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Some Double Integrals Stemming from the Boltzmann Equation in the Kinetic Theory of Gasses

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Abstract. The main object of this article is to revisit a certain double integral involving Kummer's confluent hypergeometric function ${}_1F_1$, which arose in the study of the collision terms of the celebrated Boltzmann equation in the kinetic theory of gases. Here, in this article, we propose to investigate some novel extensions and generalizations of this family of double integrals. We also point out some relevant connections of the results, which are presented here, with other related recent developments in the theory and applications of hypergeometric functions.

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1. Introduction and Motivation

Introduced in the year 1872 by the Austrian physicist and philosopher, Ludwig Boltzman (1844–1906), the celebrated Boltzmann equation is known to describe the statistical behaviour of a thermodynamic system which is not in a state of equilibrium. In a recent study of the collision terms of the Boltzmann equation occurring in the kinetic theory of gases, the problem of evaluation of the following double integral arose (see, for detail, [3]):

$$\Delta := \int_0^\pi \int_0^\pi {}_1F_1 \left[\begin{array}{c} \alpha; \\ \\ \gamma; \end{array} \lambda_1 + \lambda_2 \cos\psi + \lambda \cos\theta \ \cos\psi \right] \mathrm{d}\psi \ \mathrm{d}\theta, \tag{1}$$

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where λ, λ_1 and λ_2 are constants. Also, the (Kummer's) confluent hypergeometric function ${}_1F_1$, which is involved in the integral in (1) above (see, for details, [2]), corresponds to the special case of the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ when $p = q = 1$. Indeed, in terms of the general Pochhammer symbol or the *shifted factorial* $(\kappa)_\nu$, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}),$$

which is defined (for $\kappa, \nu \in \mathbb{C}$), in terms of the (Euler's) Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa + \nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0;\ \kappa \in \mathbb{C} \setminus \{0\}) \\ \kappa(\kappa + 1) \cdots (\kappa + n - 1) & (\nu = n \in \mathbb{N};\ \kappa \in \mathbb{C}), \end{cases} \tag{2}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists, a generalized hypergeometric function, with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, p)$ and q denominator parameters $\gamma_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \cdots, q)$, is given by

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \ \cdots, \alpha_p; \\ \\ \gamma_1, \ \cdots, \gamma_q; \end{matrix} \ z \right] &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\gamma_1)_n \cdots (\gamma_q)_n} \frac{z^n}{n!} \\ &= \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} \\ &\qquad \cdot \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n) \cdots \Gamma(\alpha_p + n)}{\Gamma(\gamma_1 + n) \cdots \Gamma(\gamma_q + n)} \frac{z^n}{n!} \\ &=: {}_pF_q \left(\alpha_1, \cdots, \alpha_p; \gamma_1, \cdots, \gamma_q; z \right), \end{aligned} \tag{3}$$

under appropriate conditions for convergence of the infinite series (see, for details, [21, p. 3 *et seq.*]), given by (see also [1], [10], [11] [12], [14], [15] and [24])

(i) converges absolutely for $|z| < \infty$ if $p \leqq q$,
(ii) converges absolutely for $|z| < 1$ if $p = q + 1$, and
(iii) diverges for all z $(z \neq 0)$ if $p > q + 1$.

Under the constraint $\min\{\Re(\alpha), \Re(\gamma)\} > 1$, it was shown for the double integral in (1) that (see [3, p. 13])

$$\Delta = \frac{\pi}{R} \left(\frac{\gamma - 1}{\alpha - 1} \right) \left({}_1F_1 \left[\begin{matrix} \alpha - 1; \\ \\ \gamma - 1; \end{matrix} \ \lambda_1 + R \right] - {}_1F_1 \left[\begin{matrix} \alpha - 1; \\ \\ \gamma - 1; \end{matrix} \ \lambda_1 - R \right] \right), \tag{4}$$

where, for convenience,

$$R^2 = \lambda^2 + \lambda_2^2. \tag{5}$$

The long and involved derivation of the integral formula (4) by Deshpande [3, pp. 11–13] made use of such konen results as (for example) a contour integral representation of

Kummer's confluent hypergeometric function ${}_1F_1$ [4, p. 272], a certain Neumann expansion involving the modified Bessel function $I_\nu(z)$ and the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(z)$ (see [5, p. 98]), and the addition theorem for the Legendre (or spherical) polynomials $P_n(z)$ in terms of the associated Legendre polynomials $P_n^m(z)$ (see [7, p. 35] and [5, p. 244]). In a sequel to [3], a direct and much shorter evaluation of the double integral in (4) was given by Srivastava [16] who did actually extend the integral formula (4) to the following general form (see [16, p. 8, Eq. (22)]):

$$\begin{aligned}\Delta^* &:= \int_0^\pi \int_0^\pi {}_pF_q\left[\begin{array}{c}\alpha_1, \cdots, \alpha_p;\\ \\ \gamma_1, \cdots, \gamma_q;\end{array} \lambda_1 + \lambda_2 \cos\psi + \lambda \cos\theta \,\cos\psi\right] \mathrm{d}\psi\, \mathrm{d}\theta \\ &= \frac{\pi}{R}\left(\frac{\prod\limits_{j=1}^{q}(\gamma_j - 1)}{\prod\limits_{j=1}^{p}(\alpha_j - 1)}\right)\left({}_pF_q\left[\begin{array}{c}\alpha_1 - 1, \cdots, \alpha_p - 1;\\ \\ \gamma_1 - 1, \cdots, \gamma_q - 1;\end{array} \lambda_1 + R\right]\right. \\ &\qquad\left. - {}_pF_q\left[\begin{array}{c}\alpha_1 - 1, \cdots, \alpha_p - 1;\\ \\ \gamma_1 - 1, \cdots, \gamma_q - 1;\end{array} \lambda_1 - R\right]\right), \end{aligned} \tag{6}$$

where R is given, as before, by (5) and, for convergence of the hypergeometric series involved, we require that $p \leqq q$ or $p = q + 1$ and

$$\max\{|\lambda_1| + |\lambda_2| + |\lambda_1 \pm R|\} < 1,$$

by appealing to the principle of analytic continuation.

Our present investigation is motivated essentially by the aforementioned importance of the double integral (4), as well as by its potentially useful generalization (6). It seems to be worthwhile to explore the possibility of evaluation of some further extended versions of the double integrals (4) and (6).

2. The Hurwitz-Lerch Zeta Function and the Mittag-Leffler Type Functions

I choose first to mention my having met many times and having discussed mathematical researches, especially on various families of higher transcendental functions and related topics (including, of course, about the widely- and extensively-investigated Fox H-function and the Fox-Wright function ${}_p\Psi_q$,) with my Canadian colleague, Charles Fox (1897–1977) of birth and education in England, both at McGill University and Sir George Williams University (*now* Concordia University) in Montréal, mainly during the 1970s (see, for details, [6] and [17]). Another remarkable mathematical scientist of modern times happens to be *Sir* Edward Maitland Wright (1906–2005), with whom I had the privilege to meet and discuss researches emerging from his publications on hypergeometric and related higher transcendental functions during my visit to the University of Aberdeen in Scotland in the year 1976. We recall here a series of monumental works by Wright (see, for example, [28], [29] and [30]), in which he introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [28, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n+\beta)}\, z^n \qquad \big(\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big), \tag{7}$$

where $\phi(t)$ is a function satisfying suitable sufficient conditions.

The general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$, defined by (7), not only extends the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$, which are defined, respectively, by (see [13], [26] and [27])

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \qquad \text{and} \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{8}$$

$$\big(z,\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big),$$

but also the above-mentioned Fox-Wright function ${}_p\Psi_q$, defined by (see, for details, [4, p. 183] and [24, p. 21]; see also [9, p. 56], [8, p. 65] and [23, p. 19])

$$\begin{aligned}
{}_p\Psi_q^*&\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array}\ z\right]\\
&:= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n}\ \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n}\ \cdots (b_q)_{B_q n}}\ \frac{z^n}{n!}\\
&=: \frac{\Gamma(b_1)\ \cdots\Gamma(b_q)}{\Gamma(a_1)\ \cdots\Gamma(a_p)}\ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array}\ z\right]
\end{aligned} \tag{9}$$

$$\left(\Re(A_j)>0\ \ (j=1,\ \cdots,p);\ \Re(B_j)>0\ \ (j=1,\ \cdots,q);\ \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geqq -1\right),$$

where $(\kappa)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, which we have defined already by (2), and the equality in the convergence condition holds true only for suitably-bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

In some recent developments, which are based upon the general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$, defined by (7), Srivastava [21] introduced the following function and applied it in his study of a family of fractional-order kinetic equations (see, for details, [19] and [20]):

$$\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s\, \Gamma(\alpha n+\beta)}\, z^n \qquad \big(\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\big), \tag{10}$$

where the function $\varphi(\tau)$ and the parameters α, β, s and κ are appropriately constrained. It is not difficult to see that Srivastava's function $\mathcal{E}_{\alpha,\beta}(\varphi; z)$, defined by (10), provides a hybrid form of the Mittag-Leffler type functions, the Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ defined by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s} \tag{11}$$

$$\big(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\big),$$

as well as the following interesting and potentially useful family of the multi-parameter Hurwitz-Lerch Zeta functions

$$\Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p; \sigma_1, \cdots, \sigma_q)}(z, s, \kappa),$$

which is defined by (see [25, p. 503, Eq. (6.2)]; see also [18] and [22])

$$\Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}(z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod\limits_{j=1}^{q} (\mu_j)_{n\sigma_j}}\, \frac{z^n}{(n+\kappa)^s} \tag{12}$$

$$\Bigg(p, q \in \mathbb{N}_0;\ \ \lambda_j \in \mathbb{C}\ \ (j = 1, \cdots, p);\ \ \kappa, \mu_j \in \mathbb{C} \setminus Z_0^-\ \ (j = 1, \cdots, q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+\ \ (j = 1, \cdots, p;\ k = 1, \cdots, q);\ \Delta^{**} > -1\ \text{ when }\ s, z \in \mathbb{C};$$

$$\Delta^{**} = -1\ \text{ and }\ s \in \mathbb{C}\ \text{ when }\ |z| < \nabla^*;$$

$$\Delta^{**} = -1\ \text{ and }\ \Re(\Xi) > \frac{1}{2}\ \text{ when }\ |z| = \nabla^*\Bigg),$$

where, for convenience,

$$\Delta^{**} := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j) + \frac{p-q}{2} \tag{13}$$

and

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right), \tag{14}$$

3. A General Family of Double Integrals

Before presenting an extended version of the double integrals (4) and (6), we list here each of the following elementary results which will be needed in the derivation of our general double integral.

I. A Multiple Series Identity

$$\sum_{m_1,\cdots,m_r=0}^{\infty} f\left(m_1+\cdots+m_r\right) \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!} = \sum_{m=0}^{\infty} f(m) \frac{(z_1+\cdots+z_r)^m}{m!}, \tag{15}$$

provided that the series involved are absolutely convergent.

II. An Integral Identity

$$\begin{aligned}\int_0^{\pi} \cos^m t\, g(\sin t)\, \mathrm{d}t &= [1+(-1)^m] \int_0^{\frac{\pi}{2}} \cos^m t\, g(\sin t)\, \mathrm{d}t \qquad (m \in \mathbb{N}_0) \\ &= \begin{cases} 2\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n} t\, g(\sin t)\, \mathrm{d}t & (m=2n;\ n \in \mathbb{N}_0) \\ 0 & (m=2n+1;\ n \in \mathbb{N}_0), \end{cases}\end{aligned} \tag{16}$$

provided that each of the integrals exists.

III. A Simple Series Identity

$$\sum_{m=0}^{\infty} [1-(-1)^m] h(m) = 2 \sum_{m=0}^{\infty} h(2m+1), \tag{17}$$

provided that each of the series exists.

IV. A Trigonometric Integral

$$\int_0^{\frac{\pi}{2}} \cos^{\mu} t\, \sin^{\nu} t\, \mathrm{d}t = \frac{\Gamma\left(\frac{\mu+1}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\mu+\nu+2}{2}\right)} \qquad \left(\min\{\Re(\mu), \Re(\nu)\} > -1\right). \tag{18}$$

V. Legendre's Duplication Formula and Its Consequences

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\, \Gamma(z)\, \Gamma\left(z+\frac{1}{2}\right), \tag{19}$$

which readily yields the following simpler consequences:

$$(2m)! = 2^{2m}\ m! \left(\frac{1}{2}\right)_m \quad \text{and} \quad (2m+1)! = 2^{2m}\ m! \left(\frac{3}{2}\right)_m \qquad (m \in \mathbb{N}_0). \tag{20}$$

With a view to presenting our proposed generalization of the double integrals in (4) and (6), we first slightly modify the definition (10) as follows:

$$\mathcal{E}^*_{\alpha,\beta}(\varphi^*; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi^*(n)}{(n+\kappa)^s\ \Gamma(\alpha n + \beta)}\ \frac{z^n}{n!} \qquad \big(\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\big), \tag{21}$$

where, just as in the definition (10), the function $\varphi^*(\tau)$ and the parameters α, β, s and κ are appropriately constrained.

Now, by applying the definition (21) and the case $r = 3$ of the multiple series identity (15), we find that

$$\begin{aligned}
\Omega &:= \int_0^{\pi}\int_0^{\pi} \mathcal{E}^*_{\alpha,\beta}\big(\varphi^*; \lambda_1 + \lambda_2 \cos\psi + \lambda \cos\theta\ \cos\psi, s, \kappa\big)\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\theta \\
&= \int_0^{\pi}\int_0^{\pi} \sum_{n=0}^{\infty} \frac{\varphi^*(n)}{(n+\kappa)^s\ \Gamma(\alpha n+\beta)}\ \frac{\big(\lambda_1 + \lambda_2 \cos\psi + \lambda \cos\theta\ \cos\psi\big)^n}{n!}\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\theta \\
&= \sum_{\ell,m,n=0}^{\infty} \frac{\varphi^*(\ell+m+n)}{(\ell+m+n+\kappa)^s\ \Gamma\big(\alpha(\ell+m+n)+\beta\big)}\ \frac{\lambda_1^{\ell}}{\ell!}\ \frac{\lambda_2^{m}}{m!}\ \frac{\lambda^{n}}{n!} \\
&\qquad \cdot \left(\int_0^{\pi} \cos^m\psi\ \sin^{n+1}\psi\ \mathrm{d}\psi\right)\left(\int_0^{\pi} \cos^n\theta\ \mathrm{d}\theta\right),
\end{aligned} \tag{22}$$

which, in view of the integral formulas (16), (18), (19) and (20), readily yields

$$\begin{aligned}
\Omega &= \sum_{\ell,m,n=0}^{\infty} \frac{\varphi^*(\ell+2m+2n)}{(\ell+2m+2n+\kappa)^s\ \Gamma\big(\alpha(\ell+2m+2n)+\beta\big)}\ \frac{\lambda_1^{\ell}}{\ell!}\ \frac{\lambda_2^{2m}}{(2m)!}\ \frac{\lambda^{2n}}{(2n)!} \\
&\qquad \cdot \left(2\int_0^{\frac{\pi}{2}} \cos^{2m}\psi\ \sin^{2n+1}\psi\ \mathrm{d}\psi\right)\left(2\int_0^{\frac{\pi}{2}} \cos^{2n}\theta\ \mathrm{d}\theta\right) \\
&= 2\pi \sum_{\ell,m,n=0}^{\infty} \frac{\varphi^*(\ell+2m+2n)}{\left(\frac{3}{2}\right)_{m+n}\ (\ell+2m+2n+\kappa)^s\ \Gamma\big(\alpha(\ell+2m+2n)+\beta\big)} \\
&\qquad \cdot \frac{\lambda_1^{\ell}}{\ell!}\ \frac{\left(\frac{\lambda_2}{2}\right)^{2m}}{m!}\ \frac{\left(\frac{\lambda}{2}\right)^{2n}}{n!}.
\end{aligned} \tag{23}$$

Upon replacing n in (23) by $n - m$ $(0 \leqq m \leqq n)$, we sum the resulting binomial series and simplify the outcome by using the identity (20) once again. We thus find that

$$\Omega = \frac{2\pi}{R} \sum_{\ell,n=0}^{\infty} \frac{\varphi^*(\ell+2n)}{(\ell+2n+\kappa)^s\ \Gamma\big(\alpha(\ell+2n)+\beta\big)}\ \frac{\lambda_1^{\ell}}{\ell!}\ \frac{R^{2n+1}}{(2n+1)!}$$

or, equivalently,

$$\Omega = \frac{2\pi}{R} \sum_{\ell,n=0}^{\infty} \frac{\varphi^*\big(\ell - 1 + (2n+1)\big)}{\big(\ell - 1 + (2n+1) + \kappa\big)^s \, \Gamma\Big(\alpha\big(\ell - 1 + (2n+1)\big) + \beta\Big)} \, \frac{\lambda_1^\ell}{\ell!} \, \frac{R^{2n+1}}{(2n+1)!}, \tag{24}$$

where R is given by (5).

Finally, we apply the elementary series identity (17), together with the case $r = 2$ of the multiple series identity (15). We are thus led from (24) to the following result:

$$\begin{aligned} \Omega &:= \int_0^{\pi} \int_0^{\pi} \mathcal{E}^*_{\alpha,\beta}\big(\varphi^*; \lambda_1 + \lambda_2 \cos\psi + \lambda \cos\theta \ \cos\psi, s, \kappa\big) \ \sin\psi \ \mathrm{d}\psi \ \mathrm{d}\theta \\ &= \frac{\pi}{R} \left(\sum_{n=0}^{\infty} \frac{\varphi^*(n-1)}{(n+\kappa-1)^s \, \Gamma\big(\alpha(n-1)+\beta\big)} \, \frac{(\lambda_1 + R)^n}{n!} \right. \\ &\qquad \left. - \sum_{n=0}^{\infty} \frac{\varphi^*(n-1)}{(n+\kappa-1)^s \, \Gamma\big(\alpha(n-1)+\beta\big)} \, \frac{(\lambda_1 - R)^n}{n!} \right), \end{aligned} \tag{25}$$

provided that each member of (25) exists.

Remark. By suitably specializing the sequence $\varphi^*(n)$, one can deduce from the general result (25) the corresponding double integrals involving simpler functions of the Mittag-Leffler and Hurwitz-Lerch types. In a particular case of (6), if we first set

$$\varphi^*(n-1) = (n+\kappa-1)^s \, \Gamma\big(\alpha(n-1)+\beta\big) \, \frac{\prod_{j=1}^{p} (\alpha_j - 1)_{n-1}}{\prod_{j=1}^{q} (\gamma_j - 1)_{n-1}}$$

and then note, in view of the definition (2), that

$$\frac{\prod_{j=1}^{p} (\alpha_j - 1)_{n-1}}{\prod_{j=1}^{q} (\gamma_j - 1)_{n-1}} = \left(\frac{\prod_{j=1}^{q} (\gamma_j - 1)}{\prod_{j=1}^{p} (\alpha_j - 1)} \right) \left(\frac{\prod_{j=1}^{p} (\alpha_j - 1)_n}{\prod_{j=1}^{q} (\gamma_j - 1)_n} \right),$$

we arrive at the double integral formula (6).

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