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Variative problems in teaching mathematics

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Abstract. The article discusses problems with ambiguous conditions, the solution of which leads to several variants of answer. Such problems are called variative problem. The role of variative problems in intellectual development of schoolchildren in mathematics classes are described in this article. Proposed a methodology for including variative problems in the process of teaching mathematics.

2020 Mathematics Subject Classifications: 97G10, 97G30, 97G40

Key Words and Phrases: Variative problem, variation of problem conditions, multivariate problems.

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1. Introduction

The experience says that one of the most common drawbacks of students' thinking process is its linearity, in other words, its lack of variative perception of surrounding ideas and phenomena. This can be seen by the fact that the schoolchildren prove incapable of considering the situation under different angle, interpreting the data differently, finding alternative ways of problem solution. Studying mathematics provides large opportunities to overcome these drawbacks of thinking. Variety of problems can serve this objective, provided that they are regularly stated and their variative content is discussed with the students.

These days, there is a great demand for the capabilities of searching for different, not obvious at first sight ways to solve some problems, the capabilities of comparing different possible actions and analysis of their consequences, the capabilities of making most optimal decision in multiple choice conditions. Professionals from various fields of human activity are facing today the situations which demand the above capabilities. Teaching mathematics provides large opportunities for teacher to develop variative properties of student thinking. Let's discuss some of them.

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Let's first consider the problems with ambiguous conditions. When treating such problems, you have to consider several possible situations, which usually leads to several variants of solution. In particular, such multivariative problems can be easily created in geometric framework. It's best if such problems are discussed in classes regularly and without warning, because then the students get accustomed to the necessity of individually considering several different variants of the solution of the problem. The works [1,2,3,5] are dedicated to the practical issues of using variative problems. In [4,6,7,8], the solution of some kinds of variative problems is presented.

2. A study of the variative problems

The main characteristic of the latter is the ambiguous location of objects appearing in the conditions of the problem, which leads to the necessity of considering several situations. To solve variative problem is to consider all possible variants of location of objects, which leads to several answers. From a practical point of view, it is also important that such problems can be considered even in the lower grades. In particular, they can be easily obtained from the simplest "standard" problems by some incompleteness in the conditions.

Problem 1. The price of the good has been twice changed by 10%.By how many percent did the price change at all?

Depending on whether the price rose or dropped in two cases (total of four situations are possible), there are three variants of answer:

1) If the price of the good dropped by 10% in both cases, then the total drop of the price is 19%.

2) If the price of the good first dropped by 10%, and then rose by 10%, then the price dropped in total by 1%.

3) If the price of the good first rose by 10%, and then dropped by 10%, then the price dropped in total by 1%.

4) If the price of the good rose by 10%in both cases, then the total rise of the price is 21%.

Problem 2. The distance between two cyclists is currently 60 km. What will be the distance between them in two hours, if their speeds are 10 km/h and 15 km/h, respectively? (both cyclists are driving on the same straight driveway with no turns).

Depending on the kind of movement (cyclists driving towards each other, in different directions, chasing each other, or one lagging behind other), there are four variants of answer:

Situation 1.

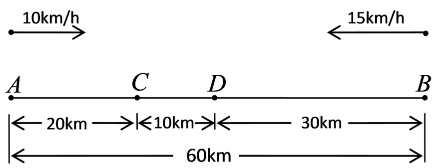

Figure 1: **Cyclists driving towards each other**

In this case, the answer is 10 km.

Situation 2. In this case, the answer is 50 km.

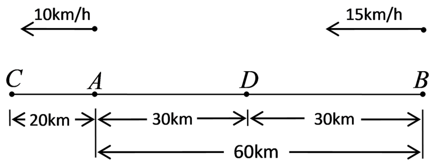

Figure 2: **Cyclists chasing each other**

Situation 3. In this case, the answer is 70 km.

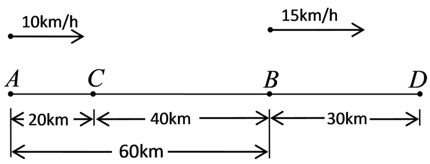

Figure 3: **One of cyclists lagging behind the other**

Situation 4. In this case, the answer is 110 km.

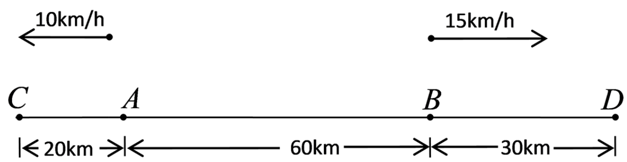

Figure 4: **Cyclists driving in different directions**

That's what's called "thinking activity"? It includes the consideration of different variants, imposition of additional conditions, statement of new problems, abstraction, and finding a general method for solving the considered problem. In this case, the removal of some condition in standard problem leads to the variative problem. Here are some other examples of variative problems.

Problem 3. Given two vertices $M(3,2)$ and $N(6,5)$, construct a square.

There is no information in this problem about whether the given vertices are consecutive or opposite to each other. Therefore, we have to consider all possible variants of locations of vertices, which leads to three answers:

1) Square $MNDC$ with the vertices $M(3,2)$, $N(6,5)$, $D(3,8)$ and $C(0,5)$.

2) Square $MNQP$ with the vertices $M(3,2)$, $N(6,5)$, $Q(9,2)$ and $P(6,-1)$.

3) Square $MENF$ with the vertices $M(3,2)$, $E(6,2)$, $N(6,5)$ and $F(3,5)$ (See Fig. 5).

Using the derivative, we solve the following multivariate problem.

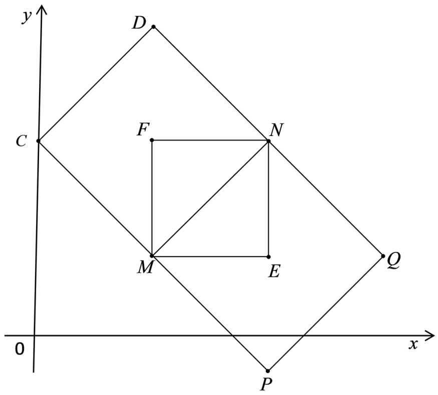

Figure 5: **Squares $MNDC$, $MNQP$, $MENF$ constructed on the given vertices M and N**

Problem 4. For each value of p find the number of roots of the equation

$$x^3 - 3x^2 - p = 0.$$

Find the intervals of increase and decrease of the function

$$f(x) = x^3 - 3x^2.$$

Since

$$f'(x) = 3x^2 - 6x = 3x\,(x-2)\,,$$

then

$$f'(x) < 0 \quad for \quad 0 < x < 2\,,$$
$$f'(x) = 0 \quad for \quad x = 0 \;\; and \;\; x = 2\,,$$
$$f'(x) > 0 \quad for \quad x < 0 \;\; and \;\; x > 2\,.$$

Thus, the continuous function $y = f(x)$ has a local minimum at point$x = 2$, and a local maximum at point $x = 0$, and $f(0) = 0, f(2) = -4,$. Moreover, the function $y = f(x)$ decreases on the interval $[0, 2]$ and increases on the intervals $(-\infty, 0]$ and $[2, +\infty)$. Consequently

$$\lim_{x\to+\infty} f(x) \quad = +\infty \quad and \quad \lim_{x\to-\infty} f(x) \quad = -\infty\,.$$

Hence it follows that in order to find out the dependence of the number of roots of the equation on the possible values of p , it is necessary to find out the relative position of the graph of the function $y = f(x)$ and the straight line p . From the properties of the function $y = f(x)$ proved above and the fact that the function $y = f(x)$ is continuous and is a polynomial of the third degree, we have that

1) for $p > 0$ the equation has one root;

2) for $p = 0$ the equation has three roots $(x_1 = x_2 = 0;\ x_3 = 3)$, among which there are two coinciding;

3) for $-4 < p < 0$ the equation has three different roots;

4) for $p = -4$ the equation has three roots $(x_1 = x_2 = 2;\ x_3 = -1)$, among which there are two coinciding;

5) for $p < -4$ the equation has one root.

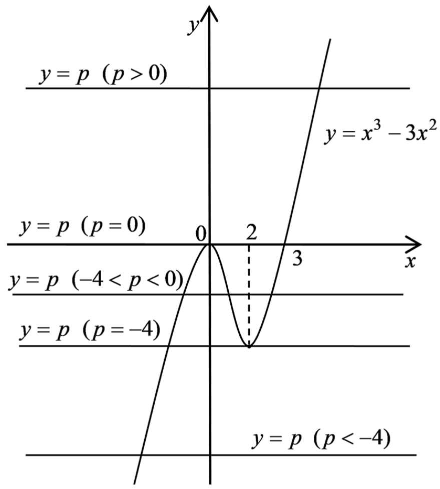

Figure 6: $y = x^3 - 3x^2$ **function graph**

Problem 5. Let two circles lie in an isosceles trapezoid. One of them, with radius 1, is inscribed in this trapezoid, while the other touches two sides of trapezoid and the first circle. The distance between the vertex of the angle created by two sides of trapezoid touching the second circle and the tangent point of two circles is twice as much as the diameter of the second circle. Find the area of the trapezoid.

It is not clear from the conditions of the problem into which angle of trapezoid the second circle is inscribed: obtuse or acute? Therefore, we have to consider both variants, and, analyzing two corresponding figures, we have to try to find out which of them satisfies the conditions of the problem.

Let O be a center of the circle inscribed in the trapezoid $ABCD$, and O_1 be a center of the second circle which touches the first circle at the point P and two sides of the trapezoid. Denote by r the radius of the second circle.

Let's join the centers of two circles and the tangent points M and N of the corresponding base BC (or AD) of trapezoid. Then we obtain two similar right triangles OMC and O_1NC (or OMD and O_1ND). As

$$OC = OP + PC = 1 + 4r,$$

$$OD = OP + PD = 1 + 4r,$$

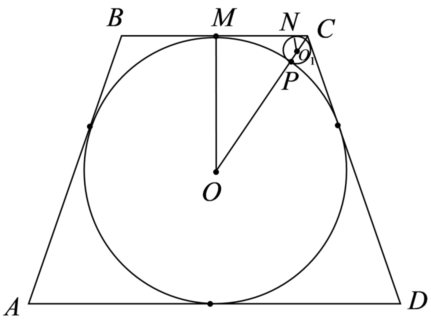

Figure 7: **Second circle inscribed in the obtuse angle of the trapezoid** $ABCD$

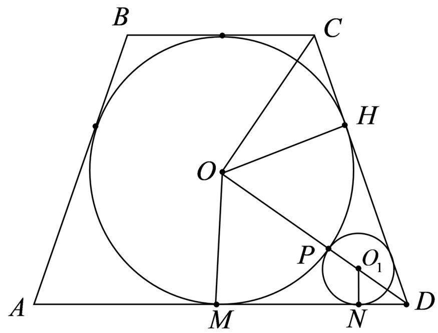

Figure 8: **Second circle inscribed in the acute angle of the trapezoid** $ABCD$

from the relations

$$\frac{OM}{O_1N} = \frac{OC}{O_1C}, \ \frac{OM}{O_1N} = \frac{OD}{O_1D}.$$

we easily obtain $r = \frac{1}{2}$. But then, by Pythagoras theorem, from the triangles OMC and OMD it follows that

$$MC = \sqrt{OC^2 - OM^2} = 2\sqrt{2},$$
$$MD = \sqrt{OD^2 - OM^2} = 2\sqrt{2}.$$

So, in the case described by Fig.7, where the second circle touches the smaller base of trapezoid, the smaller base must be equal to $4\sqrt{2}$, and in the case described by Fig.8, where the second circle touches the larger base of trapezoid, the larger base must be equal to $4\sqrt{2}$. But it is clear that if a circle is inscribed in an isosceles trapezoid, then the smaller base of trapezoid is less than the diameter of the circle, and the larger base is greater than the diameter of the circle. Consequently, only the larger base of trapezoid can be equal to $4\sqrt{2}$. In other words, the variant described by Fig. 7 contradicts the conditions of the problem and we should consider the case described by Fig.8 only.

Now it is not difficult to find the area of the trapezoid. Let's join the center O of the inscribed circle and the vertex C. The triangle COD is right. In fact,

$$\angle OCD + \angle ODC = \frac{1}{2}\angle BCD + \frac{1}{2}\angle ADC = \frac{1}{2} \cdot 180^\circ = 90^\circ,$$

i.e. $\angle COD = 90^\circ$. As the radius drawn to the tangent point H of the inscribed circle and the leg CD is a height of the triangle COD , and as, due to tangent property, the relations

$$CH = \frac{1}{2}BC \quad and \ DH = \frac{1}{2}AD$$

hold, we have

$$OH^2 = CH \cdot DH = \frac{1}{4}\, BC \cdot AD.$$

But, $OH = 1$, $AD = 4\sqrt{2}$ consequently, $BC = \frac{\sqrt{2}}{2}$. Taking into account that the height of trapezoid is equal to the diameter of the inscribed circle, we find that the area of trapezoid is equal to $\frac{9\sqrt{2}}{2}$.

Sometimes the conditions of the problem are deliberately stated somewhat vaguely, so that they geometrically admit very different figures and it is not clear which of these figures suits the problem. In this case, you first have to consider all possibilities which formally correspond to the conditions of the problem, and then to "decipher" the only true geometric configuration.

Let's mention the conditions which lead to the ambiguous interpretation of geometric problem.

1. The conditions of the problem do not define the location of points and figures with respect to each other: a) the point either belongs to the interval AB or does not, but lies on the straight line AB ; b) the points lie either on the same half-plane (half-space) with respect to the given straight line or on different ones; c) different locations of the center of inscribed circle or the orthocenter of triangle depending on the type of triangle.

2. Two tangent circles appear in the conditions of the problem, but no information about whether the tangency is internal or external. Two points are given in the problem which divide the circle into two arcs, and some straight line touches the circle. But no information about which of two arcs the tangent point lies on.

3. The problem includes objects with some properties, but there is no information about the relationship between these objects and their properties: a) it is stated that ABC is an isosceles triangle, but no information about which of its sides are equal; b) the point M divides the interval AB into the intervals of lengths a and b, but no information about which of these intervals is equal to a , and which of them is equal to b ; c) the angle between the straight lines AB and CD intersecting at the point M is equal to α, but no information about which of the angles AMD and AMC is acute.

3. Concluding remaks

Now let's list the stages of the methodology of using variative problems. In the first stage, it's best to state a "multiple choice question" problem (the possible cases "follow" from the conditions of the problem). After having solved this problem, you have to consider the generalization of all cases - a variative problem. The solution of every ”multiple choice question” problem included in school manual must be concluded with the statement of a variative problem.

In the next stage, the students must be taught how to derive a variative problem from the standard one. Students must be offered to create a problem with an ambiguous answer, to solve both definite and variative problems. Students trying to create variative problems on their own must be encouraged.

Finally, the last stage assumes the inclusion of variative problems in teaching process as one of effective tools for the realization of activity-based approach.

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