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A Pegagogical Study of Geometric Relations between the Sums of Cubic Integer Numbers and the Sums of Square Integer Numbers with the Sums of Odd Numbers

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Abstract. The geometric meaning of the sums of cubic and square numbers can be explained by expressing them as sums of consecutive odd numbers. By illustrating not only the equations but also simple diagrams, we can understand how these sums can be obtained. Geometrically viewing the relationship between series, rather than merely learning how to calculate them, helps students understand these series systematically. If natural numbers are expressed as sums of only 1, the sums of square numbers and of cubic numbers can be related to the multiplication of a vertical vector with a horizontal one whose components are all 1's. This view is educational in showing that matrix multiplication is related to the calculation of series. The notion that concepts as different as series and matrices are fundamentally and closely related is an important perspective in mathematics education.

2020 Mathematics Subject Classifications: 97H20

Key Words and Phrases: square numbers; cubic numbers; series; matrix multiplication; pictorial proof; visual thinking

1. Introduction

The calculation of $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ is a problem of counting geometrical arrangements of $(1+2+\cdots+n)^2$ points and $(2n+1)(1+2+\cdots+n)$ points, respectively [3]. Generally, $\sum_{k=0}^{n} k^\ell$ can be represented by $\sum_{k=0}^{n} k$ through recursion formulae based on the Euler-Maclaurin sum formulae which are integral representations [5]. This fact means that $\sum_{k=0}^{n} k^\ell$ can be resolved into the factor $n(n+1)$. The case of $\ell = 2$ is the simplest;

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thus, $\sum_{k=1}^{n} k^2$ can be pictorially proved by a method of calculating area called sectional mensuration [4]. The tree diagrams for expressing $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ reveal the combinatorial meaning of the recursion formulae [5]. These approaches are not necessarily practical methods of calculation, although, from a pedagogical standpoint, students can learn that the algebraic formulae of the series $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ are related to combinations and integrals.

One of the aims here is to show simple methods for calculating the series $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ by using the features of each of the integer terms, k^3 and k^2; that is, both square and cubic numbers can be represented as sums of consecutive odd numbers. By geometrically viewing the relationship between series, rather than merely learning how to calculate them, students will be able to systematically understand these series. Another aim is to show that the series $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ can also be understood in terms of the multiplication of a vertical vector with a horizontal one whose components are all 1's. From the standpoint of mathematics education, it is important for students to understand that matrix multiplication is related to how series are calculated. The notion that concepts as different as series and matrices are fundamentally and closely related is an important perspective in mathematics education.

This article is organised as follows: In Sections 2 and 3, simple methods for calculating $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ are presented in terms of the representations of k^3 and k^2 by sums of consecutive odd numbers. These approaches are useful for developing the ability to understand the characteristics of square and cubic numbers rather than mechanically calculating the sums. In Section 4, a common feature of $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ is shown with the aid of matrix multiplication.

2. Sum of Cubic Numbers

2.1. Representation Based on Sums of Odd Numbers

To adequately explain the reason that $\sum_{k=1}^{n} k^3$ can be represented as $\left(\sum_{k=1}^{n} k\right)^2$, we directly calculate the sum of consecutive odd numbers instead of using the calculation method through the formula $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. This approach can reveal the source of $\sum_{k=1}^{n} k$ in the processes of rearrangements of terms. As shown in Table 1, a cubic number, k^3, can be represented as a sum of k consecutive odd numbers.

Table 1. Cubic numbers expressed as sums of odd numbers.

$1^3 = 1.$
$2^3 = 2^2 \cdot 2 = 4 \times 2 = 4 + 4 = (4-1) + (4+1) = 3 + 5.$
$3^3 = 3^2 \cdot 3 = 9 \times 3 = 9 + 9 + 9 = (9-2) + 9 + (9+2) = 7 + 9 + 11.$
$4^3 = 4^2 \cdot 4 = 16 \times 4 = 16 + 16 + 16 + 16 = (16-3) + (16-1) + (16+1) + (16+3) = 13 + 15 + 17 + 19.$
$\cdots$
$k^3 = [k^2 - (k-1)] + \cdots + [k^2 + (k-1)].$
$\cdots$
$n^3 = [n^2 - (n-1)] + \cdots + [n^2 + (n-1)].$

Thus,

$$\begin{aligned} & 1^3 + 2^3 + 3^3 + \cdots + k^3 + \cdots + n^3 \\ = & \underbrace{1}_{1 \text{ term}} + \underbrace{(3+5)}_{2 \text{ terms}} + \underbrace{(7+9+11)}_{3 \text{ terms}} + \underbrace{(13+15+17+19)}_{4 \text{ terms}} \\ & + \cdots + \underbrace{[k^2 - (k-1)] + \cdots + [k^2 + (k-1)]}_{k \text{ terms}} \\ & + \cdots + \underbrace{[n^2 - (n-1)] + \cdots + [n^2 + (n-1)]}_{n \text{ terms}} . (1) \end{aligned}$$

Equation (1) means that $\sum_{k=1}^{n} k^3$ is the sum of consecutive odd numbers from 1 to $[n^2 + (n-1)]$, in which the number of terms is $(1 + 2 + 3 + 4 + \cdots + n)$; that is, the initial term of the arithmetic progression is 1 and the common difference between successive terms is 2. By a simple calculation of $\frac{1}{2} \times (\text{initial term} + \text{final term}) \times (\text{number of terms})$, we obtain

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \frac{1}{2} \times \left[1 + (n^2 + n - 1)\right] \times (1 + 2 + 3 + 4 + \cdots + n) \\ &= \left[\frac{n(n+1)}{2}\right]^2 . \end{aligned} \tag{2}$$

Practically, this calculation is not necessary, as shown in the following way. The second power of a natural number, k, is a sum of consecutive odd numbers in which the number of terms is k, such as $4^2 = 1 + 3 + 5 + 7$. The sum $\sum_{k=1}^{n} k^3$ is also an arithmetic series of consecutive odd numbers, one in which the number of terms is $n(n+1)/2$. Thus, we can obtain Equation (2) without tedious calculations. The sum $1+2+3+4+\cdots+n$ indicates two meanings: First, the arithmetic mean of the initial and final terms of $1+3+5+\cdots+[n^2+(n-1)]$ in Equation (1); second, the number of terms of $1+3+5+\cdots+[k^2+(k-1)]$, with k from 1 to n, in Equation (1).

2.2. Visual Thinking

Students can cultivate a sense of mathematics not only by performing calculations but also by visualising how the calculations work. The formula $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ can be proved pictorially by expressing square numbers as sums of consecutive odd numbers, with slightly different diagrams from those of well-known pictorial proofs [1, 2, 9, 10]. Two different diagrams, as shown in Figures 1 and 2, can be drawn, depending on the different ways of counting the number of squares.

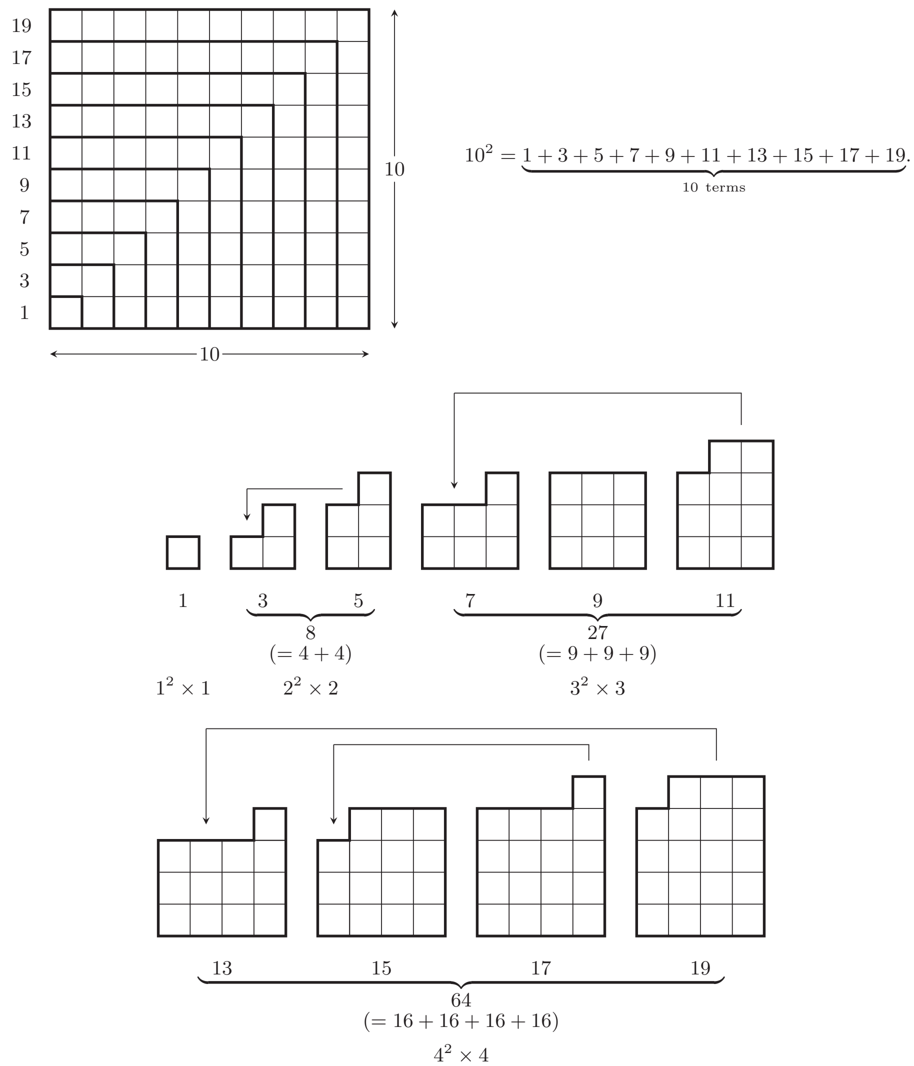

Figure 1: Visual thinking for the sum of n cubic numbers.

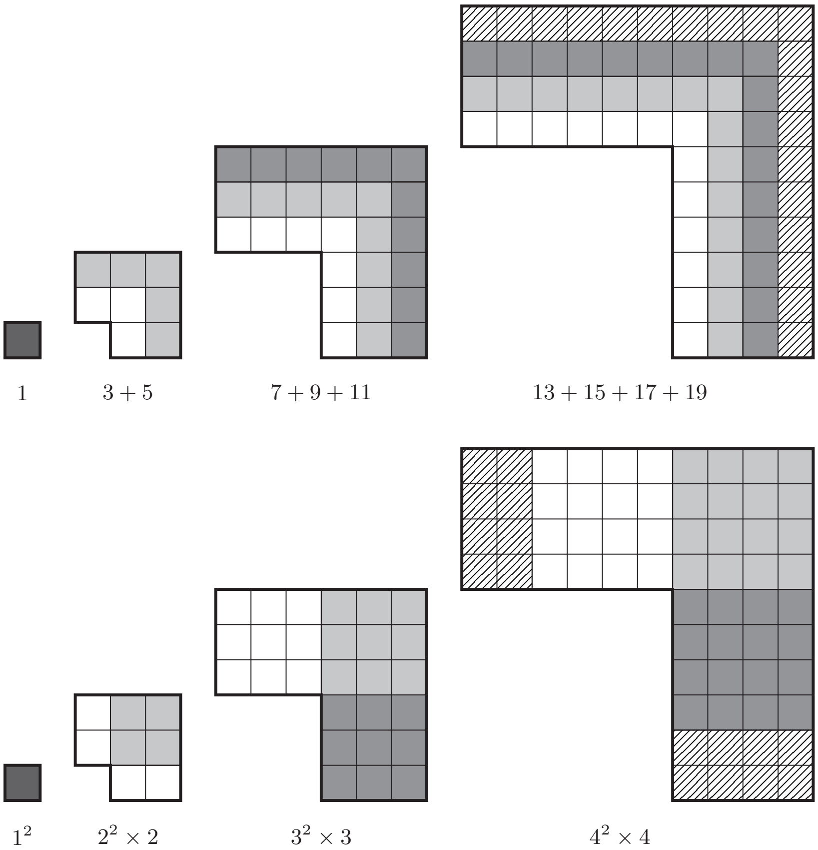

Figure 2: A sum of odd numbers can be expressed as a sum of cubic numbers. Diagrams above and below illustrate the different ways of counting the squares.

A visual clue to the fact that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ can also be presented through a similarity ratio, as shown in Figure 3. A visual proof that is based on the fact that (area ratio) = (similarity ratio)^2 for similar triangles is presented without calculating half the base and height. These diagrams illustrate that $1^2\times1+2^2\times2+3^2\times3 = (1+2+3)^2$. Right-angled isosceles triangles can also be used instead of equilateral triangles.

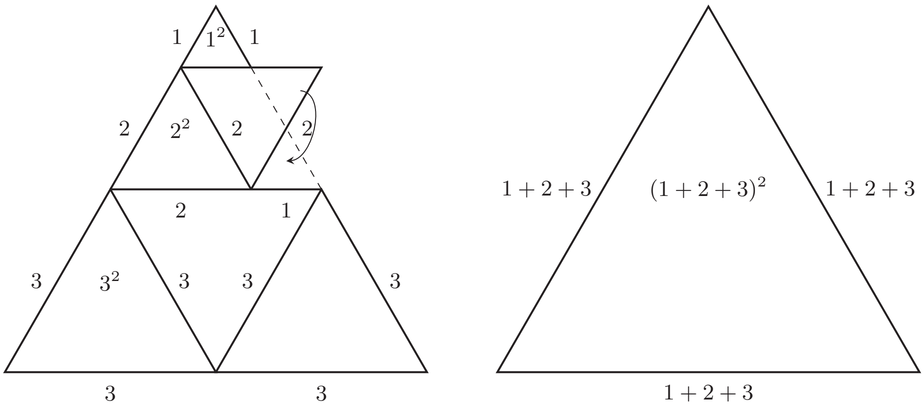

Figure 3: Visual clue to $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ based on the similarity ratio.

3. Sum of Square Numbers

3.1. Representation Based on Sums of Odd Numbers

As shown in Table 2, a square number, k^2, can be represented as the sum of the consecutive odd numbers $1, 3, 5, \ldots, 2k-1$, in which the number of terms is k. Thus,

$$\begin{aligned} & 1^2 + 2^2 + 3^2 + \cdots + k^2 + \cdots + n^2 \\ = \; & \underbrace{(1+1+1+1+\cdots+1)}_{n \text{ terms}} + \underbrace{(3+3+3+\cdots+3)}_{(n-1) \text{ terms}} + \underbrace{(5+5+\cdots+5)}_{(n-2) \text{ terms}} + \cdots \\ & + \underbrace{[(2k-1)+\cdots+(2k-1)]}_{[n-(k+1)] \text{ terns}} + \cdots + \underbrace{(2n-1)}_{1 \text{ term}}. \end{aligned} \tag{3}$$

By expressing Equation (3) with summation signs, the calculation can be simplified to obtain

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (2k-1)[n-(k-1)] \\ &= (2n+3)\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} (n+1). \end{aligned} \tag{4}$$

With slight rearrangements, Equation (4) can be rewritten as

$$3\sum_{k=1}^{n} k^2 = (2n+3)\sum_{k=1}^{n} k - n(n+1), \tag{5}$$

and thus we obtain

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{2n+3}{3}\sum_{k=1}^{n} k - \frac{2}{3}\sum_{k=1}^{n} k \\ &= \frac{2n+1}{3}\sum_{k=1}^{n} k \\ &= \frac{1}{6}n(n+1)(2n+1) \qquad (6)\end{aligned}$$

using $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Table 2. Square numbers expressed as sums of odd numbers.

$1^2 = 1$
$2^2 = 1 + 3$
$3^2 = 1 + 3 + 5$
$\cdots$
$k^2 = 1 + 3 + 5 + 7 + \cdots + (2k - 1)$
$\cdots$
$n^2 = 1 + 3 + 5 + 7 + \cdots + (2k - 1) + \cdots + (2n - 1)$

A redundant calculation without summation signs clearly shows the process of calculation, which is preferable from a pedagogical standpoint. Equation (3) can be written as

$$\begin{aligned}&1^2 + 2^2 + 3^2 + \cdots + k^2 + \cdots + n^2 \\ = \; &1 \times n + 3 \times (n-1) + 5 \times (n-2) + \cdots \\ &+ (2k-1) \times [n - (k-1)] + \cdots + (2n-1) \times [n - (n-1)]. \qquad (7)\end{aligned}$$

The right side of Equation (7) can be expressed as

$$\begin{aligned}&[1 + 3 + 5 + \cdots + (2k-1) + \cdots + (2n-1)]n \\ &-[3 \times 1 + 5 \times 2 + \cdots + (2k-1) \times (k-1) + \cdots + (2n-1) \times (n-1)]. \qquad (8)\end{aligned}$$

We can simply obtain

$$[1 + 3 + 5 + \cdots + (2k-1) + \cdots + (2n-1)]n = n^3 \qquad (9)$$

from $n^2 = 1 + 3 + 5 + 7 + \cdots + (2k-1) + \cdots + (2n-1)$, although the calculation is a little complicated:

$$\begin{aligned}&3 \times 1 + 5 \times 2 + \cdots + (2k-1) \times k + \cdots + (2n-1) \times (n-1) \\ = \; &(2+1) \times 1 + (3+2) \times 2 + \cdots + [k + (k-1)] \times (k-1) + \cdots + [n + (n-1)] \times (n-1) \\ = \; &2 \times 1 + 1^2 + 3 \times 2 + 2^2 + \cdots + k(k-1) + (k-1)^2 + \cdots + n(n-1) + (n-1)^2\end{aligned}$$

$$\begin{aligned}
&= (1+1)\times 1 + 1^2 + (2+1)\times 2 + 2^2 + \cdots + [(k-1)+1](k-1) + (k-1)^2 \\
&\quad + \cdots + [(n-1)+1](n-1) + (n-1)^2 \\
&= 2[1^2 + 2^2 + \cdots + (k-1)^2 + \cdots + (n-1)^2] + [1 + 2 + \cdots + (k-1) + \cdots + (n-1)] \\
&= 2[1^2 + 2^2 + \cdots + (k-1)^2 + \cdots + (n-1)^2 + n^2] - 2n^2 \\
&\quad +[1 + 2 + \cdots + (k-1) + \cdots + (n-1) + n] - n. \qquad (10)
\end{aligned}$$

Using Equations (9) and (10) instead of Equation (8), Equation (7) can be written as

$$\begin{aligned}
&1^2 + 2^2 + 3^2 + \cdots + k^2 + \cdots + n^2 \\
= \; & n^3 + 2n^2 + n - 2[1^2 + 2^2 + \cdots + (k-1)^2 + \cdots + (n-1)^2 + n^2] \\
& -[1 + 2 + \cdots + (k-1) + \cdots + (n-1) + n]. \qquad (11)
\end{aligned}$$

With slight rearrangements, we obtain

$$3\sum_{k=1}^{n} k^2 = n(n+1)^2 - \sum_{k=1}^{n} k. \qquad (12)$$

Equation (6) can also be derived from Equation (12) by observing that $n(n+1)^2 = 2(n+1)\sum_{k=1}^{n} k$.

3.2. Visual Thinking

It is important to illustrate how the calculations are performed in the equations and to visually perceive how they work. The relationship between the sum of square numbers and that of triangular numbers can pictorially be shown by expressing the square numbers as sums of consecutive odd numbers. Instructors should ask students to illustrate the process of deriving Equations (6) and (7) from Table 2, as shown in Figure 4.

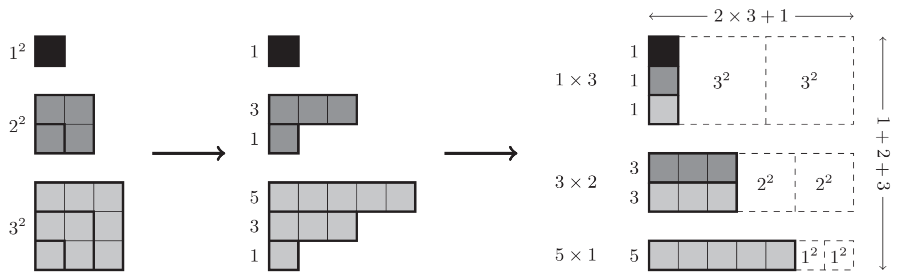

Figure 4: Pictorial proof that $3(1^2 + 2^2 + 3^2) = (2 \times 3 + 1)(1 + 2 + 3)$ based on sums of odd numbers.

The meaning of the sum of square numbers can also be illustrated with reference to Table 3:

$$\begin{aligned} 1^2+2^2+3^2+4^2 &= [1+(1+2)+(1+2+3)] \\ &\quad +[1+(1+2)+(1+2+3)+(1+2+3+4)]. \end{aligned} \tag{13}$$

The right-hand side is the sum of the sums of triangular numbers. This relationship can also be derived from other expressions of square numbers, as shown in Table 4.

$$1^2+2^2+3^2+4^2 = 2[0+1+(1+2)+(1+2+3)]+(1+2+3+4), \tag{14}$$

which coincides with Equation (13). The right-hand sides of Equations (13) and (14) can be pictorially represented, as shown in (a) and (b) in Figure 5, respectively. We can draw the usage of (b) as shown in (c), and thus

$$2[0+1+(1+2)] = \frac{2}{3}(1+2)(1+2+1). \tag{15}$$

Generalising, we can obtain

$$\begin{aligned} & 2[0+1+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+n)] \\ = & \frac{2}{3}[1+2+\cdots+(n-1)](1+n+1) \\ = & \frac{2}{3}\cdot\frac{(n-1)n}{2}\cdot(n+1), \end{aligned} \tag{16}$$

and Equation (14) can be extended to

$$\begin{aligned} 1^2+2^2+\cdots+n^2 &= \frac{2}{3}\cdot\frac{(n-1)n}{2}\cdot(n+1)+\frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned} \tag{17}$$

Table 3. Square numbers expressed as sums of two consecutive triangular numbers.

$1^2 = 1.$
$2^2 = 1+(1+2).$
$3^2 = 1+(1+2)+(2+3) = (1+2)+(1+2+3).$
$4^2 = 1+(1+2)+(2+3)+(3+4) = (1+2+3)+(1+2+3+4).$

This table is a revised version of Table 2.

Table 4. Square numbers expressed as triangular numbers.

$1^2 = 2\times 0+1.$
$2^2 = 2\times 1+2.$
$3^2 = 2\times(1+2)+3.$
$4^2 = 2\times(1+2+3)+4.$

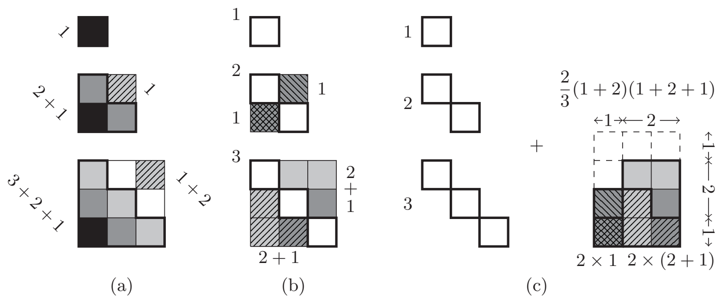

Figure 5: Square numbers expressed as triangular numbers. For simplicity, the cases of 1^2, 2^2, and 3^2 are presented. Diagrams (a) and (b) represent the sums in Tables 3 and 4, respectively. These diagrams illustrate the different ways of counting the squares. Diagram (c) shows how to find $1^2 + 2^2 + 3^2$ using (b).

4. Algebraic Meanings of $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^2$ in Terms of Multiplication of a Vertical Vector with a Horizontal One

The sum $\sum_{k=1}^{n} k^3$ can be expressed as the sum $\sum_{k=1}^{n} k$ in terms of matrix multiplication. For simplicity, we illustrate $(1+2+3)^2$ in the following. The right-hand side of

$$\begin{aligned} & (1+2+3)^2 \\ = \; & [1+(1+1)+(1+1+1)] \times [1+(1+1)+(1+1+1)] \end{aligned} \tag{18}$$

is the total of 36 1×1, which can be regarded as the sum of all the components of

$$^t(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \end{array})(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \end{array}).$$

The product of these two vectors can be expressed in a multiplication table, as shown in Figure 6. The sum of 36 1×1 can be written as

$$(1+2+3)^2 = 1^2 \times 1 + 2^2 \times 2 + 3^2 \times 3 \tag{19}$$

in the left figure, while it can be written as the sum of six consecutive odd numbers

$$(1+2+3)^2 = 1+3+5+7+9+11 \tag{20}$$

in the right figure. Thus, the sum of the cubic numbers $1^3 + 2^3 + 3^3$ is equal to the sum of $(1+2+3)$ consecutive odd numbers and the square of the sum of the natural numbers, $(1+2+3)^2$.

Another relationship between the sum of square numbers and that of natural numbers can also be interpreted with reference to the revised version of Figure 6. The sum of the

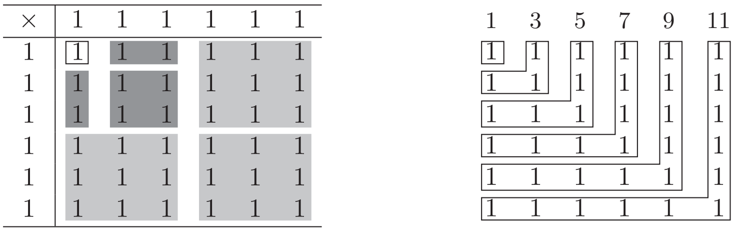

Figure 6: $(1+2+3)^2$ expressed as 36 1×1. The left and right figures illustrate the different ways of counting 1.

components of the right-hand side of

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1^2 & 1\cdot 2 & 1\cdot 3 \\ 2\cdot 1 & 2^2 & 2\cdot 3 \\ 3\cdot 1 & 3\cdot 2 & 3^2 \end{pmatrix} \tag{21}$$

can be tabulated as shown in Figure 7, and thus

$$1^2+2^2+3^2 = (1+2+3)^2 - 2(1\cdot 2 + 2\cdot 3 + 3\cdot 1). \tag{22}$$

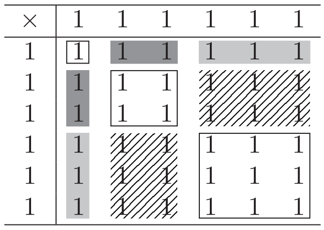

Figure 7: $((1+2+3)^2$ expressed as 36 1×1. The sum of the numbers shaded in grey is 1×2, the sum of the numbers in the shaded area is 2×3, and the sum of the numbers shaded in light grey is 3×1.

The sum $\sum_{k=1}^{n} k^2$ can also be represented by the sum $\sum_{k=1}^{n} k$ in terms of matrix multiplication. For simplicity, we illustrate $1^2+2^2+3^2$ in the following. To adopt a heuristic method [6], we replace the square-shaped table in Figure 6 with a rectangular one, as shown in Figure 8, which can be regarded as the sum of all the components of ${}^t\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. The product of these two matrices can be expressed in a multiplication table, as shown in Figure 8. The sum of 42 1×1 can be written as

$$\begin{aligned} 42\times(1\times 1) &= 3\times[1+(1+3)+(1+3+5)] \\ &= 3\times(1^2+2^2+3^2) \end{aligned} \tag{23}$$

in the left figure, while it can be written as a sum of the sums of natural numbers, $1+2+3$,

$$42 \times (1 \times 1) = (2 \times 3 + 1) \times (1 + 2 + 3), \tag{24}$$

in the right figure. Thus, the sum of the square numbers, $1^2 + 2^2 + 3^2$, is equal to $\frac{1}{3}(2 \times 3 + 1) \times (1 + 2 + 3)$. Adding two rows and four columns to the multiplication tables represents the case of $1^2 + 2^2 + 3^2 + 4^2$. In this case, the vertical direction in the left figure can be trisected, in contrast to $1^2 + 2^2 + 3^2$.

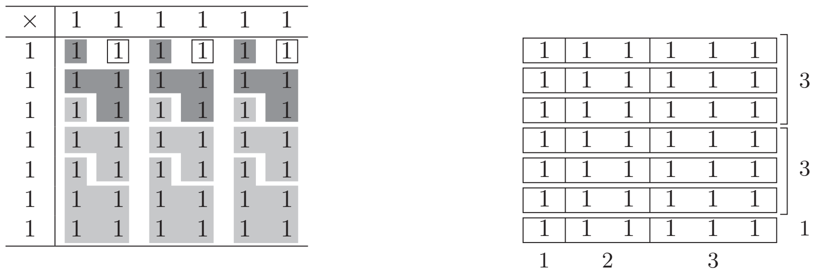

Figure 8: $((2 \times 3 + 1) \times (1 + 2 + 3)$ expressed as 42 1×1. The left and right figures show the different ways of counting 1.

The meaning of $\left(\sum_{k=1}^{n} k\right)^2$ can also be interpreted in terms of combinatorics because the left figure can be regarded as a league matchup chart [7, 8].

5. Concluding Remarks

The fundamental idea discussed in this paper is to express square and cubic numbers as sums of odd numbers. The important point is to reassemble these sums for computation and devise ways of counting. To realise this idea, it is essential not only to manipulate the equations, but also to pictorially illustrate the calculation process. Although different from the theme of this paper, the idea of the sectional quadrature method, for example, is similar to this one. In Cartesian terms, the idea of decomposing into odd numbers and then summing corresponds to the procedures of analysis and synthesis.

If natural numbers are expressed as sums of only 1, the sums of square numbers and of cubic numbers can be related to the multiplication of a vertical vector with a horizontal one whose components are all 1's. The sums of square and cubic numbers are the sums of the components of the matrices that represent the products of these multiplications. If natural numbers are expressed as sums of only 1 in the analytical approach, the sums of square numbers and of cubic numbers can be related to the multiplication of a vertical vector with a horizontal one whose components are all 1's. The sums of square and of cubic numbers are the sums of the components of the matrices that represent the products of the above multiplications. If we regard these products as multiplication tables, we can compute the sum of the components by devising a way to determine the sum of the odd numbers. It is educationally effective in broadening the view of mathematics to grasp that matrix operations in linear algebra are related to the computation of series.

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