EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 44-61
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On The Global Distance Roman Domination of Some Graphs

Giovannie Entero[1,*], Stephanie Espinola[2]

[1] *Department of Mathematics and Statistics, College of Arts and Sciences, University of Southeastern Philippines, Bo. Obrero, Davao City, Davao del Sur, Philippines*
[2] *Department of Mathematics and Statistics, Faculty, College of Arts and Sciences, University of Southeastern Philippines, Bo. Obrero, Davao City, Davao del Sur, Philippines*

Abstract. Let $k \in \mathbb{Z}^+$. A $k-$ *distance Roman dominating function* ($kDRDF$) on $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that for every vertex v with $f(v) = 0$, there is a vertex u with $f(u) = 2$ with $d(u, v) \leq k$. The function f is a *global* $k-$ *distance Roman dominating function* ($GkDRDF$) on G if and only if f is a $k-$ *distance Roman dominating function* ($kDRDF$) on G and on its complement $\overline{G}$. The *weight* of the *global* $k-$ *distance Roman dominating function* ($GkDRDF$) f is the value $w(f) = \sum_{x\in V} f(x)$. The minimum *weight* of the *global* $k-$ *distance Roman dominating function* ($GkDRDF$) on the graph G is called the *global* $k-$ *distance Roman domination number* of G and is denoted as $\gamma^k_{gR}(G)$. A $\gamma^k_{gR}(G) - function$ is the *global* $k-$ *distance Roman dominating function* on G with weight $\gamma^k_{gR}(G)$. Note that, the *global* $1-$ *distance Roman domination number* $\gamma^1_{gR}(G)$ is the usual *global Roman domination number* $\gamma_{gR}(G)$, that is, $\gamma^1_{gR}(G) = \gamma_{gR}(G)$. The authors initiated this study. In this paper, the authors obtained and established the following results: preliminary results on global distance Roman domination; the global distance Roman domination on $\overline{K_n}$, K_n, P_n, and C_n; and, some bounds and characterizations of global distance Roman domination over any graphs.

2020 Mathematics Subject Classifications: 05A18, 05C07, 05C12, 05C22, 05C38, 05C40, 05C62, 05C69, 05C70, 05C76, 05C78, 97K30

Key Words and Phrases: Classical Domination, Roman Domination, Distance Domination, Global Domination, Distance Roman Domination, Global Roman Domination

1. Introduction

Mathematics plays a vital role in various fields. One of the important areas in mathematics is graph theory which is mainly used in structural models. Graph theory is an interesting branch of mathematics when it comes to research. In mathematics and computer science, graph theory is the study of graphs which are mathematical structures used

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4478

Email addresses: giovannieentero.ge@gmail.com (G. Entero), saomega@usep.edu.ph (S. Espinola)

to model pairwise relations between objects. There are several areas in graph theory in which extensive research activities grow fast−one of which is *Domination*. Domination is a classical and an interesting topic in the theory of graphs as well as one of the most active areas of research in this discipline. The increasing interest in this area is partly explained by the diversity of its applications to both theoretical and real-world problems. Domination comes with one of its most famous variants called *Roman Domination*−the defense strategy (a.k.a. protection strategy) employed by Emperor Constantine the Great to defend the Roman Empire when it was under a certain attack.

It was traced back that, in the 4^{th} century A. D., when the Roman Empire was under attack during the period of Emperor Constantine the Great, he had the requirement that any army or a legion could be sent from its home to defend a neighboring location only if there was a second army which would stay and protect the home [12]. Thus, there are two types of armies − traveling and stationary. The first type of legion was particularly skilled agile combatants who could be promptly deployed to an adjacent province for defending against any potential attack. The latter would behave as a local force permanently located in the given province. In addition, no legion could ever depart a province in order to defend another one if such action leaves the base province unprotected [7]. Translating this strategy into the language of graph theory, each vertex with no army must have a neighboring vertex with a traveling army. Stationary armies then dominate their own vertices and a vertex with two armies is dominated by its stationary army and its open neighborhood is dominated by the traveling army [12].

Cockayne et al. [4] introduced a variant of domination called *Roman domination* suggested by the recent article in Scientific American by Ian Stewart, entitled *"Defend the Roman Empire"* as mentioned in the previous paragraph. According to the mentioned authors, the *Roman dominating function* (RDF) on the graph $G = (V, E)$ is the function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. Roman domination also comes with various varieties, one of its variants that caught the authors' attention is *Distance Roman Domination* initiated by Aram et al. [2].

Aram et al. [2] defined that the $k-$ *distance Roman dominating function (kDRDF)* on the graph $G = (V, E)$ is the function $f : V \to \{0, 1, 2\}$ satisfying the conditions that for every vertex v for which $f(v) = 0$, there is a vertex u for which $f(u) = 2$ and $d(u, v) \leq k$, where $d(u, v)$ is the distance from u to v. Additionally, the *weight* of kDRDF f is the value $w(f) = \sum_{u \in V} f(u)$ and the minimum *weight* of the kDRDF on G will be the $k-$ *distance Roman domination number* and is denoted by $\gamma_R^k(G)$, where $k \in \mathbb{Z}^+$.

All graphs considered in this paper are all finite, simple, and undirected. Let $G = (V, E)$ be a finite, simple, and undirected graph. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$. Further, let the order of the graph G be p, that is, $|V| = |V(G)| = p$ and the size be q, that is, $|E| = |E(G)| = q$.

The authors defined the *Global Distance Roman Domination* on graphs as follows: The function f is a *global* $k-$ *distance Roman dominating function (GkDRDF)* on G if and only if f is a $k-$ *distance Roman dominating function (kDRDF)* on G and on its complement $\overline{G}$. The weight of the *global* $k-$ *distance Roman dominating function*

(GkDRDF) f is the value $w(f) = \sum_{x\in V} f(x)$. The minimum weight of the *global* k $-$ *distance Roman dominating function (GkDRDF)* on the graph G is called the *global* k $-$ *distance Roman domination number* of G and is denoted by $\gamma_{gR}^k(G)$. A $\gamma_{gR}^k(G)-function$ is the *GkDRDF* on G with weight $\gamma_{gR}^k(G)$. The *GkDRDF* $f : V \to \{0,1,2\}$ can be represented by the ordered partition (V_0^f, V_1^f, V_2^f) of V induced by f, where f is the given function and $V_i^f = \{v \in V | f(v) = i$ and $i = 0,1,2\}$. Observe that, there is a one-to-one correspondence between the function $f : V \to \{0,1,2\}$ and the ordered partition (V_0^f, V_1^f, V_2^f) of V induced by f. Hence, we may write $f = (V_0^f, V_1^f, V_2^f)$. In this representation, its weight can be computed as $w(f) = |V_1^f| + 2|V_2^f|$. Note that, the *global* 1 $-$ *distance Roman domination number* $\gamma_{gR}^1(G)$ is the usual *global Roman domination number* $\gamma_{gR}(G)$, that is, $\gamma_{gR}^1(G) = \gamma_{gR}(G)$. It is worth noting that, since we are dealing with simple graphs, the distance of each vertex, say $u \in V$, to itself is zero, that is, $d(u,u) = 0$ while the distance of two different vertices say $u, v \in V$, coming from different components of graph G is assigned to be ∞, that is, $d(u,v) = \infty$, where u and v belong to different components of G.

2. Terminologies and Notations

To better understand the scope of this study, we will be needing the following definitions and some related literature.

The *distance* between vertices u and v in graph G, denoted by $d(u,v)$, is the length of the shortest path from vertex u to vertex v in graph G. The *eccentricity* of vertex u on graph G is the maximum distance from vertex u to any other vertex, say vertex v, in graph G and is denoted by $ecc(u) = \max\{d(u,v) : v \in V(G)\}$. The *radius* of graph G is the minimum eccentricity taken over all vertices of graph G and is denoted as $rad(G) = \min\{ecc(u) : u \in V(G)\}$ and the *diameter* of graph G is the maximum eccentricity taken over all vertices of graph G and is denoted as $diam(G) = \max\{ecc(u) : u \in V(G)\}$. [8]

The *degree* of a vertex v of the graph G is the number of edges incident with v in G and is denoted by $deg(v)$. The *maximum degree* of the graph G, denoted by $\Delta(G)$, is the *maximum degree* for every vertex in G, that is, $\Delta(G) = \max\{deg(v) : v \in V(G)\}$. The *minimum degree* of the graph G, denoted by $\delta(G)$, is the *minimum degree* for every vertex in G, that is, $\delta(G) = \min\{deg(v) : v \in V(G)\}$. [3]

The *neighbourhood* (or *open neighbourhood*) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v, that is, $N(v) = \{x \in V(G) : vx \in E(G)\}$. The *closed neighbourhood* of a vertex v, denoted by $N[v]$, is simply the set $\{v\} \cup N(v)$. Given a set S of vertices, we define the *neighbourhood* of S, denoted by $N(S)$, to be the union of the *neighbourhoods* of the vertices in S. Similarly, the *closed neighbourhood* of S, denoted by $N[S]$, is defined to be $S \cup N(S)$. [11]

Let $k \in \mathbb{Z}^+$. The k $-$ *degree* of a vertex v in graph G, denoted as $deg_{k,G}(v)$, is defined to be $deg_{k,G}(v) = |\{u \in V(G) | u \neq v$ and $d(u,v) \leq k\}|$. Let $k \in \mathbb{Z}^+$. The *maximum* k $-$ *degree* of the graph G, denoted by $\Delta_k(G)$, is the *maximum* k $-$ *degree* taken over all vertices of graph G, that is, $\Delta_k(G) = \max\{deg_{k,G}(v) : v \in V(G)\}$. The *minimum*

$k-$ *degree* of the graph G, denoted by $\delta_k(G)$, is the *minimum* $k-$ *degree* taken over all vertices of graph G, that is, $\delta_k(G) = \min\{deg_{k,G}(v) : v \in V(G)\}$. [2]

The $k-$ *neighbourhood* (or *open* $k-$ *neighbourhood*) of a vertex v in graph G, denoted by $N_{k,G}(v)$, is the set of vertices (different from v) adjacent to v in G within distance k, that is, $N_{k,G}(v) = \{u \in V(G) : u \neq v \text{ and } d(u,v) \leq k\}$, where $k \in \mathbb{Z}^+$. The *closed* $k-$ *neighbourhood* of a vertex v, denoted by $N_{k,G}[v]$, is simply the set $\{v\} \cup N_{k,G}(v)$. Given a set S of vertices, we define the $k-$ *neighbourhood* of S, denoted by $N_{k,G}(S)$, to be the union of the $k-$ *neighbourhoods* of the vertices in S within distance k with respect to graph G, where $k \in \mathbb{Z}^+$. Similarly, the *closed* $k-$ *neighbourhood* of S, denoted by $N_{k,G}[S]$, is defined to be $S \cup N_{k,G}(S)$. [2]

Let $v \in S \subseteq V$. Then, u is called a private neighbour of v with respect to S, denoted by u is an $S-pn$ of v, if $u \in N(v) - N(S - \{v\})$. An $S-pn$ of v is external if it is in $V - S$. The $pn(v,S) = N(v) - N(S-\{v\})$ of all $S-pn$'s of v is called the private neighbourhood set of v with respect to S. Equivalently, $pn(v,S) = \{u \in V | N(u) \cap S = \{v\}\}$. [4]

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. This means that both G and its complement $\overline{G}$ have the same vertices, but G has precisely the edges that $\overline{G}$ lacks. [8]

The join of two graphs G and H, denoted by $G+H$, is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. [8]

Let G and H be graphs of orders p_1 and p_2, respectively. Then the graph obtained by taking one copy of G of order p_1 and p_1 copies of H and then connecting the *ith* vertex of G to every vertex of the *ith* copy of H (*ith* means first, second, third and so on) is called corona, denoted by $G \circ H$. The order and the size of the corona $G \circ H$ are $p_1 + p_1p_2$ and $q_1 + p_1q_2 + p_1p_2$, respectively, where q_1 and q_2 are the sizes of graphs G and H, respectively. [5]

A cartesian product, denoted by $G \times H$, of two graphs G and H, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H)$ satisfying the following conditions: $(u_1,v_1)(u_2,v_2) \in E(G \times H)$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. [3]

The lexicographic product of graphs G and H is the graph $G[H]$ with a vertex set $V(G[H]) = \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}$ and with edge set $E(G[H]) = \{(u,v)(w,x) : uw \in E(G) \text{ or } u = w \text{ and } vx \in E(H)\}$. [6]

3. Basic Concepts

Definition 3.1. [5] A set of vertices $S \subseteq V(G)$ is a *dominating set* for graph $G = (V(G), E(G))$ if every vertex not in S is adjacent to at least one vertex in S. The *domination number* of graph G is the cardinality of any *minimum (smallest) dominating set* in G and is denoted by $\gamma(G)$.

Definition 3.2. [14] Let $k \in \mathbb{Z}^+$. A set $D \subseteq V(G)$ is a *distance* $k-$ *dominating set* of G if each $x \in V(G) \backslash D$ is within distance k from some vertex of D. The minimum cardinality taken over all distance $k-$ dominating sets of graph G is called the *distance $k-$ domination number* of G and is denoted by $\gamma_k(G)$.

Definition 3.3. [4] A *Roman dominating function (RDF)* on a graph $G=(V,E)$ is a function $f:V\to\{0,1,2\}$ satisfying the condition that every vertex v for which $f(v)=0$ is adjacent to at least one vertex u for which $f(u)=2$. The *weight* of the Roman dominating function (RDF) f is the value $w(f)=\sum_{x\in V}f(x)$. The minimum *weight* of the Roman dominating function of the graph G is called the *Roman domination number* of G and is denoted as $\gamma_R(G)$.

Definition 3.4. [9] A dominating set D of $G=(V,E)$ is a *global dominating set* if D is also a dominating set of the complement $\overline{G}$ of G. The minimum cardinality taken over all global dominating sets of G is called the *global domination number* of G and is denoted by $\gamma_g(G)$.

Definition 3.5. [1] A *$k-$ distance Roman dominating function (kDRDF)* on $G=(V,E)$ is a function $f:V\to\{0,1,2\}$ such that for every vertex v with $f(v)=0$, there is a vertex u with $f(u)=2$ with distance of at most k from each other. The *weight* of the distance Roman dominating function f is the value $w(f)=\sum_{x\in V}f(x)$. The minimum *weight* of a $k-$ distance Roman dominating function on the graph G is called the *$k-$ distance Roman domination number* of G and is denoted as $\gamma_R^k(G)$.

Definition 3.6. [12] A *global Roman dominating function (GRDF)* on the graph $G=(V,E)$ is a function $f:V\to\{0,1,2\}$ such that f is an RDF for both G and its complement $\overline{G}$. The *weight* of the global Roman dominating function f is the value $w(f)=\sum_{x\in V}f(x)$. The minimum *weight* of the global Roman dominating function on the graph G is called the *global Roman domination number* of G and is denoted as $\gamma_{gR}(G)$.

Definition 3.7. Let $k\in\mathbb{Z}^+$. A *$k-$ distance Roman dominating function (kDRDF)* on $G=(V,E)$ is a function $f:V\to\{0,1,2\}$ such that for every vertex v with $f(v)=0$, there is a vertex u with $f(u)=2$ such that $d(u,v)\leq k$. The function f is a *global $k-$ distance Roman dominating function (GkDRDF)* on G if and only if f is a *$k-$ distance Roman dominating function (kDRDF)* on G and on its complement $\overline{G}$.

Definition 3.8. Let $k\in\mathbb{Z}^+$. The *weight* of the global $k-$ distance Roman dominating function $(GkDRDF)$ f is the value $w(f)=\sum_{x\in V}f(x)$. The minimum *weight* of the global $k-$ distance Roman dominating function $(GkDRDF)$ on the graph G is called the *global $k-$ distance Roman domination number* of G and is denoted as $\gamma_{gR}^k(G)$. A $\gamma_{gR}^k(G)-$ function is a $GkDRDF$ with weight γ_{gR}^k(G).

Remark 3.9. [12] For any $n-$ vertex graph G, $2\leq\gamma_{gR}(G)\leq n$.

Theorem 3.10. [10] Let G be a simple graph of order n. Then $\gamma(G)=n$ if and only if $G\equiv\overline{K_n}$.

Proposition 3.11. [15] For any graph G of order n, $\gamma(G)=\gamma_k(G)$ if and only if every vertex in G has degree 0. (Such a graph is denoted as $G=\overline{K_n}$, the complement of the complete graph of order n)

Remark 3.12. [2] Let $k \geq 1$ be an integer. For $n-$ vertex graphs, always $\gamma_R^k(G) \leq n$, with equality when $G \cong \overline{K_n}$.

Proposition 3.13. [13]

(i) For a graph G with p vertices, $\gamma_g(G) = p$ if and only if $G = K_p$ or $\overline{K_p}$.

(ii) $\gamma_g(K_{m,n}) = 2$ for all $m, n \geq 1$.

(iii) $\gamma_g(C_4) = 2$, $\gamma_g(C_5) = 3$ and $\gamma_g(C_n) = \{\frac{n}{3}\}$, for $n \geq 6$.

(vi) $\gamma_g(P_n) = 2$ for $n = 2, 3$ and $\gamma_g(P_n) = \{\frac{n}{3}\}$ for $n \geq 4$.

Proposition 3.14. [12] Let G be any graph. Then $\gamma_g(G) = \gamma_{gR}(G)$ if and only if $G = K_n$.

4. Results and Discussions

All throughout this paper, we will be using either of the notations f_k or f to denote a function. However, for general cases where the distance $k \in \mathbb{Z}^+$ is explicit, we will be using the notation f to denote a function. Nevertheless, for the cases where the distance $k \in \mathbb{Z}^+$ must be specified, we will be using the notations f_k to refer to a function with respect to such distances. Additionally, for each $k \in \mathbb{Z}^+$, we may have at least one f_k (resp., f) on just a single graph, that is, for a particular distance, we can define several global distance Roman dominating functions over a given graph.

4.1. Preliminary Results on Global Distance Roman Domination

Remark 4.1. Let $k \in \mathbb{Z}^+$. Given the *global $k-$ distance Roman dominating function (GkDRDF)* $f_k : V \to \{0, 1, 2\}$ on graph $G = (V, E)$, for all $k \in \mathbb{Z}^+$, we have the following facts:

(i) function f_k can be represented by the ordered partition $(V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$ of V induced by f_k, where $V_i^{f_k} = \{v \in V | f_k(v) = i \text{ and } i = 0, 1, 2\}$;

(ii) from (i), there is a one-to-one correspondence between $f_k : V \to \{0, 1, 2\}$ and the ordered partition $(V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$ of V induced by f_k; and,

(iii) from (ii), f_k can be written as $f_k = (V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$.

Proof. Let $k \in \mathbb{Z}^+$. Suppose we have the global $k-$ distance Roman dominating function (GkDRDF) $f_k : V \to \{0, 1, 2\}$ on graph $G = (V, E)$.

- There is nothing to prove in part (i).
- For part (ii), we note that f_k can be expressed as follows

$$f_k = \{(u_j, f(u_j)) : u_j \in V \text{ and } f(u_j) \in \{0, 1, 2\}\}.$$

Now, let us partition (ordered partition) the function f_k in terms of images of each $u_j \in V$ treating f_k as the set of ordered pairs. So, we can have the cells f_k^0, f_k^1, and f_k^2 in which $f_k = f_k^0 \cup f_k^1 \cup f_k^2$, where

$$f_k^0 = \{(u_j, 0) : u_j \in V\},\ f_k^1 = \{(u_j, 1) : u_j \in V\},\ \text{and}\ f_k^2 = \{(u_j, 2) : u_j \in V\}.$$

and

$$f_k^i \cap f_k^l = \emptyset \text{ for all } i \neq l \text{ and } i, l \in \{0, 1, 2\}$$

which means that $f_k^0 \cap f_k^1 \cap f_k^2 = \emptyset$. Now, for the ordered partition $(V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$ of V induced by f_k, where $V_i^{f_k} = \{v \in V | f_k(v) = i \text{ and } i = 0, 1, 2\}$, we have

$$V_0^{f_k} = \{u_j \in V : f_k(u_j) = 0\},\ V_1^{f_k} = \{u_j \in V : f_k(u_j) = 1\},\ \text{and}$$
$$V_2^{f_k} = \{u_j \in V : f_k(u_j) = 2\}.$$

which also means that $V_i^{f_k} \cap V_l^{f_k} = \emptyset$ for all $i \neq l$ and $i, l \in \{0, 1, 2\}$ and $V_0^{f_k} \cap V_1^{f_k} \cap V_2^{f_k} = \emptyset$. Hence, by matching class f_k^0 to class $V_0^{f_k}$, class f_k^1 to class $V_1^{f_k}$, and class f_k^2 to class $V_2^{f_k}$, we can see that there is a one-to-one correspondence between $f_k : V \to \{0, 1, 2\}$ and the ordered partition $(V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$ of V induced by f_k. This concludes the proof of part (ii).

- For part (iii), the proof is straightforward.

Therefore, we can say now that Remark 4.1 is true and valid. □

Remark 4.2. For any graph $G = (V, E)$ of order n, there exists a function $f : V \to \{0, 1, 2\}$ satisfying the conditions that for every vertex v for which $f(v) = 0$ there exists at least one vertex u for which $f(u) = 2$ with $d(u, v) \leq k$ such that f is a $k-$ distance Roman dominating function ($kDRDF$) on G and on its complement $\overline{G}$, where $k \in \mathbb{Z}^+$. Then such a function f on G with minimum weight also exists. We call the function $f : V \to \{0, 1, 2\}$ as the global $k-$ distance Roman dominating function ($GkDRDF$) on G.

Proof. Suppose that G is a graph of order n. Assume that the vertex set of graph G is $V(G) = \{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\}$. Let $f : V(G) \to \{0, 1, 2\}$ be a function on G and let $k \in \mathbb{Z}^+$. For all $k \in \mathbb{Z}^+$, since $f = (V_0^f(G), V_1^f(G), V_2^f(G))$, we let $V_0^f(G) = V_2^f(G) = \emptyset$ and $V_1^f(G) = \{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\} = V(G)$, which means that, $f(u_i) = 1$, where $u_i \in V(G)$ and $i = 1, 2, 3, \ldots, n-1, n$, for all $k \in \mathbb{Z}^+$. Hence, by Definition 3.7, the function f is a global $k-$ distance Roman dominating function ($GkDRDF$) on G. Furthermore, since the existence of the $GkDRDF$ on any graph is now guaranteed and since the order of graph G is finite, it follows that the existence of the $GkDRDF$ with minimum weight on any graph is also guaranteed. This proves Remark 4.2. □

Remark 4.3. For all $k \in \mathbb{Z}^+$, if f_k can be written as $f_k = (V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$, then its weight $w(f_k)$ can be computed as $w(f_k) = |V_1^{f_k}| + 2|V_2^{f_k}|$.

Proof. Let $k \in \mathbb{Z}^+$. We suppose that f_k can be written as $f_k = (V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$. Now, for all $k \in \mathbb{Z}^+$, since $V = V_0^{f_k} \cup V_1^{f_k} \cup V_2^{f_k}$ with $|V| = |V_0^{f_k}| + |V_1^{f_k}| + |V_2^{f_k}|$ and since, from Definition 3.8, $w(f_k) = \sum_{u_j \in V} f_k(u_j)$, we have

$$\begin{aligned} w(f_k) &= \sum_{u_j \in V} f_k(u_j) \\ &= \sum_{u_j \in V_0^{f_k}} f_k(u_j) + \sum_{u_j \in V_1^{f_k}} f_k(u_j) + \sum_{u_j \in V_2^{f_k}} f_k(u_j) \\ &= 0 + (1)(|V_1^{f_k}|) + (2)(|V_2^{f_k}|) \\ &= |V_1^{f_k}| + 2|V_2^{f_k}|. \end{aligned}$$

that is, $w(f_k) = |V_1^{f_k}| + 2|V_2^{f_k}|$, for all $k \in \mathbb{Z}^+$. Therefore, for all $k \in \mathbb{Z}^+$, if f_k can be written as $f_k = (V_0^{f_k}, V_1^{f_k}, V_2^{f_k})$, then its weight $w(f_k)$ can be computed as $w(f_k) = |V_1^{f_k}| + 2|V_2^{f_k}|$. □

Remark 4.4. Let $k \in \mathbb{Z}^+$. For any graph G, $\gamma_R^k(G) \leq \gamma_{gR}^k(G)$.

Proof. Let f be a $\gamma_{gR}^k(G)$ − function of G and let $k \in \mathbb{Z}^+$. Then f is a k − distance Roman dominating function of G. Thus, $\gamma_R^k(G) \leq \gamma_{gR}^k(G)$ for all $k \in \mathbb{Z}^+$. □

Remark 4.5. For any graph G, $\gamma_{gR}^k(G) = \gamma_{gR}^k(\overline{G})$, for all $k \in \mathbb{Z}^+$.

Proof. By saying global, it constitutes the given graph, say graph G, together with its complement $\overline{G}$. Thus, for all $k \in \mathbb{Z}^+$, $\gamma_{gR}^k(G)$ accounts $\gamma_R^k(G)$ and $\gamma_R^k(\overline{G})$ simultaneously and also, since $\overline{\overline{G}} = G$, $\gamma_{gR}^k(\overline{G})$ accounts $\gamma_R^k(\overline{G})$ and $\gamma_R^k(\overline{\overline{G}}) = \gamma_R^k(G)$ simultaneously. Therefore, for any graph G, $\gamma_{gR}^k(G) = \gamma_{gR}^k(\overline{G})$, for all $k \in \mathbb{Z}^+$. □

5. The Global Distance Roman Domination on Special Graphs

5.1. The Global Distance Roman Domination on Empty Graph $\overline{K_n}$ and on Complete Graph K_n

Theorem 5.1. Let $G \cong \overline{K_n}$, where $\overline{K_n}$ is the null graph (empty graph) of order n. Then, for all $k \in \mathbb{Z}^+$, $\gamma_{gR}^k(G) = n$.

Proof. Assume that $G \cong \overline{K_n}$ is of order n. Suppose that we have a function f_k mapping the vertex set $V(G)$ of graph G to the set $\{0, 1, 2\}$, that is, $f_k : V(G) \longrightarrow \{0, 1, 2\}$, for all $k \in \mathbb{Z}^+$. Let f_k be defined by $f_k(u_i) = 1$, for all $u_i \in V(G)$. This mapping will give a weight of $w(f_k) = n$, for all $k \in \mathbb{Z}^+$ and this is minimum. This can be easily verified. □

Corollary 5.2. Let $G \cong K_n$, where K_n is the complete graph of order n. Then, for any $k \in \mathbb{Z}^+$, $\gamma_{gR}^k(G) = n$.

Proof. The proof follows from Theorem 5.1. □

5.2. The Global Distance Roman Domination on Path Graph P_n

Proposition 5.3. Let $G \cong P_n$, where P_n is the path graph of order n. For all $k \in \mathbb{Z}^+$,

$$\gamma_{gR}^k(G) = \begin{cases} \text{for } \ n \leq 4: \\ \quad \begin{cases} n, & \text{if } \ n = 1, 2, 3 \ \text{ and } \ \forall \ k \in \mathbb{Z}^+ \\ \lfloor \frac{4}{2k} \rfloor + 2, & \text{if } \ n = 4 \ \text{ and } \ \forall \ k \in \mathbb{Z}^+ \end{cases} \\ \text{for } \ n > 4: \\ \quad \begin{cases} 2\lfloor \frac{n}{\Delta_k+1} \rfloor, & \text{if } \ n \equiv 0 \pmod{(\Delta_k+1)} \text{ and } 1 \leq k < rad(G) \\ 2\lfloor \frac{n}{\Delta_k+1} \rfloor + 1, & \text{if } \ n \equiv 1 \pmod{(\Delta_k+1)} \text{ and } 1 \leq k < rad(G) \\ 2\lfloor \frac{n}{\Delta_k+1} \rfloor + 2, & \text{if } \ n \not\equiv 0, 1 \pmod{(\Delta_k+1)} \text{ and } 1 \leq k < rad(G) \\ 2, & \text{if } \ k \geq rad(G). \end{cases} \end{cases}$$

5.3. The Global Distance Roman Domination on Cycle Graph C_n

Proposition 5.4. Let $G \cong C_n$, where C_n is the cycle graph of order $n \geq 3$. For all $k \in \mathbb{Z}^+$,

$$\gamma_{gR}^k(G) = \begin{cases} \text{for } \ n \leq 5: \\ \quad \begin{cases} n, & \text{if } \ n = 3, 4 \ \text{ and } \ \forall \ k \in \mathbb{Z}^+ \\ n, & \text{if } \ n = 5 \ \text{ and } \ k = 1 \\ 2, & \text{otherwise} \end{cases} \\ \text{for } \ n > 5: \\ \quad \begin{cases} \frac{2n}{2k+1}, & \text{if } \ n \equiv 0 \pmod{(2k+1)} \ \text{ and } \ 1 \leq k < diam(G) \\ 2\lfloor \frac{n}{2k+1} \rfloor + 1, & \text{if } \ n \equiv 1 \pmod{(2k+1)} \ \text{ and } \ 1 \leq k < diam(G) \\ 2\lfloor \frac{n}{2k+1} \rfloor + 2, & \text{if } \ n \not\equiv 0, 1 \pmod{(2k+1)} \ \text{ and } \ 1 \leq k < diam(G) \\ 2, & \text{if } \ k \geq diam(G). \end{cases} \end{cases}$$

6. Some Bounds of the Global Distance Roman Domination

Lemma 6.1. Let $k \in \mathbb{Z}^+$. Given any graph G, as $k \longrightarrow \infty$, $\gamma_{gR}^k(G)$ is decreasing, that is, $\gamma_{gR}^1(G) \geq \gamma_{gR}^2(G) \geq \gamma_{gR}^3(G) \geq \cdots \geq \gamma_{gR}^{k-1}(G) \geq \gamma_{gR}^k(G) \geq \gamma_{gR}^{k+1}(G) \geq \ldots$.

Proof. Let $k \in \mathbb{Z}^+$. Note that, if $k = 1$, then it is clear that $\gamma_{gR}^1(G) = \gamma_{gR}(G)$. Now, given $k > k-1$, we have $\Delta_k(G) \geq \Delta_{k-1}(G)$ and it follows that, $\gamma_{gR}^k(G) \leq \gamma_{gR}^{k-1}(G)$. Hence, in general, given $\cdots > k+1 > k > k-1 > \cdots > 3 > 2 > 1$, we have $\cdots > \Delta_{k+1}(G) \geq \Delta_k(G) \geq \Delta_{k-1}(G) > \cdots > \Delta_3(G) \geq \Delta_2(G) \geq \Delta_1(G) \geq$ and so, we can say that,

$$\gamma_{gR}^1(G) \geq \gamma_{gR}^2(G) \geq \gamma_{gR}^3(G) \geq \cdots \geq \gamma_{gR}^{k-1}(G) \geq \gamma_{gR}^k(G) \geq \gamma_{gR}^{k+1}(G) \geq \ldots .$$

This completes the proof. □

Theorem 6.2. Let $k \in \mathbb{Z}^+$. For any graph G of order $n \geq 1$, $\gamma_{gR}^k(G) \leq \gamma_{gR}(G)$.

Proof. Suppose that $k \in \mathbb{Z}^+$ and let G be any graph of order $n \geq 1$. Since $\gamma_{gR}^1(G) = \gamma_{gR}(G)$ and by Lemma 6.1, we have

$$\cdots \leq \gamma_{gR}^k(G) \leq \gamma_{gR}^{k-1}(G) \leq \cdots \leq \gamma_{gR}^3(G) \leq \gamma_{gR}^2(G) \leq \gamma_{gR}^1(G) = \gamma_{gR}(G),$$

that is,

$$\cdots \leq \gamma_{gR}^k(G) \leq \gamma_{gR}^{k-1}(G) \leq \cdots \leq \gamma_{gR}^3(G) \leq \gamma_{gR}^2(G) \leq \gamma_{gR}(G),$$

and hence, by transitivity, we have $\gamma_{gR}^k(G) \leq \gamma_{gR}(G)$. So, for all $k \in \mathbb{Z}^+$, $\gamma_{gR}^k(G) \leq \gamma_{gR}(G)$. This completes the proof. □

Remark 6.3. The upper bound in Theorem 6.2 is sharp and the strict inequality is attained.

Proof. The proof is clear. □

Theorem 6.4. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order $|V(G)| \geq 2$, $2 \leq \gamma_{gR}^k(G) \leq 2|V(G)|$.

Proof. Let $G = (V(G), E(G))$ be any graph of order $|V(G)| \geq 2$ and let $k \in \mathbb{Z}^+$. We let f_k to be an arbitrary global $k-$ distance Roman dominating function ($GkDRDF$) on graph G, where $k \in \mathbb{Z}^+$. For all $k \in \mathbb{Z}^+$, by Remark 4.3, $w(f_k) = |V_1^{f_k}(G)| + 2|V_2^{f_k}(G)|$. Since we are talking about number of elements, $|V(G)|, |V_0^{f_k}(G)|, |V_1^{f_k}(G)|, |V_2^{f_k}(G)| \geq 0$ and so as the weight $w(f_k)$ of f_k. Now, since $|V(G)| \geq 2$, it follows that,

$$\gamma_{gR}^k(G) \geq 2 \qquad (1)$$

and since $\gamma_{gR}^k(G)$ is the minimum weight taken over all $GkDRDF$ on graph G, for all f_k on G and for all $k \in \mathbb{Z}^+$, we have $\gamma_{gR}^k(G) \leq w(f_k) \leq 2|V(G)|$ and thus, by transitivity, we have

$$\gamma_{gR}^k(G) \leq 2|V(G)| \qquad (2)$$

Hence, from (1) and (2), for all $k \in \mathbb{Z}^+$, we have

$$2 \leq \gamma_{gR}^k(G) \leq 2|V(G)|.$$

This completes the proof. □

Theorem 6.5. Let $k \in \mathbb{Z}^+$. For any graph G of order n, $\gamma_{gR}^k(G) \leq n$, with the equality when $G \cong \overline{K_n}, K_n$.

Proof. Let $k \in \mathbb{Z}^+$ and let G be any graph of order n. For all $k \in \mathbb{Z}^+$, by Theorem 6.2, we have

$$\gamma_{gR}^k(G) \leq \gamma_{gR}(G) \qquad (1)$$

and by Remark 3.9, we have

$$2 \leq \gamma_{gR}(G) \leq n \qquad (2)$$

Thus, for all $k \in \mathbb{Z}^+$, from (1) and (2), we obtain $\gamma_{gR}^k(G) \leq \gamma_{gR}(G) \leq n$, and by transitivity, we get $\gamma_{gR}^k(G) \leq n$. Moreover, the equality $\gamma_{gR}^k(G) = n$, when $G \cong \overline{K_n}, K_n$, is an immediate consequence of Theorem 5.1 and Corollary 5.2, and hence, the upper bound n is sharp. This completes the proof. □

7. Some Characterizations of the Global Distance Roman Domination

Theorem 7.1. Let $\mathcal{G}$ be the class of connected graphs whose complements are also connected. For any graph $G \in \mathcal{G}$ of order $n \geq 4$, $\gamma_{gR}^k(G) = 2$ if and only if $k \geq \max\{diam(G), diam(\overline{G})\}$, where $k \in \mathbb{Z}^+$.

Proof. Suppose that $\mathcal{G}$ denote the class of connected graphs whose complements are also connected. Let $G \in \mathcal{G}$ be any graph of order $n \geq 4$ and let its vertex set be $V(G) = \{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\}$. Since $G \in \mathcal{G}$, with $n = |V(G)| \geq 4$, the distances and eccentricities in graph G are finite. Thus, $diam(G), diam(\overline{G}) < \infty$. Now, let us consider the following:

($\Longrightarrow$) Assume that $\gamma_{gR}^k(G) = 2$. We let $f_k : V(G) \longrightarrow \{0, 1, 2\}$ be a $\gamma_{gR}^k(G)-$ function, where $k \in \mathbb{Z}^+$. This implies that, $w(f_k) = \sum_{u_i \in V(G)} f_k(u_i) = \gamma_{gR}^k(G) = 2$, where $i = 1, 2, 3, \ldots, n-1, n$. This implies further that, for $k \in \mathbb{Z}^+$, we may define the $\gamma_{gR}^k(G)-$ function f_k as $f_k = (V_0^{f_k}(G), V_1^{f_k}(G), V_2^{f_k}(G))$, where there exists a vertex $u_j \in V(G) = V(\overline{G})$ such that

$$V_0^{f_k}(G) = \{u_{i \neq j} \in V(G) : i \in \{1, 2, 3, \ldots, n\}\} = V(G) \backslash \{u_j\},$$
$$V_1^{f_k}(G) = \emptyset, \text{ and } V_2^{f_k}(G) = \{u_j\},$$

for all $k \in \mathbb{Z}^+$. Thus, for all $u_{i \neq j} \in V(G) = V(\overline{G})$, where $i \in \{1, 2, 3, \ldots, n\}$, and for all $k \in \mathbb{Z}^+$, we have $f_k(u_{i \neq j}) = 0$ and $f_k(u_j) = 2$ and this implies further that, for a distance $k \in \mathbb{Z}^+$, we have

$$d(u_{i \neq j}, u_j) \leq k \qquad (1)$$

which means that the distances of vertices $u_{i \neq j}$ from vertex u_j is at most k. Rewriting (1), we get

$$d(u_j, u_{i \neq j}) \leq k \qquad (2)$$

Since we are talking about distances here, we have $d(u_j, u_{i\neq j}), k > 0$, and so, from (2), with respect to G and $\overline{G}$, we have

$$\max\{d(u_j, u_{i\neq j}) : u_i \in V(G) \text{ where } i \in \{1, 2, 3, \dots, n\}\} \leq \max\{k\}$$

which implies that

$$\max\{d(u_j, u_{i\neq j}) : u_i \in V(G) \text{ where } i \in \{1, 2, 3, \dots, n\}\} \leq k \qquad (3)$$

From (3),

$$ecc(u_j) \leq k. \qquad (4)$$

Since u_j is the only vertex in the set partition $V_2^{f_k}(G)$ (note that, $V_2^{f_k}(G)$=$V_2^{f_k}(\overline{G})$) at distance $k \in \mathbb{Z}^+$, for all $i \neq j$,

$$ecc(u_{i\neq j}) \leq ecc(u_j) \qquad (5)$$

Hence, from (4) and (5), we have

$$ecc(u_{i\neq j}) \leq k \qquad (6)$$

Thus, for all $u_i \in V(G) = V(\overline{G})$, including vertex u_j, from (4) and (6), $(u_i) \leq k$, that is, with respect to graph G and $\overline{G}$,

$$ecc(u_1), ecc(u_2), ecc(u_3), \dots, ecc(u_n) \leq k \qquad (7)$$

Taking the maximum of both sides of (7), we get $\max\{ecc(u_1), ecc(u_2), ecc(u_3), \dots, ecc(u_n)\} \leq \max\{k\}$ which implies that

$$\max\{ecc(u_1), ecc(u_2), ecc(u_3), \dots, ecc(u_n)\} \leq k \qquad (8)$$

Now, from (8), we know that

$$\max\{ecc(u_1), ecc(u_2), ecc(u_3), \dots, ecc(u_n)\} = \max\{diam(G), diam(\overline{G})\} \qquad (9)$$

and hence, from (8) and (9), we have $\max\{diam(G), diam(\overline{G})\} \leq k$ which can be rewritten as $k \geq \max\{diam(G), diam(\overline{G})\}$. Hence, if $\gamma_{gR}^k(G) = 2$, then $k \geq \max\{diam(G), diam(\overline{G})\}$, where $k \in \mathbb{Z}^+$. This proves the forward part of the theorem.

$(\Longleftarrow)$ Assume that $k \geq \max\{diam(G), diam(\overline{G})\}$, where $k \in \mathbb{Z}^+$. Hence, if $k = \max\{diam(G), diam(\overline{G})\} \in \mathbb{Z}^+$, then there exists at least one vertex, say $u_j \in V(G) = V(\overline{G})$, such that

$$N_{k,G}(u_j) = \bigcup_{\substack{u_i \in V(G) \\ i \in \{1,2,\dots,n\}}} N_{k,G}(u_{i\neq j}) = V(G)\backslash\{u_j\}$$

and

$$N_{k,\overline{G}}(u_j) = \bigcup_{\substack{u_i \in V(\overline{G}) \\ i \in \{1,2,\ldots,n\}}} N_{k,\overline{G}}(u_{i\neq j}) = V(\overline{G})\backslash\{u_j\}.$$

Thus, for this case, we will define a function f_k over G and over $\overline{G}$, respectively, as

$$f_k = (V_0^{f_k}(G), V_1^{f_k}(G), V_2^{f_k}(G)) \text{ and } f_k = (V_0^{f_k}(\overline{G}), V_1^{f_k}(\overline{G}), V_2^{f_k}(\overline{G}))$$

where

$$V_0^{f_k}(G) = N_{k,G}(u_j)\backslash\{u_j\} = V(G)\backslash\{u_j\},$$
$$V_1^{f_k}(G) = \emptyset, \text{ and } V_2^{f_k}(G) = \{u_j\},$$

and

$$V_0^{f_k}(\overline{G}) = N_{k,\overline{G}}(u_j)\backslash\{u_j\} = V(\overline{G})\backslash\{u_j\},$$
$$V_1^{f_k}(\overline{G}) = \emptyset, \text{ and } V_2^{f_k}(G) = \{u_j\},$$

which means that, with respect to G and $\overline{G}$, for $i \neq j$, $f_k(u_{i\neq j}) = 0$ and $f_k(u_j) = 2$. Since all vertices $u_{i\neq j} \in V(G) = V(\overline{G})$, for $i \in \{1, 2, \ldots, n\}$, is open neighbours of $u_j \in V(G) = V(\overline{G})$, and, with respect to G and $\overline{G}$, since $f_k(u_j) = 2$, $f_k(u_{i\neq j}) = 0$ is permissible for f_k to be called as $k -$ distance Roman dominating function for G and for $\overline{G}$. So, by Definition 3.7, the function f_k is a global $k -$ distance Roman dominating function on graph G with $k = \max\{diam(G), diam(\overline{G})\} \in \mathbb{Z}^+$. Now, using Definition 3.8 to compute the weight of f_k, we have

$$\begin{aligned} w(f_k) &= \sum_{\substack{u_i \in V(G) \\ i \in \{1,2,\ldots,n\}}} f_k(u_i) \\ &= f_k(u_1) + f_k(u_2) + \cdots + f_k(u_{j-1}) + f_k(u_j) + f_k(u_{j+1}) + \cdots + f_k(u_n) \\ &= 0 + 0 + \cdots + 0 + 2 + 0 + \cdots + 0 \\ &= 2. \end{aligned}$$

Hence, if $k = \max\{diam(G), diam(\overline{G})\}$, then $\gamma_{gR}^k(G) = 2$ and this follows from Theorem 6.4. Moreover, by just following the same arguments, we may generalize this fact as $k \geq \max\{diam(G), diam(\overline{G})\}$, where $k \in \mathbb{Z}^+$, and still get $\gamma_{gR}^k(G) = 2$, for all $k \geq \max\{diam(G), diam(\overline{G})\}$. Thus, if $k \geq \max\{diam(G), diam(\overline{G})\}$, then $\gamma_{gR}^k(G) = 2$. This proves the backward part of the theorem.

Therefore, given the class $\mathcal{G}$ of connected graphs whose complement are also connected, for any graph $G \in \mathcal{G}$ of order $n \geq 4$, $\gamma_{gR}^k(G) = 2$ if and only if $k \geq \max\{diam(G), diam(\overline{G})\}$, where $k \in \mathbb{Z}^+$. □

Theorem 7.2. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma(G)$ if and only if $G \cong \overline{K_n}$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. Suppose that $\gamma_{gR}^k(G) = \gamma(G)$ and let $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ be a $\gamma_{gR}^k(G)$ − function of G, where $V_i^f(G) = \{v \in V(G) : f(v) = i$ for $i = 0, 1, 2\}$ is the partition of the vertex set $V(G)$ of G induced by the function f and $V(G) = V_0^f(G) \cup V_1^f(G) \cup V_2^f(G)$. Note that, $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$. Clearly, $V_1^f(G) \cup V_2^f(G)$ is a dominating set of G. Thus, since $\gamma(G)$ is the minimum cardinality taken over all dominating sets of G,

$$\gamma(G) \leq |V_1^f(G) \cup V_2^f(G)| = |V_1^f(G)| + |V_2^f(G)|,$$

and so, we have

$$\gamma(G) \leq |V_1^f(G)| + |V_2^f(G)|. \tag{1}$$

Since $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ is a $\gamma_{gR}^k(G)$ − function of G,

$$\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)|. \tag{2}$$

Thus, since we assumed that $\gamma_{gR}^k(G) = \gamma(G)$, from (1) and (2), we have $|V_1^f(G)| + 2|V_2^f(G)| = \gamma_{gR}^k(G) = \gamma(G) \leq |V_1^f(G)| + |V_2^f(G)|$, and hence, we have

$$|V_1^f(G)| + 2|V_2^f(G)| \leq |V_1^f(G)| + |V_2^f(G)|. \tag{3}$$

Since $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$, by addition and subtraction properties of equality, we have $|V_2^f(G)| \leq 0$, that is, $|V_2^f(G)| \leq 0$ and since $|V_2^f(G)| \geq 0$, we may conclude that $|V_2^f(G)| = 0$. Thus, $V_0^f(G) = \emptyset$, and so, we are forced to have $V_1^f(G) = V(G)$ which implies that $|V_1^f(G)| = |V(G)| = n$. Since f is a $\gamma_{gR}^k(G)$ − function of G, $\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)| = n + 0 = n$ which implies that $\gamma(G) = n$ for we assumed that $\gamma_{gR}^k(G) = \gamma(G)$. Thus, in reference to Theorem 3.10, this shows that $G \cong \overline{K_n}$. Conversely, assume that $G \cong \overline{K_n}$. By Theorem 5.1, $\gamma_{gR}^k(G) = \gamma_{gR}^k(\overline{K_n}) = n$ and since $G \cong \overline{K_n}$, it follows that, $\gamma(G) = \gamma(\overline{K_n}) = n$. Thus, since $\gamma_{gR}^k(G) = n$ and $\gamma(G) = n$, $\gamma_{gR}^k(G) = \gamma(G)$. This completes the proof. □

Theorem 7.3. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma_k(G)$ if and only if $G \cong \overline{K_n}$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. Assume that $\gamma_{gR}^k(G) = \gamma_k(G)$ and let $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ be a $\gamma_{gR}^k(G)$ − function of G, where $V_i^f(G) = \{v \in V(G) : f(v) = i$ for $i = 0, 1, 2\}$ is the partition of the vertex set $V(G)$ of G induced by the function f and $V(G) = V_0^f(G) \cup V_1^f(G) \cup V_2^f(G)$. Note that, $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$. Clearly, $V_1^f(G) \cup V_2^f(G)$ is a k − distance dominating set of G, where $k \in \mathbb{Z}^+$. Hence, since $\gamma_k(G)$ is the minimum cardinality taken over all k − distance dominating sets of G, for each $k \in \mathbb{Z}^+$,

$$\gamma_k(G) \leq |V_1^f(G) \cup V_2^f(G)| = |V_1^f(G)| + |V_2^f(G)|,$$

that is, we have
$$\gamma_k(G) \leq |V_1^f(G)| + |V_2^f(G)|. \quad (1)$$

Since $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ is a $\gamma_{gR}^k(G)$ − function of G,
$$\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)|. \quad (2)$$

Thus, since we assumed that $\gamma_{gR}^k(G) = \gamma_k(G)$, from (1) and (2), we have $|V_1^f(G)| + 2|V_2^f(G)| = \gamma_{gR}^k(G) = \gamma_k(G) \leq |V_1^f(G)| + |V_2^f(G)|$, and hence, we have
$$|V_1^f(G)| + 2|V_2^f(G)| \leq |V_1^f(G)| + |V_2^f(G)|. \quad (3)$$

Since $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$, by addition and subtraction properties of equality, we have $|V_2^f(G)| \leq 0$, that is, $|V_2^f(G)| \leq 0$ and since $|V_2^f(G)| \geq 0$, we may conclude that $|V_2^f(G)| = 0$. Hence, $V_0^f(G) = \emptyset$, and so, we are forced to have $V_1^f(G) = V(G)$ which implies that $|V_1^f(G)| = |V(G)| = n$. Since f is a $\gamma_{gR}^k(G)$ − function of G, $\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)| = n + 0 = n$ which implies that $\gamma_k(G) = n$ for we assumed that $\gamma_{gR}^k(G) = \gamma_k(G)$. Hence, in reference to Proposition 3.11, this shows that $G \cong \overline{K_n}$. Conversely, suppose that $G \cong \overline{K_n}$. By Theorem 5.1, $\gamma_{gR}^k(G) = \gamma_{gR}^k(\overline{K_n}) = n$ and since $G \cong \overline{K_n}$, it follows that, $\gamma_k(G) = \gamma_k(\overline{K_n}) = n$. Thus, since $\gamma_{gR}^k(G) = n$ and $\gamma_k(G) = n$, $\gamma_{gR}^k(G) = \gamma_k(G)$. This completes the proof. □

Theorem 7.4. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma_R(G)$ if and only if $G \cong \overline{K_n}$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. Suppose that $\gamma_{gR}^k(G) = \gamma_R(G)$. By Remark 3.12, we have $\gamma_R^k(G) \leq n$ and $\gamma_R^k(G) = n$ if and only if $G \cong \overline{K_n}$ and this is true for all $k \in \mathbb{Z}^+$. So, $\gamma_R^1(G) = \gamma_R(G) \leq n$ and $\gamma_R^1(G) = \gamma_R(G) = n$ if and only if $G \cong \overline{K_n}$. Thus, since $\gamma_{gR}^k(G) = \gamma_R(G)$ and $\gamma_R(G) = n$ if and only if $G \cong \overline{K_n}$, it follows that $\gamma_{gR}^k(G) = n$ if and only if $G \cong \overline{K_n}$. Hence, by Remark 3.12, $\gamma_{gR}^k(G) = \gamma_R(G) = n$ if and only if $G \cong \overline{K_n}$ and thus, the converse will then follows. This completes the proof. □

Theorem 7.5. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma_g(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. Suppose that $\gamma_{gR}^k(G) = \gamma_g(G)$ and let $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ be a $\gamma_{gR}^k(G)$ − function of G, where $V_i^f(G) = \{v \in V(G) : f(v) = i \text{ for } i = 0, 1, 2\}$ is the partition of the vertex set $V(G)$ of G induced by the function f and $V(G) = V_0^f(G) \cup V_1^f(G) \cup V_2^f(G)$. Note that, $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$. Clearly, $V_1^f(G) \cup V_2^f(G)$ is a global dominating set of G. Thus, since $\gamma_g(G)$ is the minimum cardinality taken over all global dominating sets of G,
$$\gamma_g(G) \leq |V_1^f(G) \cup V_2^f(G)| = |V_1^f(G)| + |V_2^f(G)|,$$

and so, we have

$$\gamma_g(G) \leq |V_1^f(G)| + |V_2^f(G)|. \qquad (1)$$

Since $f = (V_0^f(G), V_1^f(G), V_2^f(G))$ is a $\gamma_{gR}^k(G) -$ function of G,

$$\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)|. \qquad (2)$$

Thus, since we assumed that $\gamma_{gR}^k(G) = \gamma_g(G)$, from (1) and (2), we have $|V_1^f(G)| + 2|V_2^f(G)| = \gamma_{gR}^k(G) = \gamma_g(G) \leq |V_1^f(G)| + |V_2^f(G)|$, and hence, we have

$$|V_1^f(G)| + 2|V_2^f(G)| \leq |V_1^f(G)| + |V_2^f(G)|. \qquad (3)$$

Since $|V_0^f(G)|, |V_1^f(G)|, |V_2^f(G)| \geq 0$, by addition and subtraction properties of equality, we have $|V_2^f(G)| \leq 0$, that is, $|V_2^f(G)| \leq 0$ and since $|V_2^f(G)| \geq 0$, we may conclude that $|V_2^f(G)| = 0$. Hence, $V_0^f(G) = \emptyset$, and so, we are forced to have $V_1^f(G) = V(G)$ which implies that $|V_1^f(G)| = |V(G)| = n$. Since f is a $\gamma_{gR}^k(G) -$ function of G, $\gamma_{gR}^k(G) = |V_1^f(G)| + 2|V_2^f(G)| = n + 0 = n$ which implies that $\gamma_g(G) = n$ for we assumed that $\gamma_{gR}^k(G) = \gamma_g(G)$. Hence, in reference to Proposition 3.13, $G \cong K_n$. Conversely, assume that $G \cong \overline{K_n}$ or $G \cong K_n$. By Theorem 5.1 and Corollary 5.2, $\gamma_{gR}^k(G) = \gamma_{gR}^k(\overline{K_n}) = n$ and $\gamma_{gR}^k(G) = \gamma_{gR}^k(K_n) = n$, respectively. Also, by Proposition 3.13, $\gamma_g(G) = n$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$. Thus, the desired result follows. □

Theorem 7.6. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma_R^k(G)$ if and only if $G \cong \overline{K_n}$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. Suppose that $\gamma_{gR}^k(G) = \gamma_R^k(G)$. By Remark 3.12, we have $\gamma_R^k(G) \leq n$ and $\gamma_R^k(G) = n$ if and only if $G \cong \overline{K_n}$. Thus, since $\gamma_{gR}^k(G) = \gamma_R^k(G)$ and $\gamma_R^k(G) = n$ if and only if $G \cong \overline{K_n}$, it follows that $\gamma_{gR}^k(G) = n$ if and only if $G \cong \overline{K_n}$. Hence, $\gamma_{gR}^k(G) = \gamma_R^k(G) = n$ if and only if $G \cong \overline{K_n}$ and thus, the converse will then follows. This completes the proof. □

Theorem 7.7. Let $k \in \mathbb{Z}^+$. For any graph $G = (V(G), E(G))$ of order n, $\gamma_{gR}^k(G) = \gamma_{gR}(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$.

Proof. Let $k \in \mathbb{Z}^+$ and let $G = (V(G), E(G))$ be any graph of order n. By Theorem 7.5, $\gamma_{gR}^k(G) = \gamma_g(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$ and by Proposition 3.14, $\gamma_g(G) = \gamma_{gR}(G)$ if and only if $G \cong K_n$. Moreover, since $K_n = \overline{\overline{K_n}}$, by Proposition 3.14, we can say that $\gamma_g(G) = \gamma_{gR}(G)$ if and only if $G \cong \overline{K_n}$. Thus, from Proposition 3.14, $\gamma_g(G) = \gamma_{gR}(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$. Hence, since $\gamma_{gR}^k(G) = \gamma_g(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$ and $\gamma_g(G) = \gamma_{gR}(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$, it follows that $\gamma_{gR}^k(G) = \gamma_{gR}(G)$ if and only if $G \cong \overline{K_n}$ or $G \cong K_n$. This completes the proof. □

Acknowledgements

The authors would like to thank the Department of Science and Technology − Science Education Institute (DOST−SEI) for funding this publication.

References

[1] Aram, H. et al., *The Distance Roman Domatic Number of a Graph*, AKCE International Journal of Graphs and Combinatorics, vol. 9, no. 2, 2012, pp. 205-212.

[2] Aram, H. et al., *The Distance Roman Domination Numbers of Graphs*, Discussiones Mathematicae, January 2013.

[3] Chartrand, G. and Zhang, P., *Chromatic Graph Theory*, Discrete Mathematics And Its Applications, Taylor & Francis Group, LLC − CRC Press, Boca Raton, FL, 2009.

[4] Cockayne, E. J. et al., *Roman Domination in Graphs*, Discrete Mathematics, vol. 278, March 2004, pp. 11-12.

[5] Entero, G. M. and Pedrano, A. C., *On Connected Total Dominating Sets And Connected Total Domination Polynomial Of Corona Graphs*, Advances and Applications in Discrete Mathematics, Pushpa Publishing House, Prayagraj, India, vol. 22, no. 1, September 2019, pp. 103-115.

[6] Entero, G. M. and Pedrano, A. C., *On Connected Total Domination Polynomial Of Some Lexicographic Product Graphs*, Advances and Applications in Discrete Mathematics, Pushpa Publishing House, Prayagraj, India, vol. 27, issue 1, May 2021, pp. 147-155.

[7] Ghaffari-Hadigheh, A., *Roman Domination Problem with Uncertain Positioning and Deployment Costs*, Soft Computing, vol. 24, 2020, pp. 2637-2645.

[8] Harary, F., *Graph Theory*, Addison Wesly, Reading Mass, 1969.

[9] Harris, E. M., *Global Domination Stable Graphs*, Electronic Theses and Dissertations, 2012.

[10] Haynes, T.W., Hedetniemi, S.T., and Slater, P.J., *Fundamentals of Domination in Graphs*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1998.

[11] Hedetniemi, S. T. and Laskar, R. C., *Bibliography on Domination in Graphs and Some Basic Definitions of Domination Parameters*, Discrete Mathematics, North-Holland, vol. 86, 1990, pp. 257-277.

[12] Pushpam, L. R. P. and Padmapriea, S., *Global Roman Domination in Graphs*, Discrete Applied Mathematics, Elsevier B. V., August 2015.

[13] Sampathkumar, E., *The Global Domination Number of a Graph*, Journal of Math. Phy. Sci., vol. 23, no. 5, October 1989.

[14] Tian, F., *A Note on Distance Domination Numbers of Graphs*, Australian Journal of Combinatorics, vol. 43, 2009, pp. 181-190.

[15] Xu, L., *Roman Domination*, 2015.