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A Short Introduction to Similarity Via Ideals

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Abstract. In this work, we introduce the notion of similarity between topologies τ_1 and τ_2, on a set X via ideals. Then, we give some characterizations regarding this kind of similarity by using $*-$dense, and $\mathbf{I}-$dense subsets. We also examine the preservation of similarity with respect to the topologies τ_1^* and τ_2^*.

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1. Introduction and Preliminaries

The idea of adding the notion of ideal into the topological spaces started with the works of Kuratowski [6], and Vaidyanathaswamy [7]. After that the notion of ideal topological space and applications have been examined deeply.

An ideal $\mathbf{I}$ on a set X is a nonempty collection of subsets of X, which satisfies the following conditions:

i. If $A \in \mathbf{I}$, and $B \subset A$, then $B \in \mathbf{I}$

ii. If $A, B \in \mathbf{I}$, then $A \cup B \in \mathbf{I}$.

We denote a topological space (X, τ) with an ideal $\mathbf{I}$ defined on X by $(X, \tau, \mathbf{I})$. An ideal $\mathbf{I}$ on (X, τ) is said to be τ-codense if $\mathbf{I} \cap \tau = \{\emptyset\}$.

On the other hand, there are many papers devoted to constructing new topologies via ideals. To do that, firstly an operator called the local function is invented. Then by using this, one can get a Kuratowski closure operator.

Definition 1 ([6]). *Let (X, τ) be a topological space, and $\mathbf{I}$ be an ideal on X. Then the local function $A^*(\mathbf{I}, \tau)$ of $A \subset X$ is defined as following:*

$$A^*(\mathbf{I}, \tau) = \{x \in X \mid U \cap A \notin \mathbf{I} \text{ for every } U \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$.

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One can see that, $(.)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the conditions to make $c^*(A) = A \cup A^*(\mathbf{I}, \tau)$ a Kuratowski closure operator.

Definition 2 ([5]). *Let (X, τ) be a topological space, and $\mathbf{I}$ be an ideal on X. Since $c^*(A) = A \cup A^*(\mathbf{I}, \tau)$ is a Kuratowski closure operator, generates a topology $\tau^*(\mathbf{I}, \tau)$ on X. If there is no chance of confusion this topological space is denoted as (X, τ^*).*

Now we recall some definitions in ideal topological spaces, which are crucial in our work.

Definition 3. *Let $(X, \tau, \mathbf{I})$ be an ideal topological space and A be a subset of X. We say that A is*

i. $*$*-dense [4] if* $c^*A = X$,

ii. $\mathbf{I}$*-dense [3] if* $A^*(\mathbf{I}, \tau) = X$.

It is easy to show that $\tau \subset \tau^*$. Also note that, if $\mathbf{I} = \{\emptyset\}$ then $\tau = \tau^*$, and if $\mathbf{I} = \mathcal{P}(X)$ then $A^*(\mathcal{P}(X), \tau) = \emptyset$ which implies $\tau^* = \mathcal{P}(X)$.

Carrying general topological notions into the ideal topological spaces is a very fruitful, and generalizing process. To this end we turned our attention to similarity between the topologies defined on the same set. This topic is introduced in [2]. According to Bartoszewicz and et al. (X, τ_1) and (X, τ_2) are similar if the families of sets which have nonempty interior with respect to τ_1 and τ_2 coincide. Similarity between topological spaces is denoted by $\tau_1 \sim \tau_2$. In [2], besides some other characterizations, it is shown that two topologies are similar if and only if the families of dense subsets coincide, or $\tau_1 \setminus \{\emptyset\}$ and $\tau_2 \setminus \{\emptyset\}$ are mutually coinitial [1]. That is for all $U \in \tau_1 \setminus \{\emptyset\}$ there exists $V \in \tau_2 \setminus \{\emptyset\}$ such that $V \subset U$ and for all $U \in \tau_2 \setminus \{\emptyset\}$ there exists $V \in \tau_1 \setminus \{\emptyset\}$ such that $V \subset U$.

In this work we first define similarity with respect to an ideal, and then give some characterizations. Throughout this work, (X, τ), c_iA, and c_i^*A will denote the topological space, closure in τ_i, and in τ_i^*, respectively where $i = 1, 2$.

2. Main Results

Definition 4. *Let X be a set, τ_1, τ_2 be two given topologies, and $\mathbf{I}$ be an ideal on X. We say that τ_1 and τ_2 are similar with respect to $\mathbf{I}$ or $\mathbf{I}$-similar and denote by $\tau_1 \sim_{\mathbf{I}} \tau_2$ if, for every nonempty $U \in \tau_1$, there exists a nonempty $V \in \tau_2$ such that $V \setminus U \in \mathbf{I}$, and for every $U \in \tau_2$, there exists a nonempty $V \in \tau_1$ such that $V \setminus U \in \mathbf{I}$.*

It is clear that if τ_1 and τ_2 are similar topologies, then they are similar with respect to any ideal $\mathbf{I}$. On the other hand the following example shows that ideal similarity does not imply similarity between topologies.

Example 1. *Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{b, c\}\}$, and $\mathbf{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. τ_1 and τ_2 are $\mathbf{I}$-similar, but not similar.*

Also, if $\mathbf{I} = \{\emptyset\}$, then similarity is equivalent to $\mathbf{I}$-similarity. What is more, if τ_1 and τ_2 are $\mathbf{I}$-similar topologies and $\mathbf{J}$ is an ideal with $\mathbf{I} \subset \mathbf{J}$, then τ_1 and τ_2 are also $\mathbf{J}$-similar.

Theorem 1. *Let X be a set, τ_1, τ_2 topologies on X, and $\mathbf{I}$ be an ideal on X. Then τ_1 and τ_2 are similar with respect to $\mathbf{I}$ if and only if $*$-dense subsets coincide.*

Proof. Let $\tau_1 \sim_{\mathbf{I}} \tau_2$, a subset $A \subset X$ with $c_1^* A = X$, and $c_2^* A \neq X$ be given. By definition, there exists an element x in X, so that $x \notin A$ and $x \notin A^*(\mathbf{I}, \tau_2)$. Hence there exists $U \in \tau_2(x)$ so that $U \cap A \in \mathbf{I}$. Together with this, $c_1^* A = X$ implies $x \in A^*(\mathbf{I}, \tau_1)$. By $\mathbf{I}$-similarity there exists a set $V \in \tau_1$ so that $V \setminus U \in \mathbf{I}$. Note also that $A \cap V \neq \emptyset$. If we consider $A \cap V$, we see that $A \cap V \in \mathbf{I}$ since $A \cap V \subset (U \cap A) \cup (V \cap (X \setminus U))$. However, that contradicts the fact that $c_1^* A = X$.

On the other hand, let $U \in \tau_1 \setminus \{\emptyset\}$, and suppose $V \setminus U \notin \mathbf{I}$ for every $V \in \tau_2 \setminus \{\emptyset\}$. So, $X \setminus U$ is $*$-dense with respect to τ_2. By hypothesis $X \setminus U$ is also $*$-dense with respect to τ_1. That is $(X \setminus U) \cup (X \setminus U)^*(\mathbf{I}, \tau_1) = X$, and this implies $U \subset (X \setminus U)^*(\mathbf{I}, \tau_1)$. As a result we have the contradiction: $U \cap (X \setminus U) \notin \mathbf{I}$.

Corollary 1. *Let X be a set, τ_1, τ_2 topologies on X, and $\mathbf{I}$ be an ideal on X. Then τ_1^* and τ_2^* are similar if and only if τ_1 and τ_2 are $\mathbf{I}$-similar.*

Proof. By previous theorem, τ_1 and τ_2 are $\mathbf{I}$-similar if and only if $*$-dense subsets coincide, and by [2], *Theorem* 2.2, this is true if and only if τ_1^* and τ_2^* are similar topologies.

We have another characterization for $\mathbf{I}$-similarity.

Theorem 2. *Let X be a set, τ_1, τ_2 topologies on X, and $\mathbf{I}$ be an ideal on X. Then τ_1 and τ_2 are similar with respect to $\mathbf{I}$ if and only if $\mathbf{I}$-dense subsets coincide.*

Proof. Let $\tau_1 \sim_{\mathbf{I}} \tau_2$, $A^*(\mathbf{I}, \tau_1) = X$, and $A^*(\mathbf{I}, \tau_2) \neq X$. Then, there exists an element x in X so that $x \notin A^*(\mathbf{I}, \tau_2)$. That is, for a set $U \in \tau_2(x)$, we have $U \cap A \in \mathbf{I}$. By $\mathbf{I}$-similarity there exists a subset $V \in \tau_1 \setminus \{\emptyset\}$ such that $V \setminus U \in \mathbf{I}$ which implies $V \cap A \in \mathbf{I}$. That contradicts the fact that $x \in A^*(\mathbf{I}, \tau_1) = X$.

Let this time the families of $\mathbf{I}$-dense subsets coincide, and U be a set belonging to $\tau_1 \setminus \{\emptyset\}$. Suppose $V \setminus U \notin \mathbf{I}$ for every $V \in \tau_2 \setminus \{\emptyset\}$. Then $V \setminus U \neq \emptyset$, and hence $X \setminus U$ is $\mathbf{I}$-dense in τ_2. By hypothesis, $X \setminus U$ is also $\mathbf{I}$-dense in τ_1, which brings the contradiction that $U \cap (X \setminus U) \notin \mathbf{I}$.

Lemma 1. *Let X be a set, τ_1, τ_2 topologies on X, and $\mathbf{I}$ be an ideal on X. If τ_1 and τ_2 are similar with respect to $\mathbf{I}$, and $\mathbf{I}$ is a τ_i-codense ideal for $i = 1, 2$ then $\mathbf{I}$ is also τ_j-codense ideal for $j = 3 - i$.*

Proof. Without loss of generality, assume $\mathbf{I}$ be a τ_1-codense ideal, and let $U \in \mathbf{I} \cap \tau_2 \setminus \{\emptyset\}$. By $\mathbf{I}$-similarity, there exists a set $V \in \tau_1 \setminus \{\emptyset\}$ so that $V \setminus U \in \mathbf{I}$. However these imply $V \in \mathbf{I}$, which is impossible since $\mathbf{I} \cap \tau_1 = \{\emptyset\}$.

Now, we examine the similarity between τ_1^* and τ_2^*. One can easily show that, if τ_1 and τ_2 are similar topologies, then τ_1^*, and τ_2^* are also similar. On the other hand the converse is not true, as the following example shows:

Example 2. *Let us reconsider the Example* 1*. The topologies* τ_1^**, and* τ_2^* *satisfy:*

$$\tau_1^* = \{X, \emptyset, \{b\}, \{a,b\}, \{b,c\}\} = \tau_2^*,$$

are the same. However, as we mentioned, τ_1*, and* τ_2 *are not similar.*

Note that, the ideal of the previous example is codense with respect to both topologies, but this is not enough to have similarity between τ_1, and τ_2. This motivates the following question:

Are there any conditions can be added to an ideal **I** for carrying similarity between τ_1^* and τ_2^* to the case of τ_1 and τ_2.

3. Conclusion

In this work the notion of similarity between topological spaces, is blended with ideals on topological spaces. It is proved that, we can deduce the similarity with respect to an ideal **I**, under the condition of coinciding **I**-dense subsets, and the similarity between τ_1^*, and τ_2^* topologies is equivalent the **I**-similarity between τ_1 and τ_2 topologies. However, the question asking if there are any conditions can be added to an ideal **I** for carrying similarity between τ_1^* and τ_2^* to the spaces τ_1 and τ_2 is still open for a possible future work.

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