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Prolate Spheroidal Wavelet Coefficients,Frames and Double Infinite Matrices

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Abstract. In this paper we defined the double infinite matrix $A = a(m,n,k)$and study the action of A on $f \in L^2(R)$ and on its prolate spheroidal wavelet coefficients. We also find the frame condition for A-transform of $f \in L^2(R)$ whose wavelet series expansion is known.

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1. Introduction

The study of continuous prolate spheroidal wave functions (PSWFs) has been an active area of research in both electrical engineering and mathematics. Yet they seem to be an inexhaustible and inspirational source of new ideas and methods, both theoretical and applied. The PSWFs are those that are most highly,localized simultaneously in both the time and frequency domain. This fact was discovered by Slepian and his collaborators and was presented in a series of articles [7],[8],[12]-[14] about forty years ago.
Let us recall the connection between PSWFs and the Shannon sampling theorem (Shannon[10]) given by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}. \tag{1}$$

The above formula (1) holds for π-bandlimited signals with finite energy,that is,for continuous functions in $L^2(R)$ whose Fourier transform has support in $[-\pi,\pi]$. This theorem has became a well known part of both the mathematical and engineering literature. The sinc function $S(t) = \frac{\sin \pi t}{\pi t}$ which appears in this formula is closely related to the PSWFs $\varphi_{n,\sigma,\tau}(t)$.

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The PSWFs $\varphi_{n,\sigma,\tau}(t)$ constitute an orthogonal basis of the space of σ-band limited functions on the real line. They are maximally concentrated on the interval $[-\tau,\tau]$ and depend on parameters σ and τ. PSWFs are characterized as the eigenfunctions of an integral operator with kernel arising from the sinc functions $S(t)$:

$$\frac{\sigma}{\pi}\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}(x)S(\frac{\sigma}{\pi}(t-x))dx=\lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t),|t|\leq\tau. \tag{2}$$

It is easy to show that the symmetrical kernel $S(\frac{\sigma}{\pi}(t-x))$ is positive definite,so that from[1] we know that (2) has solutions in $L^2(-\tau,\tau)$ only for a discrete set of real positive values of $\lambda_{n,\sigma,\tau}$ say $\lambda_{0,\sigma,\tau}\geq\lambda_{1,\sigma,\tau}\geq\ldots$ and that as $n\to\infty$, $\lim\lambda_{n,\sigma,\tau}=0$. The corresponding solutions,or eigenfunctions,$\varphi_{0,\sigma,\tau}(t),\varphi_{1,\sigma,\tau}(t),\ldots$ can be chosen to be the real and orthogonal on $(-\tau,\tau)$. The variational problem that let to (2) only requires that equation to hold for $|t|\leq\tau$. With $\varphi_{n,\sigma,\tau}(x)$ on the left of (2) gives for $|x|\leq\tau$, however, the left is well defined for all t. We use this to extend the range of definition of the $\varphi_{n,\sigma,\tau}$'s and so define

$$\varphi_{n,\sigma,\tau}(t)=\frac{\sigma}{\pi\lambda_{n,\sigma,\tau}}\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}(x)S(\frac{\sigma}{\pi}(t-x))dx,|t|>\tau.$$

The eigenfunctions $\varphi_{n,\sigma,\tau}$ are now defined for all t. In addition to the equation (2), the $\{\varphi_{n,\sigma,\tau}\}$ satisfy an integral equation over $(-\infty,\infty)$

$$\frac{\sigma}{\pi}\int_{-\infty}^{\infty}\varphi_{n,\sigma,\tau}(x)S(\frac{\sigma}{\pi}(t-x))dx=(\varphi_{n,\sigma,\tau}*S_\sigma)(t)=\varphi_{n,\sigma,\tau}(t)$$

with the same kernel. This leads to a dual orthogonality

$$\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}(x)\varphi_{m,\sigma,\tau}(x)dx=\lambda_{n,\sigma,\tau}\delta_{nm},$$

$$\int_{-\infty}^{\infty}\varphi_{n,\sigma,\tau}(x)\varphi_{m,\sigma,\tau}(x)dx=\delta_{nm}$$

and the fact that they constitute an orthogonal basis of $L^2(-\tau,\tau)$, as well as an orthonormal basis of the subspaces B_σ of $L^2(-\infty,\infty)$, the Paley-Wiener space of all $\sigma-$bandlimited functions.

We are interested mainly in $\varphi_{0,\sigma,\tau}$ whose concentration on the interval $[-\tau,\tau]$is maximum. Since $\varphi_{k,\sigma,\tau}$ has exactly k zeros in the interval $[-\tau,\tau]$,so $\varphi_{0,\sigma,\tau}$ (PSWFs)are entire functions and therefore can not vanish on any interval,they can be made uniformly small outside of $[-\tau,\tau]$ for τ or σ sufficiently large, so that computationally they behave like functions with compact support. To construct PS wavelets,the scaling function $\phi=\varphi_{0,\pi,\tau}$, where τ is any positive number,was introduced by [15] and obtained a basis composed of a space $V_0\subset L^2$ (R) which turns out to be the Paley-Wiener space B_π of π - bandlimited functions.

A multiresolution analysis (MRA)are then based on this construction. The other spaces are obtained by dilation by factors of two and consist of the Paley-Wiener spaces $V_m = B_{2^m\pi}$.The sinc function is the standard scaling function of this MRA. It is well known that sinc function has very good frequency localization, but not very good time localization. It follows that this wavelet basis has limited use in comparison to the Daubechies wavelets which have compact support in the time domain. However,PSWFs are superior to sinc function and they are similar to the Daubechies wavelets for practical computations.

Using the standard wavelet approach in which dilations of $\varphi_{0,\pi,\tau}(2^m t)$ are used to get the basis $\{\varphi_{0,\pi,\tau}(2^m t - n)\}$ of V_m we get

$$\phi(2^m t) = \varphi_{0,\pi,\tau}(2^m t) = 2^{m/2}\varphi_{0,2^m\pi,2^{-m}\pi\tau}(t),$$

which show that the concentration interval becomes smaller as m increases. So we have to fix the concentration interval by taking $\{\varphi_{0,2^m\pi,\tau}(t - 2^{-m}n)\}$ instead as a possible Riesz basis of V_m. Thus our new basis for V_0 and V_m are different from the standard wavelet basis for V_0 and V_m consisting of translates of the sinc function.

2. Frames and Applications to Prolate Spheroidal Wavelets

The notion of frame goes back to Duffin and Schaeffer[6] in the early 1950s to deal with the problems in nonharmonic Fourier series. In many cases the wavelet experts prefer to work with frames instead of Riesz bases. The recent development and work on frames and related topics,(see[2][3][4]). In this paper,we will use the double infinite matrices (see [9][10]) to obtain the frame conditions and prolate spheroidal wavelet coefficients.

A sequence $\{x_n\}$ in a Hilbert space H is a frame if there exist constants c_1 and c_2, $0 < c_1 \le c_2 < \infty$, such that

$$c_1 ||f||^2 \le \sum_{n\in z} |\langle f, x_n\rangle|^2 \le c_2 ||f||^2, \tag{3}$$

for all $f \in H$. The spermium of all such numbers c_1 and infimum of all such numbers c_2 are called the frame bounds of the frame. The frame is called tight frame when $c_1 = c_2$ and is called normalized tight frame when $c_1 = c_2 = 1$. Any orthonormal basis in a Hilbert space H is a normalized tight frame. In another paper[5] we have proved that a prolate spheroidal wavelet function $\phi_m(t - 2^{-m}n) = \varphi_{0,2^m\pi,\tau}(t - 2^{-m}n) \in L^2(R)$, constitute a frame with frame bounds c_1 and c_2, if any $f \in L^2(R)$ such that

$$c_1 ||f||^2 \le \sum_{m,n=-\infty}^{\infty} |\langle f, \phi_m(t - 2^{-m}n)\rangle|^2 \le c_2 ||f||^2.$$

Let $A = a(m,n,\kappa) = \int_{-\infty}^{\infty} \phi_m(t - 2^{-m}n)\phi_m(t - 2^{-m}\kappa)dt$ be a double infinite matrix of real numbers. Then, A-transform of a double sequence $\{\phi_m^\kappa\}$ is defined as

$$A\phi_m^\kappa = \int_{-\infty}^{\infty} \phi_m(t - 2^{-m}n)\phi_m(t - 2^{-m}\kappa)\phi_m^\kappa dt$$

which is called A-means of the sequence ϕ_m^κ. This definition is due to Moricz and Rhoades [9].

A double matrix $A = a(m, n, \kappa)$ is satisfied the following conditions :

(i) $\lim_{m\to\infty} a(m, n, \kappa) = 1, n, \kappa \in z$

(ii) $||A|| = \sup_{m>0} |a(m, n, \kappa)| < \infty, n, \kappa \in z$.

The approximation of a function in $L^2(R)$ by function in V_m is given by a series of the form

$$f(t) = \sum_\kappa b_\kappa^m \phi_m(t - 2^{-m}\kappa) \tag{4}$$

where the coefficient may be obtained from the dual Riesz basis. In this case the result is the projection of a function f onto V_m. The coefficients for the projection are

$$b_\kappa^m = \int_{-\infty}^{\infty} f(t)\tilde{\phi_m}(t - 2^{-m}\kappa)dt.$$

The kernel of this projection is given by

$$q_m(x, t) = \sum_\kappa \phi_m(x - 2^{-m}\kappa)\tilde{\phi_m}(t - 2^{-m}\kappa), \tag{5}$$

here $\tilde{\phi_m}$ is biorthogonal to ϕ_m.

3. Main Results

In this section we prove the following theorems.

Theorem 1. *Let $A = a(m, n, \kappa$ be a double infinite matrix. If*

$$f(t) = \sum_\kappa b_\kappa^m \phi_m(t - 2^{-m}\kappa)$$

is a PS wavelet expansion of $f \in L^2(R)$ with wavelet coefficients

$$b_\kappa^m = \int_{-\infty}^{\infty} f(t)\tilde{\phi_m}(t - 2^{-m}\kappa)dt = \langle f, \tilde{\phi_m}(t - 2^{-m}\kappa)\rangle,$$

then the frame condition for A-transform of $f \in L^2(R)$ is

$$c_1||f||_2^2 \le \sum_\kappa |\langle Af, q_m(y, t)\rangle|^2 \le c_2||f||_2^2 \tag{6}$$

where q_m is given by (5), Af is the A-transform of f and $0 < c_1 \le c_2 < \infty$.

Proof. By (3), we can write

$$f(t) = \sum_{\kappa} b_{\kappa}^{m} \phi_m(t - 2^{-m}\kappa)$$

taking A-transform of f, we get

$$\begin{aligned} Af &= \int_{-\infty}^{\infty} \phi_m(t-2^{-m}n)\phi_m(t-2^{-m}\kappa)\Sigma_\kappa b_\kappa^m \phi_m(t-2^{-m}\kappa)dt \\ &= \Sigma_\kappa \int_{-\infty}^{\infty} \phi_m(t-2^{-m}n)\phi_m(t-2^{-m}\kappa)\int_{-\infty}^{\infty} f(y)\tilde{\phi_m}(y-2^{-m}\kappa)\phi_m(t-2^{-m}\kappa)dydt \\ &= \Sigma_\kappa \langle Af, \tilde{\phi_m}(y-2^{-m}\kappa)\rangle \phi_m(t-2^{-m}\kappa) \\ &= \langle Af, \Sigma_\kappa \tilde{\phi_m}(y-2^{-m}\kappa)\phi_m(t-2^{-m}\kappa)\rangle \\ &= \langle Af, q_m(y,t)\rangle, \end{aligned}$$

therefore

$$\begin{aligned} \Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2 &\leq \Sigma_\kappa \int_{-\infty}^{\infty} |Af q_m(y,t)|^2 dt \\ &\leq \Sigma_\kappa \int_{-\infty}^{\infty} (Af)^2 dt \int_{-\infty}^{\infty} [S_{2^m}(y-t)]^2 dt = ||A||_2^2 ||f||_2^2 \end{aligned} \tag{7}$$

(since $q_m(y,t) = \frac{\sin 2^m \pi(y-t)}{\pi(y-t)} = S_{2^m}(y-t)$)

Now, for any arbitrary $f \in L^2(R)$, define

$$\tilde{f} = [\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2]^{-1/2} f.$$

Clearly

$$\langle A\tilde{f}, q_m(y,t)\rangle = [\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2]^{-1/2} \langle Af, q_m(y,t)\rangle$$

then

$$\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2 \leq 1.$$

If there exist a positive constant α, then

$$||A\tilde{f}||_2^2 \leq \alpha,$$

so

$$[\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2]^{-1} ||Af||_2^2 \leq \alpha$$

or

$$\begin{gathered} [\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2]^{-1} |\langle Af, q_m(y,t)\rangle|^2 \leq \alpha \\ \leq [\Sigma_\kappa |\langle Af, q_m(y,t)\rangle|^2]^{-1} ||A||_2^2 ||f||_2^2 \leq \alpha \end{gathered}$$

or

$$c_1||f||_2^2 \leq \Sigma_\kappa|\langle Af, q_m(y,t)\rangle|^2. \tag{8}$$

Combining (7) and (8) we get

$$c_1||f||_2^2 \leq \Sigma_\kappa|\langle Af, q_m(y,t)\rangle|^2 \leq c_2||f||_2^2.$$

Hence the proof is completed.

Theorem 2. *If b_κ^m are the PS wavelet coefficients of $f \in L^2(R)$, that is, $b_\kappa^m = \langle f, \tilde{\phi_m}(t-2^{-m})\rangle$ then the $d_m = \langle f, q_m(y,t)\rangle$, where $\{d_m\}$ is defined as the A-transform of $\{b_\kappa^m\}$ by*

$$d_m = \sum_{n,\kappa} a(m,n,\kappa) b_\kappa^m \phi_m(t-2^{-m}\kappa). \tag{9}$$

Proof. We have

$$\sum_{n,\kappa} a(m,n,\kappa) b_\kappa^m \phi_m(t-2^{-m}\kappa) =$$

$$\sum_{n,\kappa}\langle \phi_m(t-2^{-m}n), \phi_m(t-2^{-m}\kappa)\rangle\langle f, \tilde{\phi_m}(t-2^{-m}n)\rangle \phi_m(t-2^{-m}n)$$

$$= \int_{-\infty}^{\infty}\sum_{n,\kappa}\phi_m(t-2^{-m}n)\phi_m(t-2^{-m}\kappa)dt$$

$$\times(\int_{-\infty}^{\infty} f(y)\tilde{\phi_m}(y-2^{-m}\kappa)dy)\phi_m(t-2^{-m}\kappa)$$

$$= \int_{-\infty}^{\infty}\sum_{n,\kappa}\phi_m(t-2^{-m}n)\tilde{\phi_m}(y-2^{-m}\kappa)(\phi_m(t-2^{-m}n))^2 dt \int_{-\infty}^{\infty} f(y)dy$$

$$= \int_{-\infty}^{\infty} q_m(y,t) f(y) dy \int_{-\infty}^{\infty}[\phi_m(t-2^{-m}n)]^2 dt$$

$$= \langle f, q_m(y,t)\rangle.$$

Hence the proof is completed.

Theorem 3. *Let $A = a(m,n,\kappa)$ be a double nonnegative infinite matrix then $\{q_m(y,t)\}$ constitute a frame of $L^2(R)$.*

Proof. We have

$$\begin{aligned}\sum_m |d_m|^2 &= \sum_m |\langle f, q_m(y,t)\rangle|^2 \\ &= \frac{1}{2\pi}\sum_m |\langle \hat{f}, \hat{q_m}(y,t)\rangle|^2\end{aligned}$$

$$= \frac{1}{2\pi}\sum_m \int_{-\infty}^{\infty} |\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi),\hat{q_m}(w,\xi)d\xi|^2 dw$$
$$= \frac{1}{2\pi}\sum_m \int_{-\infty}^{\infty} |\chi_{2^m\pi}(w)\hat{f}(w)|^2 dw$$
$$= \frac{1}{2\pi}\sum_m \int_{-2^m\pi}^{2^m\pi} |\hat{f}(w)|^2 dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw = ||f||_2^2$$

that is,

$$\sum_m |d_m|^2 = ||f||_2^2, f \in L^2(R).$$

Therefore, for matrix A=a(m,n,κ), we have

$$c_1||f||_2^2 \le \sum_m |d_m|^2 \le c_2||f||_2^2,$$

where

$$0 \le c_1, c_2 < \infty.$$

This completes the proof of the theorem.

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