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Forcing Connected Co-Independent Hop Domination Numbers in the Join and Corona of Graphs

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Abstract. This study deals with the forcing subsets of a minimum connected co-independent hop dominating sets in graphs. Bounds or exact values of the forcing connected co-independent hop domination numbers of graphs resulting from some binary operations such as join and corona of graphs are determined.
Some main results generated in this study include characterization of the minimum connected co-independent hop dominating sets, characterization of the forcing subsets for these types of sets, and bounds or exact values of the forcing connected co-independent hop domination numbers of the join and corona of graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Forcing subsets, connected co-independent hop domination, strictly co-independent set, co-independent set, join, corona

1. Introduction

Beginning with C. Berge [4] in 1958, the study on domination in graphs was developed. There are now a lot of studies involving domination and its variations. One of its variation is the connected co-independent domination number of graphs that was studied in [7].

Years later, a new domination parameter called hop domination was introduced in [12] by Natarajan and Ayyaswamy and were also studied in [3, 13–15]. A study in 2021 by Nanding and Rara [11] introduced a new concept of hop domination called the connected co-independent hop domination and generated some characterizations of connected co-independent hop domination in graphs.

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On the other hand, the concept of forcing numbers started from the study of molecular resonance structure which was introduced by Klein and Randic [10] in 1987. Harary et al. [16] first used the name "forcing number" and introduced the concept of the forcing of a perfect match in 1991. Chartrand et al. [5] initiated the investigation on the relation between forcing and domination concepts in 1997 and defined the term "forcing domination number". In 2017, John et al. [9] investigated the forcing connected domination number of a graph, and Armada and Canoy [1] investigated the forcing independent domination number of a graph in 2019. Furthermore, in 2018, Canoy et al. [2] investigated the forcing domination number of graphs under some binary operations.

In this study, the researchers define and establish the forcing subsets of minimum connected co-independent hop dominating sets in graphs and generate some characterizations of forcing subsets of minimum connected co-independent hop dominating sets of graphs resulting from the join and corona of two graphs and determine the values or bounds of their corresponding forcing connected co-independent hop domination numbers.

Connected co-independent hop domination in graphs can have real world applications. For an application, in [6], Desormeaux, Haynes, and Henning inspired their research on these concepts through social networking applications. They considered a factory with a large number of employees and needed to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this. In other words, the manager will select some workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible to minimize costs (extra costs for inspectors) and protect privacy (keep the inspectors' identity confidential). To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed in which each worker is represented by a vertex and an edge between two workers represents possible bias, that is, whether the two workers are close friends or enemies. Ideally, an inspector should not be adjacent to any worker who is being inspected.

In connected co-independent hop domination [11], every worker will be inspected by the nearest non-biased inspector. That is, an inspector who is a close friend (or an enemy) of a close friend (or enemy) of a worker. This is to save time and effort of locating a particular worker. Also, the inspectors should be acquainted with each other and all non-inspector workers are neither friends nor enemies, that is, they are not adjacent or there is no edge between them. The connected co-independent hop domination number will give the minimum number of inspectors needed.

In forcing subsets of connected co-independent hop domination, in each respective group of minimum number of inspectors that will inspect the workers in the designated areas of the factory, the members of that particular group of minimum number of inspectors will be assigned only to that distinct group of minimum number of inspectors, that is, it will strengthen the bond of the respective group of minimum number of non-biased inspectors with each other, since they are uniquely assigned to particular groups, and they will trust each other more doing their duties and will have a much easier time doing their job regarding with the respective workers that they are assigned to inspect. The forcing connected co-independent hop domination number will determine the minimum number of members from the respective group of minimum number of inspectors that will be assigned only to that particular group of respective minimum number of inspectors.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [8] for elementary graph theoretic concepts.

An *independent set* S in a graph G is a subset of the vertex-set of G such that no two vertices in S are adjacent in G. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a *β-set* of G.

A dominating set $D \subseteq V(G)$ is called a *connected co-independent dominating set* of G if the subgraph $\langle D\rangle$ induced by D is connected and $V(G) \setminus D$ is an independent set. The cardinality of such a minimum set D is called *connected co-independent domination number* of G denoted by $\gamma_{c,coi}(G)$. A connected co-independent dominating set D with $|D| = \gamma_{c,coi}(G)$ is called a *$\gamma_{c,coi}$-set* of G.

Let G be a connected graph. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a *γ_h-set*.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u\in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

Let G be a graph. A subset S of $V(G)$ is a *strictly co-independent set* of G if $V(G)\backslash S$ is an independent set and $N_G(v) \cap S \neq S$ for all $v \in V(G)\backslash S$. The minimum cardinality of a strictly co-independent set in G, denoted by $sci(G)$ is called the *strictly co-independent number* of G. A strictly co-independent set S with $|S| = sci(G)$ is called an *sci-set* of G.

A set $S \subseteq V(G)$ is a *co-independent* set of G if $\langle V(G) \setminus S\rangle$ is independent. The minimum cardinality of a co-independent set in G, denoted by $coi(G)$ is called the co-independent number of G. A co-independent set S with $|S| = coi(G)$ is called a *coi-set* of G.

Let G be a connected graph. A hop dominating set $S \subseteq V(G)$ is a *connected co-independent hop dominating set* of G if $\langle S\rangle$ is connected and $V(G)\backslash S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of G, denoted by $\gamma_{ch,coi}(G)$, is called the *connected co-independent hop domination number* of G. A connected co-independent hop dominating set S with $|S| = \gamma_{ch,coi}(G)$ is called a *$\gamma_{ch,coi}$-set* of G.

Let W be a $\gamma_{ch,coi}$-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique $\gamma_{ch,coi}$-set containing S. The *forcing connected co-independent hop domination number* of W is given by $f\gamma_{ch,coi}(W) = \min\{|S| : S \text{ is a forcing subset for } W\}$. The *forcing connected co-independent hop domination number* of G is given by

$$f\gamma_{ch,coi}(G) = \min\{f\gamma_{ch,coi}(W) : W \text{ is a } \gamma_{ch,coi}\text{-set of } G\}.$$

Let W be an *sci*-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique *sci*-set containing S. The *forcing strictly co-independent number* of W is given by $fsci(W) = \min\{|S| : S$ is a forcing subset for $W\}$. The *forcing strictly co-independent number* of G is given by

$$fsci(G) = \min\{fsci(W) : W \text{ is an } sci\text{-set of } G\}.$$

Let W be a *coi*-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique *coi*-set containing S. The *forcing co-independent number* of W is given by $fcoi(W) = \min\{|S| : S$ is a forcing subset for $W\}$. The *forcing co-independent number* of G is given by

$$fcoi(G) = \min\{fcoi(W) : W \text{ is a } coi\text{-set of } G\}.$$

2. Known Results

The following known results are taken from [11].

Theorem 1. *Let G and H be any two graphs. Then $S \subseteq V(G + H)$ is a connected co-independent hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$ where one of the following holds:*

(i) $S_G = V(G)$ and S_H is a strictly co-independent set of H.

(ii) $S_H = V(H)$ and S_G is a strictly co-independent set of G.

Corollary 1. *Let G and H be any two graphs where $|V(G)| = n$ and $|V(H)| = m$. Then $\gamma_{ch,coi}(G + H) = min\{n + sci(H), m + sci(G)\}$.*

Theorem 2. *Let G be a nontrivial connected graph and H be any graph. A set $S \subseteq V(G \circ H)$ is a connected co-independent hop dominating set of $G \circ H$ if and only if $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where $S_v \subseteq V(H^v)$ and $V(H^v) \setminus S_v$ is an independent subset of $V(H^v)$ for each $v \in V(G)$.*

Corollary 2. *Let G be a nontrivial connected graph of order n and H be any graph of order m. Then $\gamma_{ch,coi}(G \circ H) = n(1 + m - \beta(H))$.*

3. Forcing Connected Co-Independent Hop Domination Number of Some Special Graphs

Remark 1. Let G be a connected graph. Then
(i) $f\gamma_{ch,coi}(G) = 0$ if and only if G has a unique $\gamma_{ch,coi}$-set, and
(ii) $f\gamma_{ch,coi}(G) = 1$ if and only if G has at least two $\gamma_{ch,coi}$-sets, one of which, say B, contains an element which is not found in any $\gamma_{ch,coi}$-set of G.

Theorem 3. Let G be a connected graph. Then $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$ if and only if for all $\gamma_{ch,coi}$-set B of G and for each $v \in B$, there exists $u_v \in V(G) \setminus B$ such that $[B \setminus \{v\}] \cup \{u_v\}$ is a $\gamma_{ch,coi}$-set of G.

Proof: Suppose that $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$. Let B be a $\gamma_{ch,coi}$-set of G such that $f\gamma_{ch,coi}(G) = |B| = \gamma_{ch,coi}(G)$, that is, B is the only forcing subset for itself. Let $v \in B$. Since $B \setminus \{v\}$ is not a forcing subset for B, there exists a $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is a $\gamma_{ch,coi}$-set of G.

Conversely, suppose that every $\gamma_{ch,coi}$-set B' of G satisfies the given condition. Let B be a $\gamma_{ch,coi}$-set of G such that $f\gamma_{ch,coi}(G) = f\gamma_{ch,coi}(B)$. Suppose further that B has a forcing subset Q with $|Q| < |B|$, that is, $B = Q \cup P$ where $P = \{z \in B : z \notin Q\}$. Pick $z \in P$. By assumption, there exists $u_z \in V(G) \setminus B$ such that $\big[B \setminus \{z\}\big] \cup \{u_z\} = T$ is a $\gamma_{ch,coi}$-set of G. Hence, $T = Q \cup R$, where $R = \big[P \setminus \{z\}\big] \cup \{u_z\}$, is a $\gamma_{ch,coi}$-set containing Q, a contradiction. Hence, B is the only forcing subset for B. Therefore, $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$. □

Proposition 1. For any complete graph K_n with $n \geq 1$ vertices, $f\gamma_{ch,coi}(K_n) = 0$.

Proof: By definition of K_n, $V(K_n)$ is the only $\gamma_{ch,coi}$-set of K_n. By Remark 1(i), $f\gamma_{ch,coi}(K_n) = 0$. □

Proposition 2. For any path P_n with $n \geq 1$ vertices,

$$f\gamma_{ch,coi}(P_n) = \begin{cases} 0, & \text{if } n \neq 3, \\ 1, & \text{if } n = 3. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. Clearly, $f\gamma_{ch,coi}(P_1) = f\gamma_{ch,coi}(P_2) = 0$. Moreover, if $n = 4$, then P_n has $\gamma_{ch,coi}$-set $B_1 = \{v_2, v_3\}$ which is the only $\gamma_{ch,coi}$-set of P_n. By Remark 1(i), $f\gamma_{ch,coi}(P_n) = 0$. Suppose that $n > 4$, then clearly $B_2 = \{v_2, v_3, v_4, \ldots, v_{n-1}\}$ is the only $\gamma_{ch,coi}$-set of P_n. Thus, by Remark 1(i), $f\gamma_{ch,coi}(B_2) = 0 = f\gamma_{ch,coi}(P_n)$.

Suppose that $n = 3$. Then P_n has $\gamma_{ch,coi}$-sets $B_3 = \{v_1, v_2\}$ and $B_4 = \{v_2, v_3\}$ which are the only $\gamma_{ch,coi}$-sets of P_n with $v_1 \in B_3$ and $v_1 \notin B_4$. Hence, by Remark 1(ii), $f\gamma_{ch,coi}(B_3) = 1 = f\gamma_{ch,coi}(P_n)$. □

Proposition 3. For any cycle C_n with $n \geq 3$ vertices,

$$f\gamma_{ch,coi}(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ n-1, & \text{if } n \geq 4. \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Since $C_3 = K_3$, by Proposition 1, $f\gamma_{ch,coi}(C_3) = 0$. Suppose that $n \geq 4$. Then the $\gamma_{ch,coi}$-sets of C_n are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, $\ldots$, $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(C_n) \setminus B_j$ such that $\big[B_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a $\gamma_{ch,coi}$-set of G. Hence, by Theorem 3, $f\gamma_{ch,coi}(C_n) = n - 1$. □

4. Forcing Connected Co-Independent Hop Domination in the Join of Graphs

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \stackrel{\bullet}{\cup} V(H)$ and edge set $E(G + H) = E(G) \stackrel{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Remark 2. Let G be a connected graph. Then

(i) $fsci(G) = 0$ if and only if G has a unique *sci*-set, and

(ii) $fsci(G) = 1$ if and only if G has at least two *sci*-sets, one of which, say B, contains an element which is not found in any *sci*-set of G.

Theorem 4. Let G be a connected graph. Then $fsci(G) = sci(G)$ if and only if for all *sci*-set B of G and for each $v \in B$, there exists $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is an *sci*-set of G.

Proof: Suppose that $fsci(G) = sci(G)$. Let B be an *sci*-set of G such that $fsci(G) = |B| = sci(G)$, that is, B is the only forcing subset for itself. Let $v \in B$. Since $B \setminus \{v\}$ is not a forcing subset for B, there exists a $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is an *sci*-set of G.

Conversely, suppose that every *sci*-set B' of G satisfies the given condition. Let B be an *sci*-set of G such that $fsci(G) = fsci(B)$. Suppose further that B has a forcing subset Q with $|Q| < |B|$, that is, $B = Q \cup P$ where $P = \{z \in B : z \notin Q\}$. Pick $z \in P$. By assumption, there exists $u_z \in V(G) \setminus B$ such that $\big[B \setminus \{z\}\big] \cup \{u_z\} = T$ is an *sci*-set of G. Hence, $T = Q \cup R$, where $R = \big[P \setminus \{z\}\big] \cup \{u_z\}$, is an *sci*-set containing Q, a contradiction. Hence, B is the only forcing subset for B. Therefore, $fsci(G) = sci(G)$. □

Proposition 4. For any complete graph K_n with $n \geq 1$ vertices, $fsci(K_n) = 0$.

Proof: By definition of K_n, $V(K_n)$ is the only *sci*-set of K_n. By Remark 2(i), $fsci(K_n) = 0$. □

Proposition 5. For any path P_n with $n \geq 1$ vertices,

$$fsci(P_n) = \begin{cases} 0, & \text{if } n = 1, 2, 4 \text{ and } n > 5 \text{ is odd,} \\ 1, & \text{if } n = 3 \text{ and } n \geq 6 \text{ is even,} \\ 2, & \text{if } n = 5. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. Clearly, $fsci(P_1) = fsci(P_2) = fsci(P_4) = 0$, $fsci(P_3) = 1$ and $fsci(P_5) = 2$. If $n > 5$ and n is odd, then clearly $B = \{v_2, v_4, v_6, \ldots, v_{n-3}, v_{n-1}\}$ is the only *sci*-set of P_n. Thus, by Remark 2(i), $fsci(B) = 0 = fsci(P_n)$.

Suppose that $n \geq 6$ and n is even. Then P_n has *sci*-sets $B_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $B_2 = \{v_2, v_4, v_6, \ldots, v_n\}$. It can be verified that B_1 is the only *sci*-set of P_n containing the vertex v_1. Hence, by Remark 2(ii), $fsci(P_n) = 1$. □

Proposition 6. For any cycle C_n with $n \geq 3$ vertices,

$$fsci(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ 1, & \text{if } n > 4 \text{ and } n \text{ is even,} \\ 2, & \text{if } n > 3 \text{ and } n \text{ is odd,} \\ 3, & \text{if } n = 4. \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Since $C_3 = K_3$, by Proposition 4, $fsci(C_3) = 0$. Suppose that $n = 4$. Then the *sci*-sets of C_4 are $R_1 = \{v_1, v_2, v_3\}$, $R_2 = \{v_2, v_3, v_4\}$, $R_3 = \{v_1, v_3, v_4\}$ and $R_4 = \{v_1, v_2, v_4\}$. Clearly, for each $v_i \in R_j$ where $i, j \in \{1, 2, 3, 4\}$, there exists $v_k \in V(C_4) \setminus R_j$ such that $[R_j \setminus \{v_i\}] \cup \{v_k\}$ is an *sci*-set of G. Thus, by Theorem 4, $fsci(C_4) = 3$. Now, suppose that $n > 4$ and n is even. Then $B_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $B_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ are the only *sci*-sets of C_n with $v_1 \in B_1$ and $v_1 \notin B_2$. Hence, by Remark 2(ii), $fsci(B_1) = 1 = fsci(C_n)$.

Next, suppose that $n > 3$ and n is odd. Then

$$\begin{aligned}
S_1 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}, \\
S_2 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\}, \\
S_3 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_n\} \text{ and,} \\
S_4 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\}
\end{aligned}$$

are the *sci*-sets of C_n. Hence, no vertex of C_n is contained in a unique *sci*-set. Thus, $fsci(C_n) \geq 2$. Clearly, $\{v_1, v_n\}$ is uniquely contained in S_1. Therefore, $fsci(S_1) = 2 = fsci(C_n)$. □

In view of Theorem 1, we have the following theorem.

Theorem 5. Let G and H be any graphs. Then $S \subseteq V(G + H)$ is a connected co-independent hop dominating set of $G + H$ if and only if one of the following holds:
(i) $S = V(G) \cup S_H$ where S_H is a strictly co-independent set of H,
(ii) $S = V(H) \cup S_G$ where S_G is a strictly co-independent set of G.

As a consequence of Theorem 5, the next results follow.

Corollary 3. Let G be any graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a $\gamma_{ch,coi}$-set of $K_1 + G$ if and only if $S = \{v\} \cup T$ where T is an *sci*-set of G.

Corollary 4. Let G be any graph. Then

$$f\gamma_{ch,coi}(K_1 + G) = \begin{cases} 0, & \text{if } G \text{ has a unique } sci\text{-set,} \\ fsci(G), & \text{if } G \text{ has no unique } sci\text{-set.} \end{cases}$$

Proof: Suppose that G has a unique *sci*-set, say S_G. Then by Corollary 3, $\{v\} \cup S_G$ is a unique $\gamma_{ch,coi}$-set of $K_1 + G$. By Remark 1(i), $f\gamma_{ch,coi}(K_1 + G) = 0$. Now, suppose that G has no unique *sci*-set. Let A be an *sci*-set of G and let F be a forcing subset for A such that $fsci(G) = fsci(A) = |F|$. By Corollary 3, $S = \{v\} \cup A$ is a $\gamma_{ch,coi}$-set of $K_1 + G$. Then it can be seen that F is also a forcing subset for S. Thus,

$$f\gamma_{ch,coi}(K_1 + G) \leq f\gamma_{ch,coi}(S) \leq |F| = fsci(G).$$

Let $S_0 = \{v\} \cup A_0$ be a $\gamma_{ch,coi}$-set of $K_1 + G$ such that $f\gamma_{ch,coi}(K_1 + G) = f\gamma_{ch,coi}(S_0)$. By Corollary 3, A_0 is an *sci*-set of G. Let F_0 be a forcing subset for S_0 with $f\gamma_{ch,coi}(S_0) = |F_0.|$

Suppose F_0 is not a forcing subset for A_0. Then there exists an *sci*-set $A_0^{'}$ of G with $A_0^{'} \neq A_0$ and $F_0 \subseteq A_0^{'}$. By Corollary 3, $S_0^{'} = \{v\} \cup A_0^{'}$ is a $\gamma_{ch,coi}$-set of $K_1 + G$. Since $A_0^{'} \neq A_0$, $S_0^{'} \neq S_0$. Thus, $F_0 \subseteq S_0^{'}$, a contradiction since F_0 is a forcing subset for S_0. Hence, F_0 is a forcing subset for A_0. Thus,

$$f\gamma_{ch,coi}(K_1 + G) = f\gamma_{ch,coi}(S_0) = |F_0| \geq fsci(A_0) \geq fsci(G).$$

Therefore, $f\gamma_{ch,coi}(K_1 + G) = fsci(G)$. □

Example 1. (1.) For the fan $F_n = K_1 + P_n$, where $n \geq 2$,

$$f\gamma_{ch,coi}(F_n) = fsci(P_n) = \begin{cases} 0, & \text{if } n = 2, 4 \text{ and } n > 5 \text{ is odd,} \\ 1, & \text{if } n = 3 \text{ and } n \geq 6 \text{ is even,} \\ 2, & \text{if } n = 5. \end{cases}$$

(2.) For the wheel $W_n = K_1 + C_n$, where $n \geq 3$,

$$f\gamma_{ch,coi}(W_n) = fsci(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ 1, & \text{if } n > 4 \text{ and } n \text{ is even,} \\ 2, & \text{if } n > 3 \text{ and } n \text{ is odd,} \\ 3, & \text{if } n = 4. \end{cases}$$

(3.) For the star $S_n = K_{1,n}$ of order $n + 1$,

$$f\gamma_{ch,coi}(S_n) = \begin{cases} 0, & \text{if } n = 1, \\ 1, & \text{if } n > 1. \end{cases}$$

Another consequence of Theorem 5 is the next corollary.

Corollary 5. Let G and H be any graphs with $|V(G)| < |V(H)|$ and $sci(H) = sci(G)$ or $|V(G)| = |V(H)|$ and $sci(H) < sci(G)$. Then $S \subseteq V(G + H)$ is a $\gamma_{ch,coi}$-set of $G + H$ if and only if $S = V(G) \cup S_H$ for some *sci*-set S_H of H.

Theorem 6. For any graphs G and H with $|V(G)| < |V(H)|$ and $sci(H) = sci(G)$, or $|V(G)| = |V(H)|$ and $sci(H) < sci(G)$. Then

$$f\gamma_{ch,coi}(G + H) = \begin{cases} 0, & \text{if } H \text{ has a unique } sci\text{-set,} \\ fsci(H), & \text{if } H \text{ has no unique } sci\text{-set.} \end{cases}$$

Proof: Suppose that H has a unique *sci*-set, say S_H. Then by Corollary 5, $V(G) \cup S_H$ is a unique $\gamma_{ch,coi}$-set of $G + H$. By Remark 1(i), $f\gamma_{ch,coi}(G + H) = 0$. Now, suppose that H has no unique *sci*-set. Let A be an *sci*-set of H and let F be a forcing subset for A such that $fsci(H) = fsci(A) = |F|$. By Corollary 5, $S = V(G) \cup A$ is a $\gamma_{ch,coi}$-set of $G + H$. Suppose F is not a forcing subset for S. Then there exists a $\gamma_{ch,coi}$-set $S^{'}$ of $G + H$ such that $S' \neq S$ and $F \subseteq S'$. By Corollary 5, $S' = V(G) \cup A'$ where A' is an *sci*-set of H. Since $S' \neq S$, $A' \neq A$. On the other hand, F being a forcing subset for A which is an *sci*-set of H implies that $F \subseteq V(H)$. Thus, $F \subseteq A'$, a contradiction since F is a forcing subset for A. Hence, F is a forcing subset for S. Thus,

$$f\gamma_{ch,coi}(G+H) \leq f\gamma_{ch,coi}(S) \leq |F| = fsci(H).$$

Let $S_0 = V(G) \cup A_0$ be a $\gamma_{ch,coi}$-set of $G+H$ such that $f\gamma_{ch,coi}(G+H) = f\gamma_{ch,coi}(S_0)$. By Corollary 5, A_0 is an *sci*-set of H. Let F_0 be a forcing subset for S_0 with $f\gamma_{ch,coi}(S_0) = |F_0.|$ Suppose F_0 is not a forcing subset for A_0. Then there exists an *sci*-set $A_0' \neq A_0$ of H such that $F_0 \subseteq A_0'$. By Corollary 5, $S_0' = V(G) \cup A_0'$ is a $\gamma_{ch,coi}$-set of $G+H$ with $F_0 \subseteq S_0'$ and $S_0' \neq S_0$. This is a contradiction since F_0 is a forcing subset for S_0. Thus, F_0 is a forcing subset for A_0. Hence,

$$f\gamma_{ch,coi}(G+H) = f\gamma_{ch,coi}(S_0) = |F_0| \geq fsci(A_0) \geq fsci(H).$$

Therefore, $f\gamma_{ch,coi}(G+H) = fsci(H)$. □

Example 2. Let $G = C_3$ and $H = P_7$. Then $|V(C_3)| < |V(P_7)|$ and $sci(C_3) = 3 = sci(P_7)$. Since P_7 has a unique *sci*-set, $f\gamma_{ch,coi}(C_3 + P_7) = 0$.

Example 3. Let $G = C_3$ and $H = P_3$. Then $|V(C_3)| = |V(P_3)|$ and $sci(P_3) = 2 < 3 = sci(C_3)$. Since P_3 has no unique *sci*-sets, $f\gamma_{ch,coi}(C_3+P_3) = fsci(P_3) = 1$.

5. Forcing Connected Co-Independent Hop Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$.

Remark 3. Let G be a connected graph. Then
(i) $fcoi(G) = 0$ if and only if G has a unique *coi*-set, and
(ii) $fcoi(G) = 1$ if and only if G has at least two *coi*-sets, one of which, say B, contains an element which is not found in any *coi*-set of G.

Theorem 7. Let G be a connected graph. Then $fcoi(G) = coi(G)$ if and only if for all *coi*-set B of G and for each $v \in B$, there exists $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is a *coi*-set of G.

Proof: Suppose that $fcoi(G) = coi(G)$. Let B be a *coi*-set of G such that $fcoi(G) = |B| = coi(G)$, that is, B is the only forcing subset for itself. Let $v \in B$. Since $B \setminus \{v\}$ is not a forcing subset for B, there exists a $u_v \in V(G) \setminus B$ such that

$\big[B \setminus \{v\}\big] \cup \{u_v\}$ is a *coi*-set of G.

Conversely, suppose that every *coi*-set $B^{'}$ of G satisfies the given condition. Let B be a *coi*-set of G such that $fcoi(G) = fcoi(B)$. Suppose further that B has a forcing subset Q with $|Q| < |B|$, that is, $B = Q \cup P$ where $P = \{z \in B : z \notin Q\}$. Pick $z \in P$. By assumption, there exists $u_z \in V(G) \setminus B$ such that $\big[B \setminus \{z\}\big] \cup \{u_z\} = T$ is a *coi*-set of G. Hence, $T = Q \cup R$, where $R = \big[P \setminus \{z\}\big] \cup \{u_z\}$, is a *coi*-set containing Q, a contradiction. Hence, B is the only forcing subset for B. Therefore, $fcoi(G) = coi(G)$. □

Proposition 7. For any complete graph K_n with $n \geq 1$ vertices,

$$fcoi(K_n) = \begin{cases} 0, & \text{if } n = 1, \\ n-1, & \text{if } n \geq 2. \end{cases}$$

Proof: Suppose that $V(K_n) = \big\{v_1, v_2, \ldots, v_n\big\}$. Clearly, $fcoi(K_1) = 0$. If $n = 2$, then K_n has *coi*-set $R_1 = \{v_1\}$ and $R_2 = \{v_2\}$ which are the only *coi*-sets of K_n with $v_1 \in R_1$ and $v_1 \notin R_2$. By Remark 3(ii), $fcoi(K_n) = n - 1 = 1$.

Suppose that $n > 2$. Then the *coi*-sets of K_n are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, $\ldots$, $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(K_n) \setminus B_j$ such that $\big[B_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a *coi*-set of G. Hence, by Theorem 7, $fcoi(K_n) = n - 1$. □

Proposition 8. For any path P_n with $n \geq 1$ vertices,

$$fcoi(P_n) = \begin{cases} 0, & \text{if } n = 1, 3 \text{ and } n \geq 5 \text{ is odd}, \\ 1, & \text{if } n = 2, 4 \text{ and } n \geq 6 \text{ is even}. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. Clearly, $fcoi(P_1) = fcoi(P_3) = 0$ and $fcoi(P_2) = 1$. If $n = 4$, then P_n has *coi*-sets $B_1 = \{v_1, v_3\}$, $B_2 = \{v_2, v_4\}$ and $B_3 = \{v_2, v_3\}$ which are the only *coi*-sets of P_n with $v_4 \in B_2$ and $v_4 \notin B_1, B_3$. Thus, by Remark 3(ii), $fcoi(P_n) = 1$.

Now, suppose that $n \geq 5$ and n is odd, then clearly $B = \{v_2, v_4, v_6, \ldots, v_{n-3}, v_{n-1}\}$ is the only *coi*-set of P_n. Thus, by Remark 3(i), $fcoi(B) = 0 = fcoi(P_n)$. Next, suppose that $n \geq 6$ and n is even. Then P_n has *coi*-sets $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $S_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ which are the only *coi*-set of P_n with $v_3 \in S_1$ and $v_3 \notin S_2$. Hence, by Remark 3(ii), $fcoi(P_n) = 1$. □

Proposition 9. For any cycle C_n with $n \geq 3$ vertices,

$$fcoi(C_n) = \begin{cases} 1, & \text{if } n = 4 \text{ and } n > 4 \text{ is even}, \\ 2, & \text{if } n = 3 \text{ and } n > 3 \text{ is odd}. \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. It can be verified that $fcoi(C_4) = 1$. Suppose that $n = 3$. Then the *coi*-sets of C_3 are $Q_1 = \{v_1, v_2\}$, $Q_2 = \{v_2, v_3\}$ and $Q_3 = \{v_1, v_3\}$. Clearly, for each $v_i \in Q_j$ where $i, j \in \{1, 2, 3\}$, there exists $v_k \in V(C_3) \setminus Q_j$ such that $\big[Q_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a *coi*-set of G. Thus, by Theorem 7, $fcoi(C_3) = 2$. Now, suppose that $n > 4$ and n is even. Then $B_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $B_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ are the only *coi*-sets of C_n with $v_3 \in B_1$ and $v_3 \notin B_2$. Thus, by Remark 3(ii), $fcoi(B_1) = 1 = fcoi(C_n)$. Next, suppose that $n > 3$ and n is odd. Then

$$\begin{aligned} Q_1 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}, \\ Q_2 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\}, \\ Q_3 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_n\} \text{ and,} \\ Q_4 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\} \end{aligned}$$

are *coi*-sets of C_n. Hence, no vertex of C_n is contained in a unique *coi*-set. Thus, $fcoi(C_n) \geq 2$. Clearly, $\{v_1, v_n\}$ is uniquely contained in Q_1. Therefore, $fcoi(Q_1) = 2 = fcoi(C_n)$. □

In view of Theorem 2, we have the following theorem.

Theorem 8. Let G be a nontrivial connected graph and H be any graph. A set $S \subseteq V(G \circ H)$ is a connected co-independent hop dominating set of $G \circ H$ if and only if $S = V(G) \cup \big(\bigcup_{v \in V(G)} S_v\big)$ where S_v is a co-independent set of H^v for each $v \in V(G)$.

The next result is a restatement of Corollary 2.

Corollary 6. Let G be a nontrivial connected graph and H be any graph. A set $S \subseteq V(G \circ H)$ is a $\gamma_{ch,coi}$-set of $G \circ H$ if and only if $S = V(G) \cup \big(\bigcup_{v \in V(G)} S_v\big)$ where S_v is a *coi*-set of H^v for each $v \in V(G)$. In particular, $\gamma_{ch,coi}(G \circ H) = |V(G)|\big(1 + coi(H)\big)$.

Theorem 9. Let G be a nontrivial connected graph of order n and H be any graph. Then

$$f\gamma_{ch,coi}(G \circ H) = \begin{cases} 0, & \text{if } H \text{ has a unique } coi\text{-set}, \\ n\big[fcoi(H)\big], & \text{if } H \text{ has no unique } coi\text{-set}. \end{cases}$$

Proof: Suppose H has a unique *coi*-set. For each $v \in V(G)$, let $P_v \subseteq V(H^v)$ be the unique *coi*-set of H^v. By Corollary 6, $S = V(G) \cup \Big(\bigcup_{v \in V(G)} P_v\Big)$ is the unique $\gamma_{ch,coi}$-set of $G \circ H$. Thus, by Remark 1(i), $f\gamma_{ch,coi}(G \circ H) = 0$. On the other hand, suppose that H does not have a unique *coi*-set. For each $v \in V(G)$, let $Q_v \subseteq V(H^v)$ be a *coi*-set of H^v with $fcoi(H^v) = fcoi(Q_v)$, and let $P_{Q_v} \subseteq Q_v$ be a forcing subset for Q_v with $fcoi(Q_v) = |P_{Q_v}|$. Then by Corollary 6, $S_Q = V(G) \cup \Big(\bigcup_{v \in V(G)} Q_v\Big)$ is a $\gamma_{ch,coi}$-set of $G \circ H$. Let $C = \bigcup_{v \in V(G)} P_{Q_v}$. Then C is a forcing subset for S_Q. Thus,

$$f\gamma_{ch,coi}(G \circ H) \leq f\gamma_{ch,coi}(S_Q) \leq |C| = n\big[fcoi(H)\big].$$

Next, let S' be a $\gamma_{ch,coi}$-set of $G \circ H$ such that $f\gamma_{ch,coi}(G \circ H) = f\gamma_{ch,coi}(S')$. Then by Corollary 6, let $S' = V(G) \cup \Big(\bigcup_{v \in V(G)} R_v \Big)$ where R_v is a *coi*-set of H^v for each $v \in V(G)$. Let C' be a forcing subset for S' such that $f\gamma_{ch,coi}(S') = |C'|$. Suppose that there exists $w \in V(G)$ such that $C' \cap R_w = C_w$ is not a forcing subset for R_w. Let $R_w^{'}$ be a *coi*-set of H^w with $R_w^{'} \neq R_w$. Then

$$S'' = V(G) \cup \Big(\bigcup_{v \in V(G) \setminus \{w\}} R_v \Big) \cup R_w^{'}$$

is a $\gamma_{ch,coi}$-set of $G \circ H$ with $S' \neq S''$ and $C' \subseteq S''$, a contradiction. Thus, $S_v = C' \cap R_v$ is a forcing subset for R_v for each $v \in V(G)$. Let $S_0 = \bigcup_{v \in V(G)} S_v$. Then

$$f\gamma_{ch,coi}(G \circ H) = |C'| \geq |S_0| = \sum_{v \in V(G)} |S_v| \geq \sum_{v \in V(G)} fcoi(H^v) = |V(G)| fcoi(H).$$

Therefore, $f\gamma_{ch,coi}(G \circ H) = n\big[fcoi(H)\big]$. □

Example 4. Let $G = K_2$ and $H = P_5$. Since P_5 has a unique *coi*-set, $f\gamma_{ch,coi}(K_2 \circ P_5) = 0$.

Example 5. Let $G = C_3$ and $H = P_4$. Since P_4 has no unique *coi*-sets, $f\gamma_{ch,coi}(C_3 \circ P_4) = 3\big[fcoi(P_4)\big] = 3 \cdot 1 = 3$.

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