC*
	Close to a				
f_1	-1.6	2.1467×10^{-23}	$0. \times 10^{-126}$	3	5.80
f_2	-0.4	2.7202×10^{-68}	$0. \times 10^{-124}$	4	5.99
f_3	0.4	3.4437×10^{-58}	$0. \times 10^{-124}$	4	6.01
f_4	1.3	1.1276×10^{-25}	$0. \times 10^{-124}$	3	5.86
	Far from a				
f_1	6	1.4799×10^{-41}	$0. \times 10^{-124}$	4	6.10
f_2	6	4.7400×10^{-19}	1.4×10^{-111}	5	5.50
f_3	14	2.0648×10^{-17}	1.7×10^{-102}	4	5.66
f_4	23	1.8704×10^{-19}	6.7×10^{-116}	4	6.29
	Very far from a				
f_1	-60	1.5327×10^{-21}	$0. \times 10^{-121}$	5	5.72
f_2	-60	7.2911×10^{-19}	$0. \times 10^{-106}$	10	5.68
f_3	140	2.2199×10^{-37}	$0. \times 10^{-118}$	6	5.72
f_4	230	4.7705×10^{-81}	$0. \times 10^{-117}$	6	6.00

The above tables confirm that the new methods are stable, and always converge to the solution for any initial guess and nonlinear test function used.

4.2. Some existing fourth and sixth order methods

Consider the following fourth-order and sixth-order methods for the purpose of comparing results:
Method of Francisco I. Chicharro, Alicia Cordero, Neus Garrido and Juan Torregrosa(G1) [5]

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{f^2(x_n) + f(x_n) f(y_n) + 2f^2(y_n)}{f(x_n) f'(x_n)}
\end{aligned}
\tag{18}
$$

Method of M.Hafiz, M.Khirallaha(MHK) [9]

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= y_n - \frac{f(x_n)}{6\left(f'(x_n) + f'(y_n)\right)} \cdot \left[1 + 3\frac{f'(x_n)}{f'(y_n)} - 4\ln\left(2 - \frac{f'(x_n)}{f'(y_n)}\right)\right]
\end{aligned}
\tag{19}
$$

Method of Ekta Sharma, Sunil Panday and Mona Dwivedi (NPM)[3]

$$\begin{aligned} y_n &= x_n - \frac{2}{3}u_n \\ x_{n+1} &= x_n - \frac{4f(x_n)}{f'(x_n)+3f'(y_n)}\left(1+u_n^3\right) - \frac{9}{16}\left(\frac{\phi}{f'(x_n)}\right)^2 u_n^3 \end{aligned} \tag{20}$$

Where

$$u_n = \frac{f(x_n)}{f'(x_n)}, \qquad \phi = \frac{f'(x_n)-f'(y_n)}{u_n}$$

Method of Kalyanasundaram Madhu and Jayakumar Jayaraman (PM1)[8]

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{4f(x_n)}{f'(x_n)+3f'(y_n)}\left(1+\frac{5}{16}(\tau-1)^2\right)\left(1+\frac{1}{4}(\eta-1)^2+\frac{1}{6}(\eta-1)^3\right) \end{aligned} \tag{21}$$

Where $\tau = \dfrac{f'(y_n)}{f'(x_n)}$, $\eta = \dfrac{f'(x_n)}{f'(y_n)}$

Method of P. Maroju · Á. Magreñán· S.Motsa. Í. Sarría2 (Method(8))[13]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \left(1+\frac{f(y_n)}{f(x_n)}\frac{(f(x_n)+(\alpha-1)f(y_n))}{(f(x_n)-f(y_n))}\right)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)+g''(x_n)(z_n-x_n)} \end{aligned} \tag{22}$$

Where $\alpha = 2, \quad g''(x_n) = \dfrac{2f(y_n)f'(x_n)^2}{f(x_n)^2}$

Method of Alicia Cordero, Marlon-Martínez and Juan Torregrosa(GMT(1)) [2]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{2f[x_n,y_n]-f'(x_n)} \\ x_{n+1} &= z_n - (\alpha+(1+\alpha)u_n+(1-\alpha)v_n)\frac{f(z_n)}{f'(x_n)} \end{aligned} \tag{23}$$

Where $u_n = 1-\dfrac{f[x_n,y_n]}{f'(x_n)}$, $v_n = \dfrac{f'(x_n)}{f[x_n,y_n]}$ if $\alpha = 1$

5. Numerical examples in complex domain

The attraction basins technique is a method to show how different starting points affect the behaviour of the function. In this way, we can compare different iterative methods depending on the convergence area of the basins of attraction of the roots, where the iterative method is better if it has a larger area of convergence, which means that the number of black dots is less.

Many scholars and researchers have compared their iterative methods using the attraction basins. For example, Obadah Solaiman in [12], has compared six iterative methods with a different order for solving non-linear equations, and he concluded that getting a better attraction basin does not only depend on convergence order as there are many other factors that affect the result, like the number of arithmetic operations which is needed at each iteration, number of steps in the iterative scheme, and number of functional evaluations in each iteration. We aim here to use the attraction basins as an indicator for comparing the iterative methods which have been mentioned above with the new methods AMF1 and AMF2. We have drawn the attraction basins for the iterative methods and have also applied them to some polynomial functions in the examples from 1 to 3 as described below.

We used the rectangular D to draw the attraction basins and D is a subset of C; we take the rectangular $[-2,2]\times[-2,2]$ from 256×256 in example 1 and 3; and in example 2, we take the rectangular $[-3,3]\times[-3,3]$ where it contains all nonlinear equation roots $f(z)=0$; we assigned special and different colours for basins for better visibility; each root takes a special colour. We allocate the colour black if the method was unsuccessful to find the solution under the conditions established for convergence, such as tolerance $\varepsilon = 10^{-3}$ and the maximum of 20 iterations. In Tables 3 to 8, we compared the methods in the complex domain, where the first column represents the number of black dots (NB). The fewer black dots indicate that the method is better since the black colour denotes lack of convergence to any of the roots with 20 iterations. The second column (BI) represents the brightness indicator as the brighter the colour, the better the method, in which reaching the solution takes fewer repetitions. The third column (T) represents the time that the method had taken to reach the solution. At last, the fourth column(I/P) is the mean of iterations, measured in iterations/point, (I is for iterations and P is for points). The Figures and tables below were generated with Mathematica 11.

Example 1. *We consider the nonlinear equation*

$$f_1(z) = z^3 - z$$

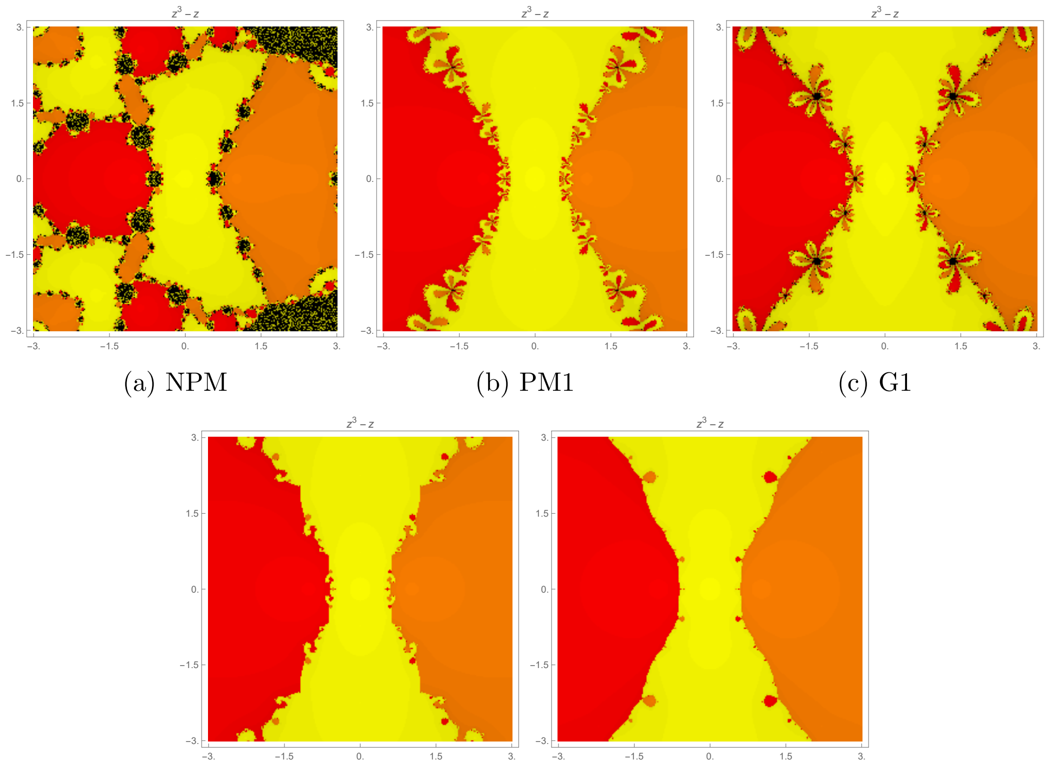
A polynomial has three roots which are $0, 1, -1$. *The fourth-order methods basins of attraction are depicted in Figure 1 and the sixth-order methods are in Figure 2. The fourth-order methods are compared in Table 3. We can notice that the best methods are AMF1 and MHK and followed by the method PM1. Table 2 compared the results of the sixth order methods and. From the results in Table 4 AMF2 is the best.*

Table 3: Comparison of different methods in complex plane in Example 1

Method	NB	BI	T	I/P
NPM	6060	0.44707	49.19	4.09
PM1	24	0.484127	39.16	3.91
G1	470	0.479182	22.39	4.68
MHK	0	0.479169	22.22	3.39
AMF1	0	0.478249	24.48	3.05

Table 4: Comparison of different methods in complex plane in Example 1

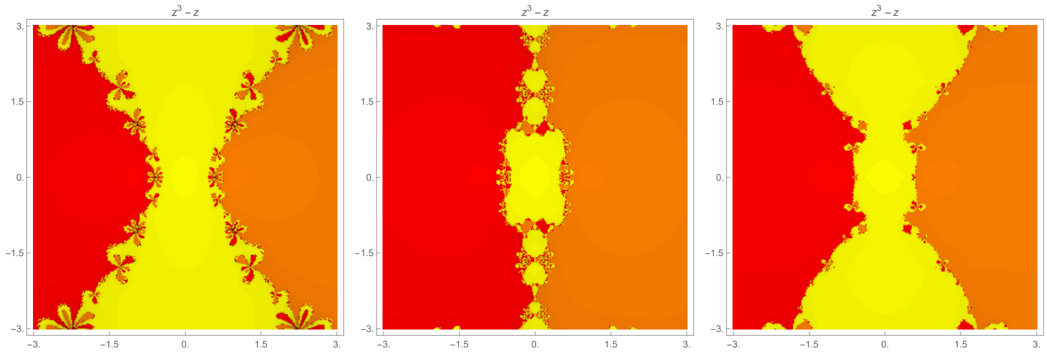
Method	NB	BI	T	I/P
$GMT(1)$	30	0.490129	27.08	3.48
$Method8$	0	0.414513	31.25	2.98
$AMF2$	0	0.4784	30.59	2.55



(a) NPM (b) PM1 (c) G1

(d) MHK (e) AMF1

Figure 1: Basins of attraction to iterative methods for $f_1(z) = z^3 - z$



(a) GMT(1) (b) Method8 (c) AMF2

Figure 2: Basins of attraction to iterative methods for $f_1(z) = z^3 - z$

Example 2: We consider the equation

$$f_2(z) = z^4 - z + i$$

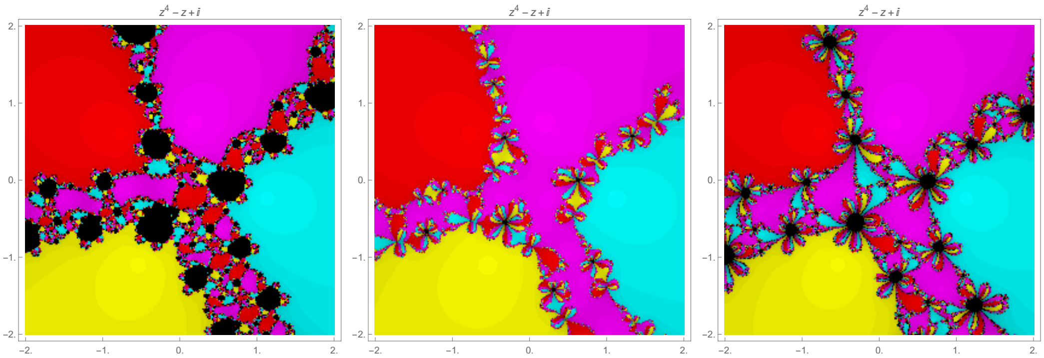
Which has four roots which are $-0.759845 + 0.592595i, -0.532605 - 1.08829i, 0.181924 + 0.732098i, 1.11052 - 0.236405i$. The basins of attraction for the fourth-order methods are depicted in Figure 3 and the sixth-order methods are presented in Figure 4. Based on comparing the fourth-order methods in Table 5, we can notice that, the best methods are MHK and AMF1 in terms of the number of black points and followed by PM1. As for the brightness index and average repetitions, the new method AMF1 is the best. From Table 6 for comparing sixth-order methods, we notice that the best method is the new method AMF2

Table 5: Comparison of different methods in complex plane in Example 2

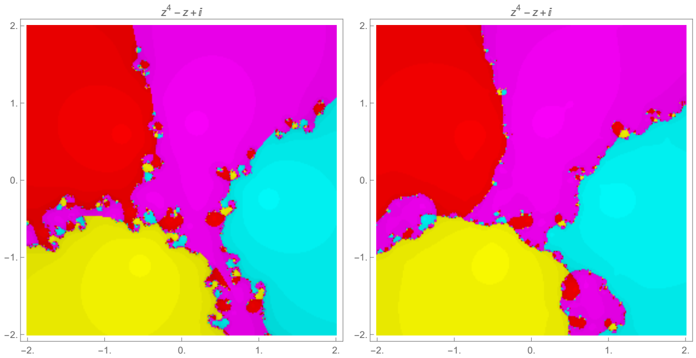
Method	NB	BI	T	I/P
NPM	8866	0.435429	89.09	3.53
PM1	203	0.515396	57.64	3.88
G1	3222	0.470937	61.08	4.98
MHK	0	0.530415	49.41	3.06
AMF1	0	0.542123	43.59	2.55

Table 6: Comparison of different methods in complex plane in Example 2

Method	NB	BI	T	I/P
$GMT(1)$	451	0.515123	52.70	3.66
$Method8$	15968	0.410117	119.8	2.17
$AMF2$	0	0.546137	52.70	2.14

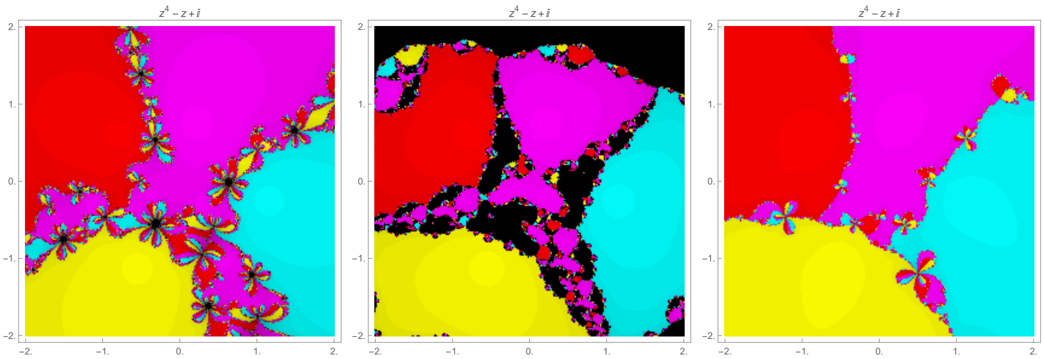


(a) NPM (b) PM1 (c) G1



(d) MHK (e) AMF1

Figure 3: Basins of attraction to iterative methods for $f_2(z) = z^4 - z + i$



(a) GMT(1) (b) Method8 (c) AMF2

Figure 4: Basins of attraction to iterative methods for $f_2(z) = z^4 - z + i$

Example 3: We consider the equation

$$f_3(z) = \left(z^5 + 10\right)\left(10z^5 - 1\right)$$

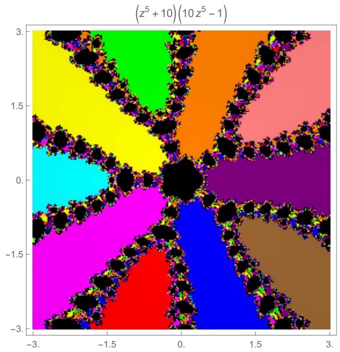
Which has ten roots which are $-1.58489, -0.51046-0.37087i, -0.51046+0.37087i, -0.48976-1.50732i, -0.48976+1.50732i,\ 0.19498-0.60008i, 0.19498+0.60008i, 0.63096, 1.28221-0.93158i, 1.28221+0.93158i$. The basins of attraction for the fourth-order methods are depicted in Figure 5 and the sixth-order methods are in Figure 6. Based on Table 7, one can notice that the best methods are the new method AMF1 and from Table 8, one notice that the new method AMF2 is the best.

Table 7: Comparison of different methods in complex plane in Example 3

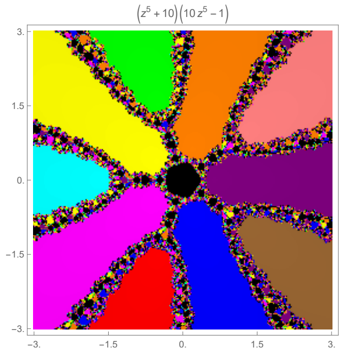
Method	NB	BI	T	I/P
NPM	13256	0.385073	157.0	5.57
PM1	6926	0.435844	110.4	5.49
G1	16627	0.363342	85.75	5.29
MHK	325	0.486447	77.29	4.97
AMF1	209	0.489694	66.88	3.99

Table 8: Comparison of different methods in complex plane in Example 3

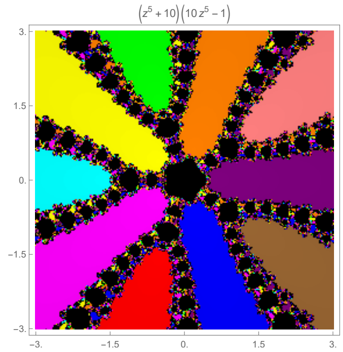
Method	NB	BI	T	I/P
$GMT(1)$	4738	0.453943	91.23	4.89
$Method8$	48555	0.168974	255.9	2.07
$AMF2$	1772	0.475139	137.2	3.84



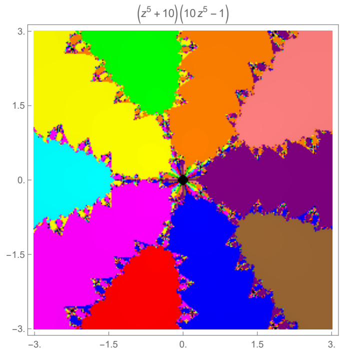
(a) NPM



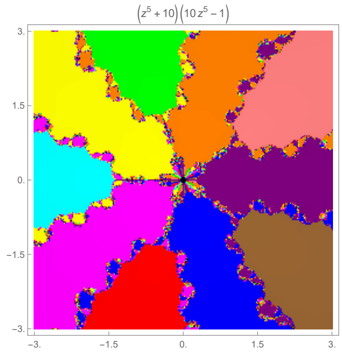
(b) PM1



(c) G1

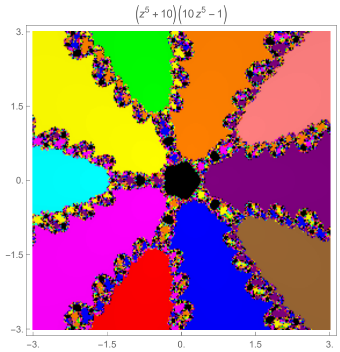


(d) MHK

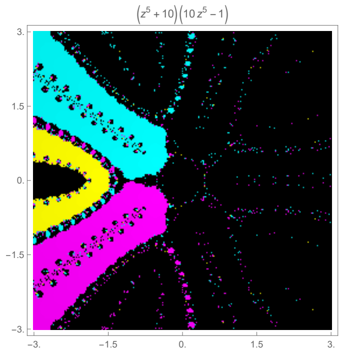


(e) AMF1

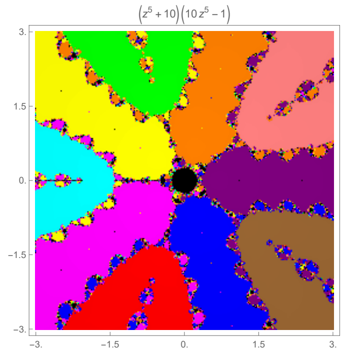
Figure 5: Basins of attraction to iterative methods for$f_3\left(z\right)=\left(z^5+10\right)\left(10z^5-1\right)$



(a) GMT(1)



(b) Method8



(c) AMF2

Figure 6: Basins of attraction to iterative methods for $f_2\left(z\right)=\left(z^5+10\right)\left(10z^5-1\right)$

6. Numerical examples in real domain

In numerical analysis, many methods produce sequences of real numbers such as the iterative methods for solving nonlinear equations. At times, the sequences' convergence is slow and their use in solving practcal problems is limited a fast convergent one [9].

We give several examples in the real domain to check the performance of our new iterative methods and to clarify their effectiveness. We will apply our new methods to the following equations:

$$f_1(x) = x^3 + 4x^2 - 10, \quad x_0 = 1 \qquad f_2(x) = sin^2 x - x^2 + 1, \quad x_0 = 2$$
$$f_3(x) = e^x - 3x, \quad x_0 = 1$$

Where x_0 is the initial approximation; we made two tables for comparisons, where the first table represents a comparison for the new fourth-order method (AMF1) with four methods of the fourth-order mentioned above and referred to by numbers (18), (19), (20) and (21), while the second table represents comparisons for the new sixth-order method (AMF2) with iterative methods from the same order mentioned above and referred to by numbers(22) and (23). All numerical computations have been carried out by using Mathematica 11, rounding to 128 significant digits. Depending on the precision of the computer, we use the stopping criteria for the iterative process:

The error $= |x_n - x_{n-1}| < \varepsilon$, where $\varepsilon = 10^{-15}$ and n is the number of iterations required for convergence. The computational order of convergence (COC)is given by :

$$COC \approx \frac{ln\,|(x_{n+1} - x_n)\,/\,(x_n - x_{n-1})|}{ln\,|(x_n - x_{n-1})\,/\,(x_{n-1} - x_{n-2})|} \tag{24}$$

where $n \in N$[15].

Tables 9, and 10, show $|x_n - x_{n-1}|$ indicating the error estimation between two consecutive iteration, the absolute values of the function $f(x_n)$ being the residual error of the nonlinear test function; the computational order of convergence (COC) and the time the iterative method take to reach the solution (Time). We note that $n = 4$ in Table 9 and $n = 3$ in Table 10. In the numerical examples presented in all Tables, if any of the used methods fail to reach convergence in a maximum of 15 iterations, it is labeled as ″NC ″.

Table 9: Comparing of the iterative methods over some the examples in real domain to order 4

Method	x_0	x_4	COC	$\|x_4 - x_3\|$	$\|f(x_4)\|$	Time
f_1	1					
NPM		1.36523001341409	4.00	3.0901×10^{-36}	$0. \times 10^{-126}$	0.0006
PM1		1.36523001341409	3.99	1.085×10^{-37}	$0. \times 10^{-126}$	0.0008
G1		1.36523001341409	3.99	2.3872×10^{-25}	$3. \times 10^{-98}$	0.001
MHK		1.36523001341409	3.99	6.528×10^{-44}	$0. \times 10^{-126}$	0.024

AMF1		1.36523001341409	3.99	2.2869×10^{-62}	$0. \times 10^{-126}$	0.0009
f_2	2					
NPM		1.40449164821534	4.02	5.0074×10^{-26}	1.3×10^{-101}	0.003
PM1		1.40449164821534	3.99	2.4206×10^{-29}	6.7×10^{-115}	0.002
G1		1.40449164821534	3.98	1.7464×10^{-23}	5.4×10^{-91}	0.001
MHK		1.40449164821534	3.99	2.0152×10^{-31}	2.1×10^{-123}	0.002
AMF1		1.40449164821534	3.99	1.9720×10^{-37}	$0. \times 10^{-126}$	0.001
f_3	1					
NPM		NC	NC	NC	NC	NC
PM1		0.61906128673594	3.93	0.6×10^{-4}	3.9×10^{-17}	0.001
G1		0.61906128673594	3.99	0.3×10^{-2}	3.9×10^{-10}	0.001
MHK		0.61906128673594	3.99	3.5218×10^{-12}	1.56×10^{-46}	0.11
AMF1		0.61906128673594	3.96	4.8330×10^{-20}	2.9×10^{-78}	0.001

Table 10: Comparing of the iterative methods over some the examples in real domain to order 6

Method	x_0	x_3	COC	$\|x_3 - x_2\|$	$\|f(x_3)\|$	Time
f_1	1					
GMT(1)		1.36523001341409	6.23	2.5423×10^{-19}	5.4×10^{-112}	0.0006
Method 8		1.36523001341409	5.96	2.2889×10^{-29}	$0. \times 10^{-125}$	0.0011
$AMF2$		1.36523001341409	6.11	1.0770×10^{-33}	$0. \times 10^{-125}$	0.0007
f_2	1.3					
$GMT(1)$		1.40449164821534	6.05	2.6192×10^{-33}	$. \times 10^{-125}$	0.0009
Method 8		1.40449164821534	6.02	9.798×10^{-42}	$0. \times 10^{-125}$	0.018
$AMF2$		1.40449164821534	6.05	1.8458×10^{-47}	$0. \times 10^{-126}$	0.001
f_3	0					
$GMT(1)$		0.61906128673594	5.99	4.1130×10^{-15}	1.7×10^{-86}	0.0007
Method 8		0.61906128673594	5.53	1.8381×10^{-17}	2.2×10^{-101}	0.0018
$AMF2$		0.61906128673594	5.72	6.0785×10^{-19}	7.2×10^{-111}	0.0007

7. Conclusion

In this paper, we have proposed a new optimal method of fourth order, and modified it to sixth-order method, for solving nonlinear equations; the error equations have been proven theoretically to show that the proposed techniques have fourth and sixth-order convergence. A stability analysis has been performed by proving the stability theories of the proposed methods and applying them to numerical examples. Finally, the proposed methods have been tested in complex and real planes on some of examples published in the literature, and the numerical results confirm that the new methods are comparable with the other methods and in most cases give better or equal results.

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