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Second Duals of Measure Algebras

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Abstract. In this paper we show that $M(G)^{**}$ determines G when G is a compact topological group. It is a new proof for theorem of Gharamani and Mcclure.

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The second dual space $\mathscr{A}^{**}$ of a Banach algebra $\mathscr{A}$ admits the Banach algebra product known as first (left) Arens product. This product extends the product of $\mathscr{A}$ as canonically embedded in $\mathscr{A}^{**}$. We briefly recall the definition of this product. For $m, n \in \mathscr{A}^{**}$, their first (left) Arens product indicated by mn is given by

$$\langle mn, f \rangle = \langle m, nf \rangle \quad (f \in \mathscr{A}^*),$$

where $nf \in \mathscr{A}^*$ is defined by

$$\langle nf, a \rangle = \langle n, fa \rangle \quad (a \in \mathscr{A}).$$

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(See [1] and [2]). Wendel in [6] proved that for locally compact groups G_1 and G_2, the group algebras $L^1(G_1)$ and $L^1(G_2)$ are isometrically isomorphic if and only if G_1 and G_2 are isomorphic in the category of topological groups. Johnson in [5] proved that the algebra $M(G)$ determines G when G is a locally compact group. In [3] Ghahramani and Lau have proved that $L^1(G)^{**}$ determines G when G is a locally compact group. Ghahramani and Mcclure in [4] proved that the algebra $(M(G))^{**}$ determines G when G is a compact topological group. In this paper we define some new ideals in Banach algebras and we apply this ideals to consider a new proof to show that $(M(G))^{**}$ determines G when G is compact. Let $\mathscr{A}$ be a Banach algebra. We consider

$$Z_l(\mathscr{A}) := \{a \in \mathscr{A} \ : \ \mathscr{A}^{**} \cdot \widehat{\dashv} \subseteq \widehat{\mathscr{A}}\}.$$

It is easy to show that $Z_l(\mathscr{A})$ is a two sided ideal of $\mathscr{A}$ so it is a left ideal of $\mathscr{A}^{**}$. Also $Z_l(\mathscr{A})$ is the union of all two sided ideals of $\mathscr{A}$ which are left ideals of $\mathscr{A}^{**}$. First we prove the following lemma.

Lemma 1. *Let $\theta : \mathscr{A} \to \mathscr{B}$ be an isometrically isomorphism between Banach algebras. Then $\theta(Z_l(\mathscr{A})) = \mathscr{Z}_{\updownarrow}(\mathscr{B})$.*

Proof. Let $\theta : \mathscr{A} \to \mathscr{B}$ be an isometrically isomorphism between Banach algebras. Then θ'' is a isometrically isomorphism between Banach algebras $\mathscr{A}^{**}$ and $\mathscr{B}^{**}$. Let $a \in Z_l(\mathscr{A})$ and $b'' \in \mathscr{B}^{**}$. Then there exists $a'' \in \mathscr{A}^{**}$ such that $b'' = \theta''(a'')$. Thus

$$b''\widehat{\theta(a)} = \theta''(a'')\widehat{\theta(a)} = \theta''(a''\widehat{a}) = \widehat{\theta(a''\widehat{\ }a)} \in \widehat{\theta(\mathscr{A})} = \widehat{\mathscr{B}}.$$

Then $\theta(Z_l(\mathscr{A})) \subset \mathscr{Z}_{\updownarrow}(\mathscr{B})$. ■

Theorem 1. *Let G be a compact group. Then $Z_l((M(G))^{**}) = \pi''(Ł^1(G))^{**}$.*

Proof. Let (e_α) be a bounded approximate identity of $L^1(G)$ with bound 1, and with cluster point $E \in L^1(G)^{**}$. We denote $\pi : L^1(G) \longrightarrow M(G)$ the inclusion map, then the map

$$m \longmapsto (\pi''(E))\widehat{m} : M(G) \longrightarrow \pi''(L^1(G)^{**})$$

is isometric embedding. We denote this map with Γ_E. Since the restriction of Γ_E to $L^1(G)$ is identity map, then $\Gamma_E(m) \in \widehat{\pi(L^1(G))}$ if and only if $m \in L^1(G)$. It is easy to show that $\Gamma_E{}''$ is isometrically embedding from $(M(G)^{**})$ into $\pi''''((L^1(G))^{****})$. The restriction of $\Gamma_E{}''$ to $\pi''(L^1(G)^{**})$ is identity map, then for every $m'' \in (M(G)^{**})$, $\Gamma_E{}''(m'') \in \pi''\widehat{(L^1(G)^{**}})$ if and only if $m'' \in \pi''\widehat{(L^1(G)^{**}})$. Let now $m'' \in Z_l((M(G)^{**}))$, then $(M(G))^{****}\widehat{m''} \subseteq (\widehat{M(G)^{**}})$. Thus

$$\pi''''(L^1(G))^{****}\widehat{m''} \subseteq (\widehat{M(G)^{**}}). \tag{1}$$

On the other hand, we have direct sum decompositions

$$(L^1(G))^{****} = \widehat{L^1(G)^{**}} \oplus \widehat{(L^1(G)^*)}^{\perp} \tag{2}$$

and

$$(M(G))^{****} = \widehat{M(G)^{**}} \oplus \widehat{(M(G)^*)}^{\perp}. \tag{3}$$

So we have

$$\pi''''(\widehat{(L^1(G)^*)}^{\perp}) \subseteq \widehat{(M(G)^*)}^{\perp}. \tag{4}$$

Since $\pi''''(L^1(G)^{****})$ is an ideal of $M(G)^{****}$, then by (2) and (4), we have $\pi''''(L^1(G))^{****}\widehat{m''} \subseteq [((\widehat{M(G)^{**}}))\cap\pi''''(L^1(G))^{****}] = \pi''\widehat{(L^1(G)^{**}})$. Therefore $\Gamma_E{}''(m'') \in \pi''\widehat{(L^1(G)^{**}})$ and $m'' \in \pi''(L^1(G)^{**})$, hence, $Z_l(M(G)^{**}) \subseteq \pi''(L^1(G)^{**})$. On the other hand since G is compact then $\pi''(L^1(G)^{**})$ is a two sided ideal of $\pi''''(L^1(G)^{****})$, so $Z_l(\pi''(L^1(G)^{**})) = \pi''(L^1(G)^{**})$ and $Z_l(\pi''(L^1(G)^{**}))$ is a two sided ideal of $M(G)^{****}$. Hence, $\pi''(L^1(G)^{**}) \subseteq Z_l(M(G)^{**})$. ■

We now apply above theorem to show that $M(G)^{**}$ determines G when G is a compact topological group. It is a new proof for the main result of [4]. By Lemma 1 and Theorem 1 we have the following.

Corollary 1 (Theorem 7 of 4)**.** . *If G_1 and G_2 are compact groups, and if θ is an isometric isomorphism from $M(G_1)^{**}$ onto $M(G_2)^{**}$, then $\theta(L^1(G_1)^{**}) = L^1(G_2)^{**}$.*

Since $L^1(G)^{**}$ determines G [3] , then we have

Corollary 2. *If G is a compact group, then $M(G)^{**}$ determines G.*

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