EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 1201-1210
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Movable Strong Resolving Domination in Graphs

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Abstract. Let G be a connected graph. A strong resolving dominating set S is a 1-*movable strong resolving dominating* set of G if for every $v \in S$, either $S \setminus \{v\}$ is a strong resolving dominating set or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a strong resolving dominating set of G. The minimum cardinality of a 1-movable strong resolving dominating set of G, denoted by $\gamma^1_{msR}(G)$ is the 1-*movable strong resolving domination number* of G. A 1-movable strong resolving dominating set with cardinality $\gamma^1_{msR}(G)$ is called a γ^1_{msR}-set of G. In this paper, we study this concept and the corresponding parameter in graphs resulting from the join, corona and lexicographic product of two graphs. Specifically, we characterize the 1-movable strong resolving dominating sets in these types of graphs and determine the exact values of their 1-movable strong resolving domination numbers.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Movable strong resolving dominating set, movable strong resolving domination number, join, corona, lexicographic product

1. Introduction

Domination in graphs was first introduced by C. Berge in 1958 [4]. There are now many studies involving domination and its variations. Domke et. al [6] introduced and investigated the concept of restrained domination in graphs. Oellermann, O. R, and Peters-Fransen, J. [11] introduced and studied the concept of strong resolving set. Slater [14] introduced and studied the concept of resolving set. Resolving sets and resolving dominating sets were also studied in [1, 10].

The concept of metric dimension has grown to become an interesting topic in graph theory. In line with this, some researchers introduced another variant, more restricted than the metric dimension, called the strong metric dimension which is the cardinality

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4440

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of a minimum strong resolving set. Furthermore, several remarkable studies are continuously appearing after its introduction by P.J. Slater who discovered its usefulness when working with the United States sonar and Coast Guard Loran (long range aids to navigation) stations. Its applications have arisen in many diverse fields including chemistry, for representing chemical compounds [7], the robot navigation [12] and geographical routing protocols [9], to name a few. In [13], an variant called the strong metric dimension, was presented where the authors illustrated its application to combinatorial search. Along with the increasing discovery of its applications, theoretical studies on this invariant also appear in several number of other papers including [3], [2], [5], [8]. This paper intends to generate additional theoretical results and help widen the pool of existing studies from where new researchers may draw new insights and directions for further investigation.

Let $G = \big(V(G), E(G)\big)$ be a graph. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is a *neighborhood* of v. An element $u \in N_G(v)$ is called a *neighbor* of v. $N_G[v] = N_G(v) \cup \{v\}$ is a *closed neighborhood* of v. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup\limits_{v \in S} N_G(v)$ and $N_G[S] = \bigcup\limits_{v \in S} N_G[v]$.

A *clique* in a graph G is a complete induced subgraph. A set $C \subseteq V(G)$ is called a *superclique* in G if $\langle C \rangle$ is a clique and for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C is maximum in G if $|C| \geq |C^*|$ for all supercliques C^* in G. The superclique number, $\omega_S(G)$, of G is the cardinality of a maximum superclique in G. A superclique C is called a dominated superclique if for every $u \in C$, there exists $v \in V(G) \setminus C$ such that $uv \in E(G)$. The dominated superclique number, $\omega_{DS}(G)$, of G is the cardinality of a maximum dominated superclique in G. A vertex u of G is maximally distant from vertex v of G, $u \neq v$, if for every vertex $w \in N_G(u), d_G(v, w) \leq d_G(u, v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are mutually maximally distant, denoted by $uMMDv$.

A vertex x of a graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, ..., x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector*

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), ..., d_G(v, x_k))$$

is called the representation of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The *metric dimension* of G, denoted by $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called *basis*.

A set $S \subseteq V(G)$ of vertices of G is a *dominating set* if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = min\{|S| : S \text{ is a dominating set of } G\}$.

A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if S is a dominating set and for every pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $u \in I_G[v, w]$ or $v \in I_G[u, w]$. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G and is denoted by $\gamma_{sR}(G)$. A strong resolving dominating set of cardinality $\gamma_{sR}(G)$ is called a γ_{sR}-set of G.

A non-empty set $S \subseteq V(G)$ of a connected graph G is a 1-*movable dominating* set of G if S is a dominating set of G and for every $v \in S$, either $S \setminus \{v\}$ is a dominating set of G or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The 1-*movable domination number* of a graph G, denoted by $\gamma_m^1(G)$ is the smallest cardinality of a 1-movable dominating set of G. A 1-movable dominating set of cardinality $\gamma_m^1(G)$ is referred to as a γ_m^1-set of G.

A resolving dominating set S of a graph G is a 1-*movable resolving dominating* set of G if for every $v \in S$, either $S \setminus \{v\}$ is a resolving dominating set or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a resolving dominating set of G. The minimum cardinality of a 1-movable resolving dominating set of G, denoted by $\gamma_{mR}^1(G)$ is the 1-*movable R-domination number* of G. A 1-movable resolving dominating set with cardinality $\gamma_{mR}^1(G)$ is called a γ_{mR}^1-set of G.

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge set $E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$. The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

2. Preliminary Results

This section introduces the movable strong resolving domination in some graphs. It also characterizes some graphs in terms of its movable strong resolving domination number.

Remark 1. [1] Any superset of a strong resolving set is a strong resolving set.

Lemma 1. [10] Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving dominating set of G if and only if $C = \varnothing$ or C is a dominated superclique in G. In particular, $\gamma_{sR}(G) = |V(G)| - \omega_{DS}(G)$.

Theorem 1. [1] Let G be a nontrivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v\rangle$. Then $S \subseteq V(K_1+G)$ is a strong resolving set of K_1+G if and only if $S = V(G)$, or $S = V(K_1 + G) \setminus C^*$ or $S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$ where C^* is a superclique in G.

Theorem 2. [1] Let G be a nontrivial connected graph of order n with $\gamma(G) \neq 1$ and $K_1 = \langle v\rangle$. Then $S \subseteq V(K_1+G)$ is a strong resolving set of K_1+G if and only if $S = V(G)$, or $S = V(G) \setminus C$, or $S = V(K_1 + G) \setminus C$ where C is a superclique in G.

Theorem 3. [1] Let $K_1 = \langle v\rangle$ and G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, m$. A proper subset S of $V(K_1 + G)$ is a strong resolving set of $K_1 + G$

if and only if $S = V(G)$, or $S = V(G) \setminus C_i$, or $S = V(K_1 + G) \setminus C_i$, where C_i is a superclique in G_i, for some $i \in \{1, 2, \ldots, m\}$.

Remark 2. *Every movable strong resolving dominating set of a connected graph G is a strong resolving dominating set in G. Hence, $\gamma_{sR}(G) \leq \gamma^1_{msR}(G)$.*

Remark 3. *The converse of Remark* 2 does not hold. To see this, the set $S = \{v_1, v_2, v_3\}$ of the path $P_4 = [v_1, v_2, v_3, v_4]$ is a strong resolving dominating set of P_4 but it is not movable strong resolving dominating set since $S \setminus \{v_1\}$ is not a strong resolving set of P_4.

Proposition 1. Any superset of a movable strong resolving dominating set is a movable strong resolving dominating set.

Proof. Let S be a movable strong resolving dominating set of G and $S \subsetneq S'$. Then by Remark 1, S' is a strong resolving dominating set of G. We show that a S' is movable strong resolving dominating set of G. Let $x \in S'$. If $x \in S$, then $S \setminus \{x\} \subseteq S' \setminus \{x\}$. Since S is a movable strong resolving dominating set of G, either $S \setminus \{x\}$ is strong resolving dominating set of G or $\exists y \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{y\}$ is strong resolving dominating set of G. If $S \setminus \{x\}$ is a strong resolving dominating set of G, then $S' \setminus \{x\}$ is also strong resolving dominating set of G by Remark 1. If $\exists y \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{y\}$ is a strong resolving dominating set, then

$$(S \setminus \{x\}) \cup \{y\} \subseteq (S' \setminus \{x\}) \cup \{y\}$$

$(S' \setminus \{x\}) \cup \{y\}$ is a strong resolving dominating set of G. Therefore, S' is a movable strong resolving dominating set of G. □

Proposition 2. Let $P_n = [v_1, v_2, \ldots, v_n]$ where $n \geq 1$. If a set $S \subseteq V(P_n)$ is a movable strong resolving dominating set of P_n, then S is a dominating set containing the vertices v_1 and v_n.

Proof. Suppose S is a movable strong resolving dominating set of P_n and suppose that S does not contain v_1 or v_n, say v_1. Since v_1MMDv_n, $S \cap \{v_1, v_n\} \neq \varnothing$. Hence $v_n \in S$. This implies that $S \setminus \{v_n\}$ and $(S \setminus \{v_n\}) \cup \{v_{n-1}\}$ if $v_{n-1} \notin S$ are not strong resolving sets of P_n, a contradiction. Therefore, S contains v_1 and v_n. □

Lemma 2. Let G be a nontrival connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a movable strong resolving dominating set of G if and only if $C = \varnothing$ or C is a dominated superclique in G and either for each $x \in S$, $C \cup \{x\}$ is a dominated superclique or there exists $y \in C \cap N_G(x)$ such that $(C \setminus \{y\}) \cup \{x\}$ is a dominated superclique in G.

Proof. Suppose $S = V(G) \setminus C$ is a movable strong resolving dominating set of G. Then S is a strong resolving dominating set of G . By Lemma 1, $C = \varnothing$ or C is a dominated superclique in G. Let $x \in S$. Since S is a movable strong resolving dominating set, either $S \setminus \{x\}$ is a strong resolving dominating or there exists $y \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{y\}$ is a strong resolving dominating set of G. Since $S \setminus \{x\} = V(G) \setminus (C \cup \{x\})$

and $(S \setminus \{x\}) \cup \{y\} = V(G) \setminus ((C \setminus \{x\}) \cup \{y\})$, by Lemma 1 $C \cup \{x\}$ is a dominated superclique or $(C \setminus y) \cup \{x\}$ is a dominated superclique in G.

For the converse, suppose $C = \varnothing$. Then, $S = V(G)$ is strong resolving dominating set of G. Thus, $S \setminus \{x\} = V(G) \setminus \{x\}$ is strong resolving dominating since $\{x\}$ is a dominated superclique for each $x \in V(G)$. So, suppose C is a dominated superclique in G and for each $x \in S$ either $C \cup \{x\}$ is a dominated superclique or there exists $y \in C \cap N_G(x)$ such that $(C \setminus \{y\}) \cup \{x\}$ is a dominated superclique. Hence, for each $x \in S$, $(S \setminus \{x\}) \cup \{y\} = V(G) \setminus (C \setminus \{y\}) \cup \{x\}$ is a strong resolving dominating set of G. Therefore, S is a movable strong resolving set of G. □

Lemma 3. Let G be a nontrivial connected graph. A set $C \subseteq V(G)$ is a superclique in G and for each $x \in V(G) \setminus C$ either $C \cup \{x\}$ is a superclique or there exists $y \in C \cap N_G(x)$ such that $(C \setminus \{y\} \cup \{x\})$ is a superclique in G if and only if $|C| = 1$ and $deg_G(v) = |V(G)| - 1$ for $v \in C$.

Proof. Suppose C is a superclique in G satisfying the given condition and let $|C| > 1$. Let $u, v \in C$. Then $uv \in E(G)$. Let $x \in V(G) \setminus C$. If $C \cup \{x\}$ is a superclique, then $x \in N_G(u) \cap N_G(v)$, a contradiction since C is a superclique. On the other hand, suppose there exists $y \in C \cap N_G(x)$ such that $(C \setminus \{y\}) \cup \{x\}$ is superclique. If $y = u$, then $x \in N_G(v)$ and there exists $z \in (N_G(x) \setminus N_G(v)) \cap (V(G) \setminus C)$. This is a contradiction since $z \in C \cup \{x\}$ or $z \in (C \setminus \{w\}) \cup \{x\}$ for some $w \in C \cap N_G(x)$, that is $z \in N_G(x) \cap N_G(v)$. Hence, $|C| = 1$. Now, suppose $deg_G(v) < |V(G)| - 1$ where $v \in C$. Then there exists $y \in V(G) \setminus N_G(v)$ and $C \cup \{y\}$ is not a superclique in G, a contradiction. Thus, $deg_G(v) = |V(G)| - 1$, showing that C is a γ-set of G.

For the converse, suppose C is γ-set of G. Then C is a superclique in G and for every $x \in V(G) \setminus C$, $C \cup \{x\}$ is a superclique in G since $\langle C \cup \{x\} \rangle$ is a path in G. □

As a consequence of Lemma 2 and Lemma 3, the next result follows.

Theorem 4. Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a movable strong resolving dominating set of G if and only if $C = \varnothing$ or C is a γ-set of G if $\gamma(G) = 1$.

3. $\gamma^1_{msR}(G + H)$ in the Join of Graphs

This section gives characterization of the movable strong resolving dominating sets in the join of graphs as well as its movable strong resolving domination number.

Theorem 5. Let G be a nontrivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a movable strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$, or $S = V(K_1 + G) \setminus C$ where C is a γ-set of G.

Proof. Suppose S is a movable strong resolving dominating set of $K_1 + G$. Then, by Theorem 1, $S = V(G)$, or $S = V(K_1 + G) \setminus C^*$, or

$$S = (V(G) \setminus C) \cup \{x \in C : deg_G(x) = n - 1\}$$

where C is a superclique in G. Since S is a dominating set, C is a dominated superclique in G and C^* is a superclique in G. By Lemma 2 and Lemma 3, C is a γ-set of G. Hence, $S = V(G)$ or $S = V(K_1 + G) \setminus C$ where C is a γ-set of G.

Conversely, suppose $S = V(G)$ or $S = V(K_1 + G) \setminus C$ where C is a γ-set of G. Then, by Theorem 1, S is a strong resolving dominating set of K_1+G. If $S = V(G)$, then $S \setminus \{x\}$ is a strong resolving dominating set of $K_1 + G$ since $\{x\}$ is a dominated superclique in G by Lemma 2 and Lemma 3 and Theorem 1. Hence, S is a movable strong resolving dominating set of $K_1 + G$. Similarly, if $S = V(K_1 + G) \setminus C$ where C is a γ-set of G, then S is a movable strong resolving set of $K_1 + G$ by Lemma 2, Lemma 3 and Theorem 1. □

Theorem 6. Let G be a nontrivial connected graph with $\gamma(G) \neq 1$. Then, $S \subseteq V(K_1+G)$ is a movable strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$.

Proof. Suppose S is a movable strong resolving dominating set of K_1+G. By Theorem 2 $S = V(G)$ or $S = V(G) \setminus C$, or $S = V(K_1 + G) \setminus C$ where C is a superclique in G. Since $\gamma(H) \neq 1$, by Lemma 2, Lemma 3 and Theorem 2, $S \neq V(G) \setminus C$ and $S \neq V(K_1 + G) \setminus C$. Hence, $S = V(G)$. The converse is clear. □

Corollary 1. Let G be a nontrivial connected graph of order n. Then $\gamma^1_{msR}(K_1+G) = n$.

Proof. Let S be a γ^1_{msR}-set of K_1+G. If $\gamma(G) = 1$, then $S = V(G)$ or $S = V(K_1+G)\setminus C$ where C is a γ-set of G. Hence,

$$\begin{aligned}
\gamma^1_{msR}(K_1 + G) = |S| &= |V(G)| \\
&= |V(K_1 + G)| - |C| \\
&= n + 1 - 1 \\
&= n.
\end{aligned}$$

On the other hand, if $\gamma(G) \neq 1$, then $S = V(G)$. Thus,

$$\gamma_{msR}(K_1 + G) = |S| = |V(G)| = n.$$

□

Theorem 7. Let $K_1 = \langle v \rangle$ and G be a disconnected graph whose components are G_i for $i = 1, 2, \dots, m$. A proper subset S of $V(K_1+G)$ is a movable strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$ or $S = V(K_1 + G) \setminus C_i$ where C_i is γ-set of G_i if $\gamma(G_i) = 1$ for all $i \in \{1, 2, \dots, m\}$.

Proof. Suppose S is a movable strong resolving dominating set of $K_1 + G$. Then, by Theorem 3, $S = V(G)$ or $S = V(G) \setminus C_i$ or $S = V(K_1 + G) \setminus C_i$ where C_i is a superclique in G_i for some $i \in \{1, 2, \dots, m\}$. If $\gamma(G_i) = 1$, then by Lemma 2, Lemma 3 and Theorem 5, C_i is a γ-set of G_i. Thus , $S \neq V(G)\setminus C_i$ showing that $S = V(G)$ or $S = V(K_1+G)\setminus C_i$ where C_i is a γ-set of G_i for some $i \in \{1, 2, \dots, m\}$. If $\gamma(G_i) \neq 1$ for all $i \in \{1.2. \dots, m\}$, then $S = V(G)$ by Lemma 2, Lemma 3, and Theorem 6.

Conversely, the case when $S = V(G$ is trivial. Suppose $S = V(K_1 + G) \setminus C_i$ where C_i is a γ-set of G_i if $\gamma(G_i) = 1$ for some $i \in \{1, 2, \ldots, m\}$. Then by Theorem 3, S is a strong resolving dominating set of $K_1 + G$. By Lemma 2, Lemma 3 and Theorem 5, S is a movable strong resolving set of $K_1 + G$. □

Corollary 2. Let G_i be connected graphs of order n_i and G be a disconnected graph whose components are G_i for $i \in \{1, 2, \ldots, m\}$. Then, $\gamma^1_{msR}(K_1 + G) = |V(G)|$.

Theorem 8. Let G and H be nontrivial connected graphs of orders m and n, respectively. A proper subset S of $V(G + H)$ is a movable strong resolving dominatiing set of $G + H$ if and only if at least one of the following is satisfied.

(i) $S = V(G + H) \setminus C_G$ where C_G is a γ-set of G if $\gamma(G) = 1$.

(ii) $S = V(G + H) \setminus C_H$ where C_H is a γ-set of H if $\gamma(H) = 1$.

Proof. Suppose S is a movable strong resolving dominating set of $G + H$. By Theorem [10], Lemma 2, and Lemma 3, *(i)* or *(ii)* holds.

The converse is clear by Theorem [10], Lemma 2, and Lemma 3. □

Corollary 3. Let G and H be nontrivial connected graphs of orders m and n, respectively. Then, $\gamma^1_{msR}(G + H) = m + n - 1$.

4. $\gamma^1_{msR}(G \circ H)$ in the Corona of Graphs

This section gives characterization of the movable strong resolving dominating sets in the corona of graphs as well as its movable strong resolving domination number.

Theorem 9. Let G be a nontrivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a movable strong resolving dominating set of $G \circ H$ if and only if $S = A \bigcup \left(\bigcup_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$.

Proof. Suppose that a proper subset S of $V(G \circ H)$ is a movable strong resolving dominating set of $G \circ H$. Since S is a strong resolving dominating set of $G \circ H$ (i) or (ii) of Theorem [10] holds. If (i) holds, then $S = A \bigcup \left(\bigcup_{u \in V(G)} V(H^u) \right)$, where $A \subseteq V(G)$. Suppose (ii) holds. Let $x \in V(H^w)$ for some $w \in V(G)$. Then $S \setminus \{x\} = A \bigcup \left(\bigcup_{u \in V(G) \setminus \{w,v\}} V(H^u) \right) \bigcup \left(V(H^w) \setminus \{x\} \right) \bigcup B_v$ is not a strong resolving set by Theorem [10]. Hence, $S = A \bigcup \left(\bigcup_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$.

For the converse, suppose $S = A\bigcup\left(\bigcup_{u\in V(G)} V(H^u)\right)$. By Theorem [10], S is a strong resolving dominating set. Let $p \in S$. If $p \in A$, then $S\setminus\{p\} = (A\setminus\{p\})\cup\left(\bigcup_{u\in V(G)} V(H^u)\right)$ is a strong resolving dominating set. If $p \in V(H^u)$ for each $u \in V(G)$, then $S \setminus \{p\} = A \cup \left(\bigcup_{u\in V(G)} V(H^u) \setminus \{p\}\right)$ is a strong resolving dominating set since $\{p\}$ is a dominated superclique in H^u. Accordingly, S is a movable strong resolving dominating set in $G \circ H$. □

Corollary 4. Let G be a connected graph of order $m > 1$ and H be any graph of order n. Then $\gamma^1_{msR}(G \circ H) = mn$.

Proof. Let C be a γ^1_{msR}-set of $G \circ H$. Then by Theorem 9, $S = A\bigcup\left(\bigcup_{u\in V(G)} V(H^u)\right)$ where $A \subseteq V(G)$. Thus,

$$\begin{aligned}\gamma^1_{msR}(G \circ H) &= |C| \\ &= |A| + \left|\bigcup_{u\in V(G)} V(H^u)\right| \\ &\geq |V(G)||V(H)| \\ &= mn.\end{aligned}$$

Let $A = \varnothing$. Then $S^* = A\bigcup\left(\bigcup_{u\in V(G)} V(H^u)\right)$ is a movable strong resolving dominating set of $G \circ H$ by Theorem 9. Hence,

$$\begin{aligned}\gamma^1_{msR}(G \circ H) &\leq |S^*| \\ &= \left|\bigcup_{u\in V(G)} V(H^u)\right| \\ &= mn.\end{aligned}$$

Therefore, $\gamma^1_{msR}(G \circ H) = mn$. □

5. $\gamma^1_{msR}(G[H])$ in the Lexicographic Product of Graphs

This section gives characterization of the movable strong resolving dominating sets in the lexicographic product of graphs as well as its movable strong resolving domination number.

Lemma 4. Let $G = K_n$ for $n > 1$ and H be a nontrivial connected graph with $\gamma(H) = 1$. Then $A \times C$ is a γ-set of $G[H]$ if and only if A is a singleton subset of $V(G)$ and C is a γ-set of H.

Proof. Suppose $A \times C$ is a γ-set of $G[H]$. Since $G = K_n$ for $n > 1$ and $\gamma(H) = 1$, $\gamma(G[H]) = 1$. Hence, A is a singleton subset of $V(G)$ and C is a singleton subset of H. We claim that C is a γ-set of H. Suppose C is not a γ-set of H. Then there exists $y \in (V(H) \setminus C)$ such that $N_H(y) \cap C = \varnothing$. Hence, $N_{G[H]}((a, y)) \cap \big(\{a\} \times C\big) = \varnothing$ for all $a \in A$. This contradicts the assumption that $A \times C$ is a γ-set of $G[H]$. Thus, C is a γ-set of H.

Conversely, suppose that A is a singleton subset of $V(G)$ and C is a γ-set of H. We claim that $A \times C$ is a γ-set of $G[H]$. Let $(a, x) \in V(G[H]) \setminus (A \times C)$. Then ($a \in A$ and $x \notin C$) or ($a \notin A$ and $x \in C$) or ($a \notin A$ and $x \notin C$). If $a \in A$ and $x \notin C$, then there exists $y \in C \cap N_H(x)$ since C is a γ-set of H. Hence, $(a, y) \in (A \times C) \cap N_{G[H]}((a, x))$. Suppose $a \notin A$ and $x \in C$. Since A is a singleton subset of $G = K_n$ for $n > 1$, there exists $b \in A \cap N_G(a)$. Hence, $(b, x) \in (A \times C) \cap N_{G[H]}((a, x))$. Also, if $a \notin A$ and $x \notin C$, then there exists $(b, y) \in (A \times C) \cap N_{G[H]}((a, x))$. Therefore, $A \times C$ is a γ-set of $G[H]$. □

Theorem 10. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph. A subset S of $V(G[H])$ is a movable strong resolving dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$, where A is a subset of $V(G)$ and $C = \varnothing$ or A is a singleton subset of $V(G)$ and C is a γ-set of H if $\gamma(H) = 1$.

Proof. Suppose S is a movable strong resolving dominating set of $G[H]$. Since $diam(G[H]) = 2$, by Theorem 4, $S = V(G[H]) \setminus (A \times C)$ where $A \times C = \varnothing$ or $A \times C$ is a γ-set of $G[H]$ if $\gamma(G[H]) = 1$. By Lemma 4, A is a singleton subset of $V(G)$ and C is a γ-set of H if $\gamma(H) = 1$.

For the converse, suppose $S = V(G[H]) \setminus (A \times C)$, where $A \subseteq V(G)$ and $C = \varnothing$ or A is a singleton subset of $V(G)$ and C is a γ-set of H if $\gamma(H) = 1$. If $A \subseteq V(G)$ and $C = \varnothing$, then $A \times C = \varnothing$. Hence, $S = V(G[H])$ is a movable strong resolving dominating set of $G[H]$. On the other hand, if A is a singleton subset of $V(G)$ and C is γ-set of H if $\gamma(H) = 1$, then $A \times C$ is a γ-set of $G[H]$ if $\gamma(G[H]) = 1$. By Theorem 4, $S = V(G[H]) \setminus (A \times C)$ is a movable strong resolving dominating set of $G[H]$. □

As a consequence of Theorem 10, the next result follows.

Corollary 5. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph of order m. Then

$$\gamma^1_{msR}(G[H]) = \begin{cases} mn\ , & \text{if} \quad \gamma(H) \neq 1 \\ mn - 1\ , & \text{if} \quad \gamma(H) = 1. \end{cases}$$

Acknowledgements

This research is funded by the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines.

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