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On the Planarity of a Directed Pathos Total Digraph of Some Special Arborescence Graphs

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Abstract. An arborescence graph is a directed graph in which, for a vertex u called the root, and any other vertex v, there is exactly one directed path from u to v. The directed pathos of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r. In [6], for an arborescence A_r, a directed pathos total digraph $Q = DPT(A_r)$ has vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; uv such that $u, v \in V(A_r)$ and u is adjacent to v; $au(ua)$ such that $a \in A(A_r)$ and $u \in V(A_r)$ and the head (tail) of a is u; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, and it is also possible to reach the head of P_i from the tail of P_j.

In this paper, the concept of planarity of the directed pathos total digraph (that is, as an acyclic directed graph which can be drawn with non crossing arcs oriented in one direction) is being discussed and applied to a directed pathos total digraph of an arborescence A_r ($DPT(A_r)$). Further, the internal vertices of these directed pathos total digraph of A_r are cconsidered.

Finally, the planarity of an arborescence resulting from the vertex-gluing of two directed paths is presented and corresponding internal vertex number is obtained.

2020 Mathematics Subject Classifications: 05C20

Key Words and Phrases: Directed path, arborescence graph, directed pathos total digraph of an arborescence, internal vertex number of a directed pathos total digraph of an arborescence

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1. Introduction

There are many graph valued functions (or graph operators) for which one can construct a new graph from a given graph, such as the line graphs, the total graphs, and their generalizations. The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a common vertex [8]. Harary and Norman [5] extended the concept of line graph of a graph and introduced the concept of line digraph of a directed graph. The line digraph $L(D)$ of a digraph D has the arcs of D as vertices. There is an arc from D-arc pq towards D-arc uv if and only if $q = u$. Behzad [1] introduced the concept of total graph of a graph. The total graph of a graph G, written $T(G)$, is the graph whose vertices can be put in one-to-one correspondence with the vertices and edges of G in such a way that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent, where the vertices and edges of G are called its members. Gary Chatrand and James Stewart [2] extended the concept of total graph of a graph to the directed case thereby introducing the total digraph. The total digraph of a directed graph D, written $T(D)$, is the digraph whose vertices are in one-to-one correspondence with the vertices and arcs of D and such that the vertex u is adjacent to the vertex v in $T(D)$ if and only if the element corresponding to u is adjacent to the element corresponding to v in D.

The concept of pathos of a graph G was introduced by Harary [4] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T. Stanton and Cowan [7] calculated the path number of certain classes of graphs like trees and complete graphs. Gudagudi [3] extended the concept of pathos of graphs to trees thereby introducing the concept called pathos line graph of a tree. A pathos line graph of a tree T , written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T , with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos. Since the pattern of pathos for a tree is not unique, the corresponding pathos line graph is also not unique. The present study is on the directed pathos of total arborescence graphs denoted by $DPT(A_r)$, where an arborescence graph A_r is a directed graph for which from an initial vertex u there is only one directed path going to another vertex v.

2. Preliminary Concepts and Results

In this section, some concepts relating to directed pathos total digraph of an arborescence are defined and $DPT(A_r)$ is discussed.

Definition 2.1. *A vertex $u \in V(D)$ of a directed graph D is a **root vertex** if u is only an initial vertex, that is, $d^-(u) = 0$.*

Definition 2.2. [6] *An* ***arborescence****, denoted by A_r, is a directed graph in which, from a root vertex u and for any other vertex v, there is exactly one directed path from u to v.*

Example 2.3. *The graph T in Figure 1 is an example of an arborescence graph where vertex a is its root, that is, $d^-(a) = 0$ and there is exactly one directed path from a to other vertices $b, c, d, e, f, g, h, i, j, k, l$, thus, $T = A_r$.*

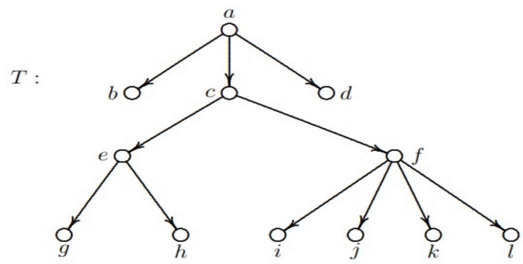

Figure 1: An arborescence graph T

Definition 2.4. [4] *The* ***pathos*** *of a graph G is a collection of minimum number of edge disjoint open paths whose union is G. The* ***path number*** *of a graph G is the number of paths in a pathos.*

Definition 2.5. *The* ***directed pathos*** *of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.*

Definition 2.6. [6] *For an arborescence A_r, a* ***directed pathos total digraph*** *$Q = DPT(A_r)$ has vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs:*
(i) *ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b;*
(ii) *uv such that $u, v \in V(A_r)$ and u is adjacent to v or an arc from u to v exists;*
(iii) *$au(ua)$ such that $a \in A(A_r)$ and $u \in V(A_r)$ and the head (tail) of a is u;*
(iv) *Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; and*
(v) *P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.*

Definition 2.7. *A vertex $v \in V(G)$ is said to be an* ***inner vertex of a planar digraph*** *G if vertex v does not belong to the boundary of the exterior region in any embeddings of G in the plane. The* ***inner vertex number*** *$i(G)$ is the maximum number of inner vertices of a planar digraph G.*

Example 2.8. *Consider the graphs in Figure 2 where the directed pathos total digraph of A_r is shown. Also, $i(Q) = 2$.*

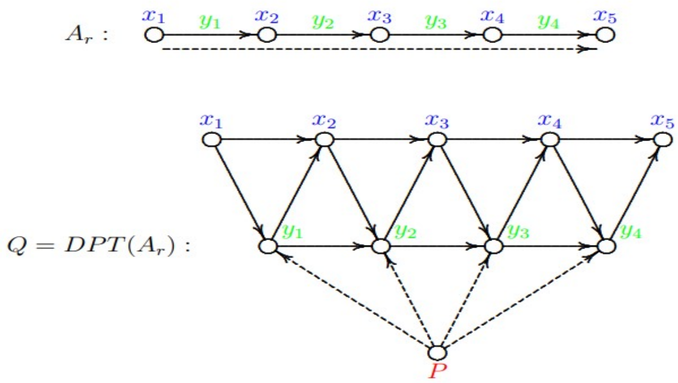

Figure 2: Directed pathos total digraph of A_r

In the context of directed graphs, a digraph G is outerplanar if $i(G) = 0$ and it is minimally non outerplanar if $i(G) = 1$.

We now enumerate some of the characterizations of the planarity of the $DPT(A_r)$ [6].

Theorem 2.9. *Every $DPT(A_r)$ is either strictly unilateral or strictly weak.*

Theorem 2.10. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is planar if and only if the underlying graph of A_r is a star graph $K_{1,n}$ on $n \leq 3$ vertices.*

Theorem 2.11. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is outerplanar if and only if A_r is either $\vec{P}_2$ or $\vec{P}_3$.*

Theorem 2.12. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is maximal outerplanar if and only if A_r is $\vec{P}_3$.*

Theorem 2.13. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is minimally non outerplanar if and only if A_r is $\vec{P}_4$.*

Theorem 2.14. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r has crossing number one if and only if the underlying graph of A_r is $K_{1,4}$.*

3. Main Results

This section presents some properties and characterizations of a directed pathos total digraph of an arborescence graph. The first result is intended for the arborescence which is a directed path $\vec{P}_n$.

Theorem 3.1. *If $A_r = \vec{P}_n$, then the directed pathos total digraph $DPT(A_r)$ of A_r is planar.*

Proof: Suppose that $A_r = \vec{P}_n$. Let $V(\vec{P}_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $A(\vec{P}_n) = \{e_1, e_2, e_3, \ldots, e_{n-1}\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of $\vec{P}_n$, respectively and $e_i = (v_i, v_{i+1})$ for $2 \leq i \leq n-1$. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ are the vertices of $T(A_r)$. Also, (v_i, v_{i+1}), (v_i, e_i), (e_i, v_{i+1}), (e_i, e_{i+1}), for $1 \leq i \leq n-1$ are the arcs of $T(A_r)$. Let $P(A_r) = \{P_1\}$ be a directed pathos of A_r such that P_1 lies on the arcs $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $\ldots$, $e_{n-1} = (v_{n-1}, v_n)$. The directed pathos vertex P is a neighbor of the vertices $e_1, e_2, \ldots, e_{n-1}$. This shows that the crossing number of $DPT(\vec{P}_n)$ is zero, that is, $cr(DPT(\vec{P}_n)) = 0$ (see Figure 3). Hence, $DPT(\vec{P}_n)$ is planar. □

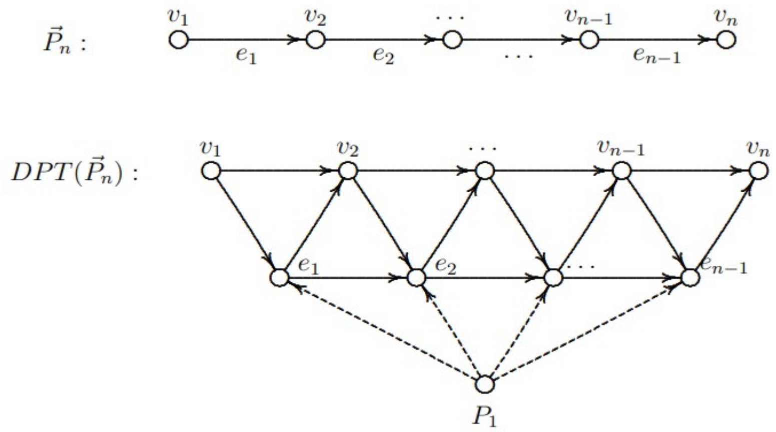

Figure 3: Directed pathos total digraph of $\vec{P}_n$

Theorem 3.2. *For an arborescence graph $\vec{P}_n$ with $n \geq 2$ vertices and $n-1$ arcs, $\left|A(DPT(A_r))\right| = 5(n-1) - 1 = 5n - 6$.*

Proof: We do this by induction. For $n = 2$, note that $DPT(\vec{P}_2)$ consists of vertices v_1, v_2, e_1, and P and arcs $(v_1, v_2), (v_1, e_1), (e_1, v_2)$, and (P, e_1). (See Figure 4). Thus,

$$\begin{aligned} \left|A(DPT(\vec{P}_3))\right| &= 4 \\ &= 5(2-1) - 1 \\ &= 5(n-1) - 1. \end{aligned}$$

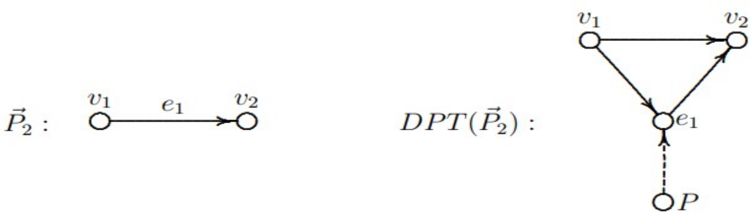

Figure 4: Digraph $\vec{P}_2$ and its directed pathos total digraph

For $n = 3$, $DPT(\vec{P_3})$ consists of vertices v_1, v_2, v_3, e_1, e_2, and P and arcs (v_1, v_2), (v_2, v_3), (v_1, e_1), (v_2, e_2), (e_1, v_2), (e_2, v_3), (e_1, e_2), (P, e_1), and (P, e_2). (See Figure 5). Thus,

$$\begin{aligned} \left|A(DPT(\vec{P_3}))\right| &= 9 \\ &= 5(3-1)-1 \\ &= 5(n-1)-1. \end{aligned}$$

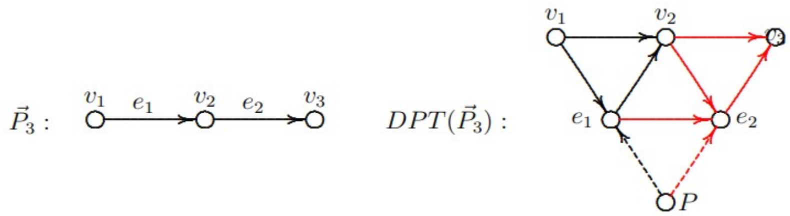

Figure 5: Digraph $\vec{P_3}$ and its directed pathos total digraph

Assume that for $\vec{P}_{n-1}$ with $n-1$ vertices, $\left|A(DPT(\vec{P}_{n-1}))\right| = 5\big((n-1)-1\big)-1$. That is,

$$\begin{aligned} \left|A(DPT(\vec{P}_{n-1}))\right| &= 5((n-1)-1)-1 \\ &= 5(n-2)-1 \\ &= 5n-10-1 \\ &= 5(n-1)-6 \\ &= 5n-11. \end{aligned}$$

That is, $A\big(DPT(\vec{P}_{n-1})\big)$ consists of the arcs (v_1, v_2), (v_2, v_3), $\ldots$, (v_{n-2}, v_{n-1}), (v_1, e_1), $(v_2, e_2), \ldots, (v_{n-2}, e_{n-1})$, (e_1, v_2), $(e_2, v_3), \ldots, (e_{n-2}, v_{n-1})$, (e_1, e_2), $(e_2, e_3), \ldots, (e_{n-3}, e_{n-2})$, (P, e_1), (P, e_2), $\ldots$, (P, e_{n-2}). (See Figure 6).

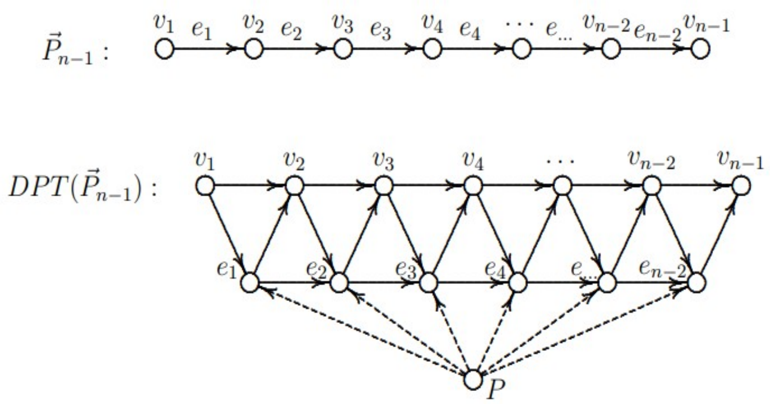

Figure 6: Digraph $\vec{P}_{n-1}$ and its directed pathos total digraph

Adding one vertex to $\vec{P}_{n-1}$ results into a directed path $\vec{P}_n$, with the additional arc (v_{n-1}, v_n). (See Figure 7).

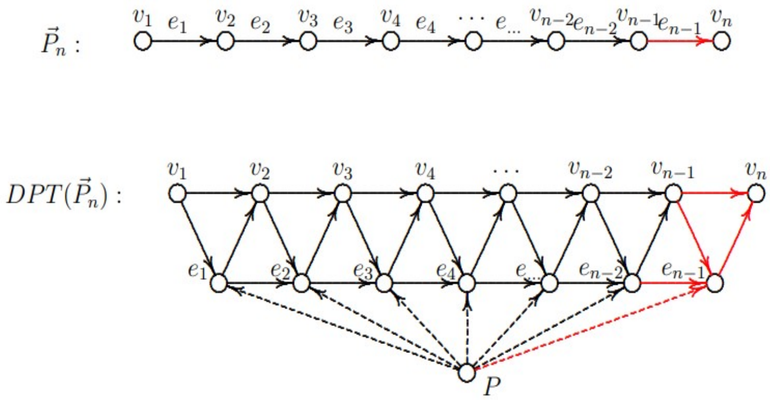

Figure 7: Digraph $\vec{P}_n$ and its directed pathos total digraph

Hence, $DPT(\vec{P}_n)$ contains the arcs in $DPT(\vec{P}_{n-1})$ and the arcs (v_{n-1}, v_n), (e_{n-1}, v_n), (v_{n-1}, e_{n-1}), (e_{n-2}, e_{n-1}), and (P, e_{n-1}). Therefore,

$$\begin{aligned}|A(DPT(\vec{P}_n))| &= 5n - 11 + 5 \\ &= 5n - 6 \\ &= 5n - 5 - 1 \\ &= 5(n-1) - 1. \quad \square\end{aligned}$$

We present a closely similar result from [6] in the next theorem using the usual notation of a path and taking into consideration a directed path as an arborescence.

Theorem 3.3. *The* $i(DPT(\vec{P}_n)) = n - 3$ *if and only if* $n \geq 4$.
Proof: Suppose that $i(DPT(\vec{P}_n)) = n - 3$, where $n < 4$. Suppose $n = 3$ and $A_r = \vec{P}_3$. Thus we have v_1, v_2, v_3 as the vertices of A_r and $e_1 = (v_1, v_2)$ and $e_2 = v_2, v_3$ as the arcs of A_r. Then the vertices of $T(A_r)$ are $\{v_1, v_2, v_3, e_1, e_2\}$ and the arcs are $(v_1, v_2), (v_2, v_3), (v_1, e_1), (e_1, v_2)$, $(v_2, e_2), (e_2, v_3), (e_1, e_2)$. Let $P(A_r) = \{P_2\}$ where $P_2 = [v_1v_2, v_2v_3]$. Therefore, $DPT(A_r)$ is an outerplanar. A contradiction since $DPT(A_r)$ should contain an internal vertex. (See Figure 5).

Conversely, suppose that $A_r = \vec{P}_n$ for $n \geq 4$. We will show $i(DPT(\vec{P}_n)) = n - 3$ by induction. Let $A_r = \vec{P}_4$ and let $V(\vec{P}_4) = \{v_1, v_2, v_3, v_4\}$. Thus, $V(DPT(\vec{P}_4)) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, P\}$ where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, and $e_3 = (v_3, v_4)$ as the arcs of $\vec{P}_4$, and P is the pathos of $\vec{P}_4$. Hence, $A(DPT(\vec{P}_4)) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, e_1),$ (e_1, v_2), (v_2, e_2), (e_2, v_3), (v_3, e_3), (e_3, v_4), (e_1, e_2), (e_2, e_3), (P, e_1), (P, e_2), $(P, e_3)\}$. (See Figure 8).

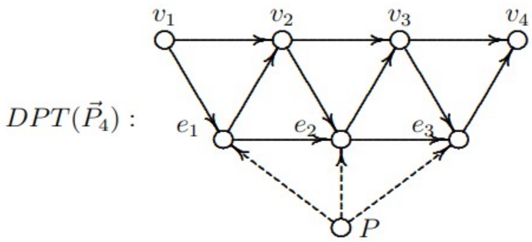

Figure 8: $DPT(\vec{P}_4)$ of $\vec{P}_4$

So e_2 is the only internal vertex of $DPT(\vec{P}_4)$. That is, $i(DPT(\vec{P}_4)) = 1 = 4 - 3 = n - 3$. Assume that for $n > 4$, $i(DPT(\vec{P}_{n-1})) = n - 1 - 3 = n - 4$. That is, $V(DPT(\vec{P}_{n-1})) = \{v_1, v_2, \ldots, v_{n-1}, e_1, e_2, \ldots, e_{n-2}, P\}$ where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $\ldots$, $e_{n-2} = (v_{n-2}, v_{n-1})$ and P is the pathos of $\vec{P}_{n-1}$. Also, $A(DPT(\vec{P}_{n-1})) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-2}, v_{n-1}), (v_1, e_1), (e_1, v_2), (v_2, e_2), (e_2, v_3), \ldots, (v_{n-2}, e_{n-2}), (e_{n-2}, v_{n-1}), (e_1, e_2), (e_2, e_3), \ldots, (e_1, e_2), (e_2, e_3), \ldots, (e_{n-3}, e_{n-2}), (P, e_1), (P, e_2), \ldots, (P, e_{n-2})\}$. (See Figure 9).

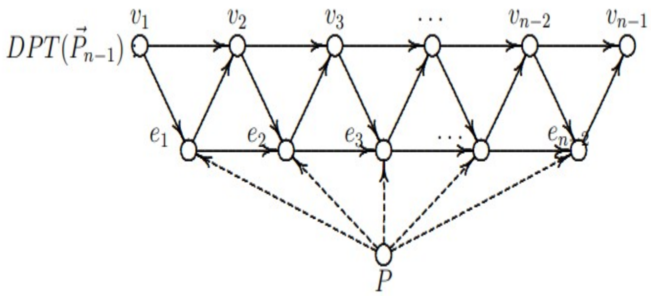

Figure 9: $DPT(\vec{P}_{n-1})$ of $\vec{P}_{n-1}$

It follows that $e_2, e_3, \ldots, e_{n-3}$ are the internal vertices of $DPT(\vec{P}_{n-1})$. That is, $i(DPT(\vec{P}_{n-1})) = n - 3 - 2 + 1 = n - 4$. Now, adding one vertex to the right side of v_{n-1} of $\vec{P}_{n-1}$ to obtain $\vec{P}_n$, we will have $V(\vec{P}_n) = \{v_1, v_2, \ldots, v_n\}$ and $A(\vec{P}_n) = \{e_1, e_2, \ldots, e_{n-1}\}$ where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $\ldots$, $e_{n-2} = (v_{n-2}, v_{n-1})$, $e_{n-1} = (v_{n-1}, v_n)$. Thus, an addition of the vertices v_n and e_{n-1} to the $V(\vec{P}_n)$ results in additional arcs $(v_{n-1}, v_n), (v_{n-1}, e_{n-1}), (e_{n-1}, v_n), (e_{n-2}, e_{n-1})$, and (P, e_{n-1}). (See Figure 10).

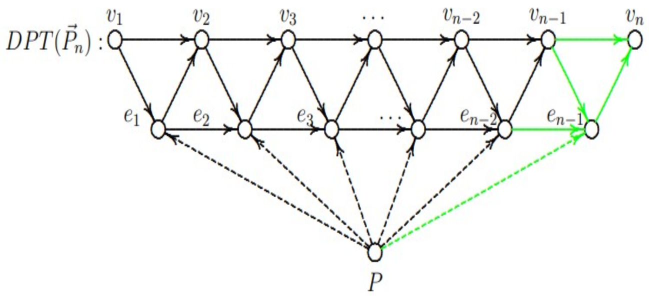

Figure 10: $DPT(\vec{P}_n)$ of $\vec{P}_n$

Therefore, the vertices $e_2, e_3, \ldots, e_{n-2}$ are the internal vertices of $DPT(\vec{P}_n)$. That is, $i(DPT(\vec{P}_n)) = n - 2 - 2 + 1 = n - 3$. □

In view of Theorem 3.3, a directed pathos total digraph $DPT(\vec{P}_n)$ is outerplanar for $n = 3$ and minimally non outerplanar for $n = 4$.

Theorem 3.4. *For an arborescence $\vec{P}_n$, $DPT(\vec{P}_n)$ is strictly weak.*
Proof: Suppose that $A_r = \vec{P}_n$. Let $V(\vec{P}_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set and let $A(\vec{P}_n) = \{e_1, e_2, e_3, \ldots, e_{n-1}\}$ be the arc set of $\vec{P}_n$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of $\vec{P}_n$, respectively, and $e_i = (v_i, v_{i+1})$ for $2 \leq i \leq n-1$. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}, P$ are the vertices of $T(A_r)$ where $P(A_r) = P$ is the directed pathos of A_r such that P lies on the arcs $x_1 = (v_1, v_2), x_2 = (v_2, v_3), \ldots, x_{n-1} = (v_{n-1}, v_n)$. Also, (v_i, v_{i+1}), (v_i, e_i), (e_i, v_{i+1}), (e_i, e_{i+1}), for $1 \leq i \leq n-1$ are the arcs of $T(A_r)$ and since P lies on the arcs $x_1, x_2, \ldots, x_{n-1}$, the directed pathos vertex P is a neighbor of the vertices $x_1, x_2, \ldots, x_{n-1}$. Note that from v_1, there is a semi-directed path to vertices $v_2, v_3, \ldots, v_n$ and from v_1, also there is a semi-directed path to vertices $e_1, e_2, \ldots, e_{n-1}$. However, there is no semi-directed path from v_1 to P. From Theorem 2.9, a directed pathos total digraph of an arborescence is either strictly unilateral or strictly weak. Thus, for any arborescence directed planar graph, its $DPT(A_r)$ is strictly weak. □

Corollary 3.5. *For any arborescence graph A_r containing $K_{1,4}$, its $DPT(A_r)$ is non-planar.*
Proof: This follows from Theorem 2.14. □

Theorem 3.6. *For an arborescence graph $A_r = \vec{S}_{1,2(n)}$, the directed pathos total digraph $DPT(A_r)$ is nonplanar if and only if $n \geq 3$.*
Proof: Suppose $A_r = \vec{S}_{1,2(n)}$, where $n \leq 2$. Then A_r is just a path. By Theorem 3.1, $DPT(A_r)$ is planar.

Conversely, suppose that $A_r = \vec{S}_{1,2(n)}$, where $n \geq 3$. For $n = 3$, let $V(A_r) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the vertex set and $A(A_r) = \{e_1 = (v_1v_2),\ e_2 = (v_2v_3),\ e_3 = (v_3v_4),\ e_4 = (v_4v_5),\ e_5 = (v_3v_6),\ e_6 = (v_6v_7)\}$ be the arc set of A_r such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively. Then we have the following vertices for $T(A_r)$, that is $V(T(A_r)) = \{v_1, v_2, \ldots, v_7, e_1, e_2, e_3, \ldots, e_6\}$ and arcs (v_i, e_i) for $1 \leq i \leq 4, 6$, (v_3, e_5), (e_i, v_{i+1}) for $1 \leq i \leq 6$, (v_i, v_{i+1}) for $1 \leq i \leq 4, 6$, (v_3, v_6), (e_1, e_2), (e_2, e_3), (e_3, e_4), (e_5, e_6), (e_2, e_5). Let $P(A_r) = \{P_1, P_2\}$ be a directed pathos set of A_r such that P_1 lies on the arcs $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)$; P_2 lies on $(v_3, v_6), (v_6, v_7)$. Thus the directed pathos vertex P_1 is a neighbor of the vertices $v_1v_2, v_2v_3, v_3v_4, v_4v_5$; P_2 is a neighbor of v_3v_6, v_6v_7. This shows that $cr(DPT(A_r)) = 1$, a nonplanar. For $n \geq 4$, it is nonplanar since $K_{1,4}$ is its subdigraph. This shows that $cr(DPT(A_r)) \neq 0$ by Theorem 2.14. Hence, $DPT(A_r)$ is nonplanar. □

Theorem 3.7. *For an arborescence $\vec{P}_n$, $DPT(\vec{P}_n)$ contains a $K_{1,n-1}$ graph.*
Proof: From Theorem 3.1, note that P lies on the arcs $e_1e_2, e_2e_3, \ldots, e_{n-2}e_{n-1}$. Thus, P is a neighbor of $e_1e_2, e_2e_3, \ldots, e_{n-2}e_{n-1}$. Therefore, $K_{1,n-1}$ is a subdigraph of $DPT(\vec{P}_n)$. □

Theorem 3.8. *For an arborescence graph A_r which is an n-pan, the directed pathos total digraph $DPT(A_r)$ has $cr(n\text{-}pan) = 1$ if and only if $n \geq 3$.*
Proof: Suppose that A_r is an n-pan with $n < 2$ and $cr(A_r) = 1$. Let $V(A_r) = \{v_1, v_2, v_3\}$ be the vertex set and $A(A_r) = \{e_1, e_2\}$ be the arc set of A_r such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively. Thus, $A_r \cong \vec{P}_3$. By Theorem 3.1, all path graphs are planar, thus $cr(A_r) = 0$, a contradiction.

Conversely, suppose that A_r is an n-pan graph with $n \geq 3$ vertices. We consider the following cases.

Case 1: Suppose that A_r is an n-pan graph with $n = 3$. Then $V\big(T(A_r)\big) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4\}$ is the vertex set of $T(A_r)$ and its arcs are $(v_i, v_{i+1}), (e_i, e_{i+1})$, (e_i, v_{i+1}) for $1 \leq i \leq 3$, (v_i, e_i) for $1 \leq i \leq 4$, (v_4, v_2), and (e_4, e_2). Let $P(A_r) = \{P_1\}$ be a directed pathos set of A_r such that P_1 lies on the arcs (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_2). Then the directed pathos vertex P_1 is a neighbor of the vertices $v_1v_2, v_2v_3, v_3v_4, v_4v_2$. This shows that the crossing number of $DPT(A_r)$ is one, that is, $cr(DPT(A_r)) = 1$ where (e_4, e_2) crosses (P_1, e_1).

Case 2: Suppose that the underlying graph of A_r is an n-pan graph with $n = 4$. Then $V\big(T(A_r)\big) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$ is the vertex set of $T(A_r)$ and (v_i, v_{i+1}), (e_i, e_{i+1}), (e_i, v_{i+1}), (v_i, e_i), (v_4, v_2), and (e_4, e_2) are the arcs. Let $P(A_r) = \{P_1\}$ be a directed pathos set of A_r such that P_1 lies on the arcs $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_i, v_{i+1})$,

(v_4, v_2) for $1 \leq i \leq n-1$. Then the directed pathos vertex P_1 is a neighbor of the vertices $v_1v_2, v_2v_3, v_3v_4, \ldots, v_iv_{i+1}, v_4v_2$ for $1 \leq i \leq n-1$. This shows that the crossing number of $DPT(A_r)$ is one, that is, $cr(DPT(A_r)) = 1$. For all n, (e_4, e_2) crosses (P_1, e_1). □

Theorem 3.9. *If $A_r = \vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}}$ then the directed pathos total digraph $DPT(A_r)$ of A_r is planar where x_1 and y_1 are the initial vertices of $\vec{P}_{m_{(x_1)}}$ and $\vec{P}_{n_{(y_1)}}$, respectively.*

Proof: Suppose that $A_r = \vec{P}_m \bullet \vec{P}_n$. Let $V(\vec{P}_m) = \{x_1, x_2, x_3, \ldots, x_m\}$ and let $A(\vec{P}_m) = \{a_1, a_2, a_3, \ldots, a_{m-1}\}$ such that x_1 and $a_1 = (x_1, x_2)$ are the root and root arc of $\vec{P}_m$, respectively and $a_i = (x_i, x_{i+1})$ for $2 \leq i \leq m-1$. Then $x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_{m-1}$ are the vertices of $T(\vec{P}_m)$. Also, (x_i, x_{i+1}), (x_i, a_i), (a_i, x_{i+1}), (a_i, a_{i+1}), are the arcs of $T(\vec{P}_m)$. Let $V(\vec{P}_n) = \{y_1, y_2, y_3, \ldots, y_n\}$ and let $A(\vec{P}_n) = \{b_1, b_2, b_3, \ldots, b_{n-1}\}$ such that y_1 and $b_1 = (y_1, y_2)$ are the root and root arc of $\vec{P}_n$, respectively and $b_j = (y_j, y_{j+1})$ for $2 \leq j \leq n-1$. Then $y_1, y_2, \ldots, y_n, b_1, b_2, \ldots, b_{n-1}$ are the vertices of $T(\vec{P}_n)$. Also, (y_j, y_{j+1}), (y_j, b_j), (b_j, y_{j+1}), (b_j, b_{j+1}), are the arcs of $T(\vec{P}_n)$. Let $P(A_r) = \{P_1, P_2\}$ such that P_1 lies on the arcs $a_1 = (x_1, x_2)$, $a_2 = (x_2, x_3)$, $\ldots$, $a_{m-1} = (x_{m-1}, x_m)$ and P_2 lies on the arcs $b_1 = (y_1, y_2)$, $b_2 = (y_2, y_3)$, $\ldots$, $b_{n-1} = (y_{n-1}, y_n)$. The directed pathos vertex P_2 is a neighbor of the vertices $b_1, b_2, \ldots, b_{n-1}$. Note that x_1 and y_1 are the initial vertices of graphs $\vec{P}_m$ and $\vec{P}_n$, respectively, so $x_1 = y_1$ in $\vec{P}_m \bullet \vec{P}_n$. This shows that the $cr(DPT(\vec{P}_n)) = 0$ (see Figure 11). Hence, $DPT(\vec{P}_n)$ is planar. □

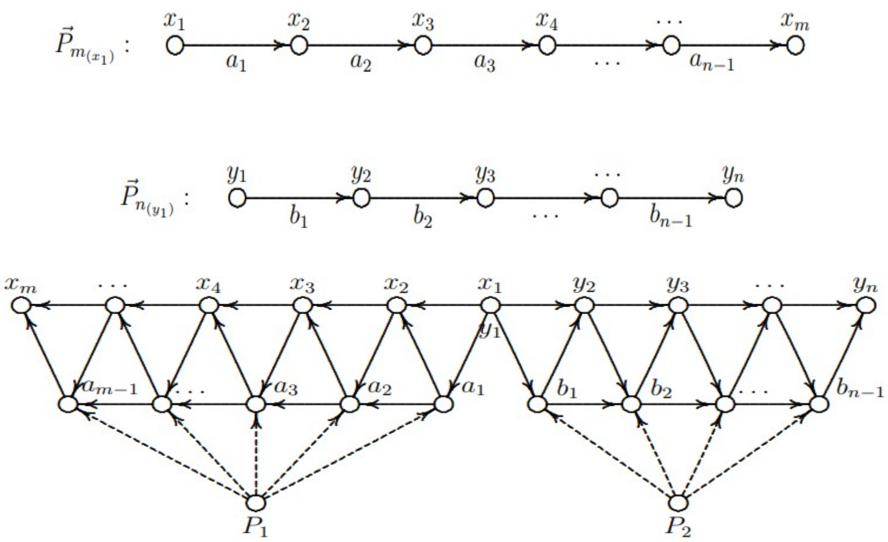

Figure 11: Directed pathos total digraph of $\vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}}$

Theorem 3.10. *For an $A_r = \vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}}$, $i(DPT(A_r)) = (m+n) - 6$ if and only if $m, n \geq 4$.*

Proof: Suppose that $i(DPT(A_r)) = (m+n) - 6$, where $m, n < 4$. Let $m, n = 3$. Thus, we have $v_1, v_2, v_3, v_4, v_5, v_6$ as the vertices of A_r and $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_1, v_4)$ $e_4 = (v_4, v_5)$ as the arcs of A_r. Then the vertices of $T(A_r)$ are $\{v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4\}$

and the arcs are $(v_1, v_2), (v_2, v_3), (v_1, v_4), (v_4, v_5), (v_1, e_1), (e_1, v_2), (v_2, e_2), (e_2, v_3), (v_1, e_3),$ $(e_3, v_4), (v_4, e_4), (e_4, v_5)$. Let $P(A_r) = \{P_1, P_2\}$ where $P_1 = [v_1v_2, v_2v_3]$ and $P_2 = [v_1v_4, v_4v_5]$. Therefore, $DPT(A_r)$ is an outerplanar. A contradiction since $DPT(A_r)$ should contain an internal vertex.

Conversely, suppose that $A_r = \vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}}$ for $m, n \geq 4$. We will show $i(DPT(\vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}})) = (m + n) - 6$ by induction. Let $A_r = \vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}}$ and let $V(\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Thus, $V(DPT(\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}})) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, e_1, e_2, e_3, e_4, e_5, e_6, P_1, P_2\}$ where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$, $e_4 = (v_1, v_5)$, $e_5 = (v_5, v_6)$, and $e_6 = (v_6, v_7)$ as the arcs of $\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}}$, and $\{P_1, P_2\}$ is the pathos set of $\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}}$. Hence, $A(DPT(\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}}))$=$\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_5), (v_5, v_6), (v_6, v_7), (v_1, e_1), (e_1, v_2), (v_2, e_2), (e_2, v_3), (v_3, e_3), (e_3, v_4), (v_1, e_4), (e_4, v_5), (v_5, e_6), (e_6, v_7), (e_1, e_2), (e_2, e_3), (e_4, e_5), (e_5, e_6), (P_1, e_1), (P_1, e_2), (P_1, e_3), (P_2, e_4), (P_2, e_5), (P_2, e_6)\}$. So e_2 and e_5 are the only internal vertices of $DPT(\vec{P}_{4_{(x_1)}} \bullet \vec{P}_{4_{(y_1)}})$. That is, $i(DPT(A_r)) = 2 = 4 + 4 - 6 = m + n - 6$.

Assume that for $m, n > 4$, $i(DPT(A_r)) = (m-1)+(n-1)-6$. That is, $V(DPT(A_r)) = \{x_1,\ x_2\ ,x_3,\ \ldots,\ x_{m-1},\ y_1,\ y_2,\ y_3,\ \ldots,\ y_{n-1},\ a_1,\ a_2,\ a_3,\ \ldots,\ a_{m-2},\ b_1,\ b_2,\ b_3,\ \ldots,\ b_{n-2},\ P_1,\ P_2\}$. Also, $A(DPT(A_r)) = \{(x_1, x_2),\ (x_2, x_3),\ \ldots,\ (x_{m-2}, x_{m-1}),\ (y_1, y_2),\ (y_2, y_3),\ \ldots,\ (y_{n-2}, y_{n-1}),\ (x_1, a_1),\ (a_1, x_2),\ \ldots,\ (x_{m-2}, a_{m-2}),\ (a_{m-2}, x_{m-1}),\ (y_1, b_1),\ (b_1, y_2),\ \ldots,\ (y_{n-2}, b_{n-2}),\ (b_{n-2}, y_{n-1}),\ (a_1, a_2),\ \ldots,\ (a_{m-3}, a_{m-2}),\ (b_1, b_2),\ \ldots,\ (b_{n-3}, b_{n-2}),\ (P_1, a_1),\ \ldots,\ (P_1, a_{m-2}),\ (P_2, b_1),\ \ldots,\ (P_2, b_{n-2})\}$. It follows that $a_2, a_3,\ \ldots,\ a_{m-3}$ and $b_2, b_3,\ \ldots,\ b_{n-3}$ are the internal vertices of $DPT(A_r)$. That is $i(DPT(\vec{P}_{m-1_{(x_1)}} \bullet \vec{P}_{n-1_{(y_1)}})) = m-3-1+n-3-1 = m+n-4$. Now, adding one vertex to the right sides of x_{m-1} and y_{n-1} of $\vec{P}_{m-1}$ and P_{n-1} to obtain $\vec{P}_m \bullet \vec{P}_n$, we will have $V(DPT(\vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}})) = \{x_1,\ x_2,\ x_3,\ \ldots,\ x_{m-1},\ x_m,\ y_1,\ y_2,\ y_3,\ \ldots,\ y_{n-1},\ y_n,\ a_1,\ a_2,\ a_3,\ \ldots,\ a_{m-2},\ a_{m-1},\ b_1,\ b_2,\ b_3,\ \ldots,\ b_{n-2},\ b_{n-1},\ P_1,\ P_2\}$. Also, $A(DPT(A_r)) = \{(x_1, x_2),\ (x_2, x_3),\ \ldots, (x_{m-2}, x_{m-1}),\ (x_{m-1}, x_m),\ (y_1, y_2),\ (y_2, y_3),\ \ldots, (y_{n-2}, y_{n-1}),\ (y_{n-1}, y_n),\ (x_1, a_1),\ (a_1, x_2), \ldots,\ (x_{m-2}, a_{m-2}),\ (a_{m-2}, x_{m-1}),\ (y_1, b_1),\ (b_1, y_2),\ \ldots, (y_{n-2}, b_{n-2}),\ (b_{n-2}, y_{n-1}),\ (a_1, a_2), \ldots,\ (a_{m-3}, a_{m-2}),\ (a_{m-2}, a_{m-1}),\ (b_1, b_2), \ldots,\ (b_{n-3}, b_{n-2}),\ (b_{n-2}, b_{n-1}),\ (P_1, a_1), \ldots,\ (P_1, a_{m-2}),\ (P_1, a_{m-1}),\ (P_2, b_1),\ \ldots,\ (P_2, b_{n-2}), (P_2, b_{n-1})\}$. Therefore, the vertices $a_2, a_3, \ldots, a_{m-2}$ and $b_2, b_3, \ldots, b_{n-2}$ are the internal vertices of $DPT(A_r)$. That is, $i(DPT(\vec{P}_{m_{(x_1)}} \bullet \vec{P}_{n_{(y_1)}})) = m - 3 + n - 3 = m + n - 6$. $\square$

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