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On b-open sets via infra soft topological spaces

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Abstract. This work aims to present the concept of infra soft b-open sets (IS-b-open sets) as a generalized new class of infra open sets (IS-open sets). We first investigate their basic properties and study their behaviours under infra soft homeomorphism maps. Then, we establish some soft operators such as interior, closure, limit and boundary using IS-b-open sets and IS-b-closed sets. The relationships between them are illustrated and discussed. Finally, we display some soft maps (S-map) defined using IS-b-open and IS-b-closed sets and scrutinize their master properties.

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1. Introduction

Molodtsov [62] proposed the idea of soft set (S-set) as a new mathematical tool to deal with vagueness. He presented some of its applications to some areas. Since the advent of S-set, they have been applied to address some problems and phenomena in different disciplines such as information system [9], economy [14], linear equations [27], computer science [50] and decision-making problems [53].

The main operations and operators via S-set theory such as the difference, intersection, and union between two S-sets, and a complement of an S-set were introduced by Maji et

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al. [61]. Then, new operations and operators between S-sets were presented in [27, 44]. Some extensions of S-sets were proposed with the goal of expanding the applications of S-sets such as bipolar S-sets [8] and double-framed S-sets [38].

Recently, topology has been applied to model some real-life issues as showed in [1, 11, 15, 16, 32, 46, 56, 64]. To study topology via S-set theory, Çağman et al. [51] and Shabir and Naz [65], in 2011, introduced the concept of soft topology(ST). They followed different techniques for studying ST. This article follows Shabir and Naz' technique which is defined an ST over a fixed set of universe and a fixed set of parameters. The basic concepts and notions of classical topology have been studied in ST such as caliber and chain conditions [43], compactness [2, 26, 28, 29, 40, 49], local compactness [47] separation axioms [18, 21, 22, 42, 45, 52], fixed point theorem [7, 19], connectedness [54, 57, 60], mappings [20, 25, 30, 58], bioperators [48], covering properties [34, 35, 59], sum of topologies [31, 36] and generalized open sets [3]. Additionally, STs and supra STs were discussed in ordered settings as given in [24]. Al-shami and Kočinac [33] elucidated the conditions under which the soft operators and classical operators of interior and closure are interchangeable. It should be noted that some classical topological properties were generalized to STs without consideration for the divergences between STs and classical topologies, which causes some incorrect forms of some results; so some articles were conducted to put forward the correct frame of these results via soft structures; see [4–6].

In 2021, Al-shami [13] familiarized the structure of infra soft topologies(ISTs) [13] and showed the motivations for studying this structure. He with his coauthors continued investigating several topological concepts and properties via this structure such as compactness [12], homeomorphisms [10], connectedness [17], separation axioms [37, 39], infra soft semi-open (IS-semi-open) [23] and infra soft pre-open (IS-pre-open) sets [41]. In this article, we display the notion of soft b-open sets (S-b-open sets) and applied to initiate new operators and mappings via infra soft structures.

The structure of this article is designed as follows. In Sect. 2, we recall the main ideas and findings that make this work self-contained. In Sect. 3, we introduce the notion of infra soft b-open sets(IS-b-open sets) and establish its master characterizations. In Sect. 4, we define new operators and discuss their main properties. In Sect. 5, we explore novel kinds of mappings and demonstrated their features. Ultimately, we give the main contributions of the article and propose some future works.

2. Preliminaries

2.1. Soft set theory

Definition 1. *[62] A mapping $\mathcal{H}$ from a set of parameters $\mathcal{O}$ into 2^X, where 2^X is the power set of X, is called an S-set denoted by $(\mathcal{H}, \mathcal{O})$, and it can written as follows $(\mathcal{H}, \mathcal{O}) = \{(o, \mathcal{H}(o)) : o \in \mathcal{O}$ and $\mathcal{H}(o) \in 2^X\}$.*

$C(X_{\mathcal{O}})$ refers to the class of all S-sets over X with the set of parameters $\mathcal{O}$.

Definition 2. *[44] A complement of an S-set $(\mathcal{H},\mathcal{O})$, denoted by $(\mathcal{H}^c,\mathcal{O})$, provided that a map $\mathcal{H}^c : \mathcal{O} \to 2^X$ is given by $\mathcal{H}^c(o) = X \setminus \mathcal{H}(o)$ for each $o \in \mathcal{O}$.*

Definition 3. *[61] If $\mathcal{H}(o) = \emptyset$ (resp., $\mathcal{H}(o) = X$) for all $o \in \mathcal{O}$, then $(\mathcal{H},\mathcal{O})$ is called a null S-set (resp., an absolute) S-set over X.*

Φ and $\widetilde{X}$ are the symbols of null S-set and absolute S-set,respectively.

Definition 4. *[63] $(\mathcal{H},\mathcal{O})$ is called a soft point (S-point) on X if there is $o \in \mathcal{O}$ such that $\mathcal{H}(o) = x \in X$ and $\mathcal{H}(o') = \emptyset$ for each $o' \neq o$. The symbol of an S-point will be δ_o^x.*

Definition 5. *[44] The intersection of S-sets $(\mathcal{H},\mathcal{O})$ and $(\mathcal{F},\Delta)$ on X, symbolized by $(\mathcal{H},\mathcal{O})\widetilde{\cap}(\mathcal{F},\Delta)$, is an S-set $(\mathcal{G},T)$, where $T = \mathcal{O} \cap \Delta \neq \emptyset$, and a map $\mathcal{G} : T \to 2^X$ is given by $\mathcal{G}(o) = \mathcal{H}(o) \cap \mathcal{F}(o)$ for each $o \in T$.*

Definition 6. *[61] The union of S-sets $(\mathcal{H},\mathcal{O})$ and $(\mathcal{F},\Delta)$ on X, symbolized by $(\mathcal{H},\mathcal{O})\widetilde{\cup}(\mathcal{F},\Delta)$, is an S-set $(\mathcal{G},T)$, where $T = \mathcal{O} \cup \Delta$ and a map $T : \mathcal{O} \to 2^X$ is given as follows:*

$$\mathcal{G}(o) = \begin{cases} \mathcal{H}(o) & : \ o \in \mathcal{O} \setminus \Delta \\ \mathcal{F}(o) & : \ o \in \Delta \setminus \mathcal{O} \\ \mathcal{H}(o) \cup \mathcal{F}(o) & : \ o \in \mathcal{O} \cap \Delta \end{cases}$$

Definition 7. *[55] A S-set $(\mathcal{H},\mathcal{O})$ is a subset of an S-set $(\mathcal{F},\Delta)$, symbolized by $(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{F},\Delta)$, if $\mathcal{O} \subseteq \Delta$ and $\mathcal{H}(o) \subseteq \mathcal{F}(o)$ for all $o \in \mathcal{O}$. If $(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{F},\Delta)$ and $(\mathcal{F},\Delta)\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$, then $(\mathcal{H},\mathcal{O})$ and $(\mathcal{F},\Delta)$ are called soft equal.*

The definition of soft maps(S-map) in [58] was adjusted as follows.

Definition 8. *[10] Let $f : X \to \mathcal{S}$ and $\psi : \mathcal{O} \to \Delta$ be two maps. A S-map f_ψ of $C(X_{\mathcal{O}})$ into $C(\mathcal{S}_\Delta)$ is a relation such that any S-point in $C(X_{\mathcal{O}})$ is related to one and only one S-point in $C(\mathcal{S}_\Delta)$ such that*

$$f_\psi(\delta_o^x) = \delta_{\psi(o)}^{f(x)} \text{ for any } \delta_o^x \in C(X_{\mathcal{O}}).$$

In addition, $f_\psi^{-1}(\delta_\gamma^y) = \bigsqcup_{\substack{\lambda \in \psi^{-1}(\gamma) \\ x \in f^{-1}(y)}} \delta_\lambda^x$ for any $\delta_\gamma^y \in C(\mathcal{S}_\Delta)$.

Definition 9. *[63] For an S-map $f_\psi : C(X_{\mathcal{O}}) \to C(\mathcal{S}_\Delta)$, if f and ψ are injective (resp., surjective, bijective), then f_ψ is called injective (resp., surjective, bijective).*

2.2. Infra soft topological spaces

Definition 10. *[13] A subfamily μ of $C(X_{\mathcal{O}})$ is called an infra soft topology(IST) on X if it contains Φ and it is closed under finite intersection.*

The triple $(X,\mu,\mathcal{O})$ is called an ISTS. The elements of μ are called IS-open sets and their complements are called IS-closed sets.

Definition 11. *[13] Let $(\mathcal{H},\mathcal{O})$ be a subset of $(X,\mu,\mathcal{O})$.*

(i) *the IS-closure points of* $(\mathcal{H}, \mathcal{O})$*, denoted by* $cl(\mathcal{H}, \mathcal{O})$*, is the intersection of all IS-closed subsets of* $(X, \mu, \mathcal{O})$ *containing* $(\mathcal{H}, \mathcal{O})$.

(ii) *the IS-interior points of* $(\mathcal{H}, \mathcal{O})$*, denoted by* $int(\mathcal{H}, \mathcal{O})$ *is the union of all IS-open subsets of* $(X, \mu, \mathcal{O})$ *which are contained in* $(\mathcal{H}, \mathcal{O})$.

Proposition 1. *[13] Let* $(\mathcal{H}, \mathcal{O})$ *and* $(\mathcal{F}, \mathcal{O})$ *subsets of an ISTS* $(X, \mu, \mathcal{O})$*. Then*

(i) $cl[(\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{F}, \mathcal{O})] = cl(\mathcal{H}, \mathcal{O})\widetilde{\cup}cl(\mathcal{F}, \mathcal{O})$*, and*

(ii) $int[(\mathcal{H}, \mathcal{O})\widetilde{\cap}(\mathcal{F}, \mathcal{O})] = int(\mathcal{H}, \mathcal{O})\widetilde{\cap}int(\mathcal{F}, \mathcal{O})$.

Proposition 2. *[13] Let* $(\mathcal{H}, \mathcal{O})$ *be an IS-open set. Then*

$$(\mathcal{H}, \mathcal{O})\widetilde{\cap}cl(\mathcal{F}, \mathcal{O})\widetilde{\subseteq}cl[(\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{F}, \mathcal{O})] \text{ for any } (\mathcal{F}, \mathcal{O}) \text{ in } (X, \mu, \mathcal{O}).$$

Proposition 3. *[13] Let* $(\mathcal{H}, \mathcal{O})$ *be an IS-closed set. Then*

$$int[(\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{F}, \mathcal{O})]\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})\widetilde{\cup}int(\mathcal{F}, \mathcal{O}) \text{ for any } (\mathcal{H}, \mathcal{O}) \text{ in } (X, \mu, \mathcal{O}).$$

Definition 12. *[10] A bijective S-map* $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ *is said to be an IS-homeomorphism if it is IS-open (i.e,the image of any IS-open set is IS-open), and IS-continuous (i.e,the pre-image of any IS-open set is IS-open).*

We call a property which is kept by any IS-homeomorphism an IS-topological property.

Definition 13. *[10] Let* $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ *be an S-map and* $\mathcal{M} \neq \emptyset$ *be a subset of* X*. A S-map* $f_{\psi_{|\mathcal{M}}} : (\mathcal{M}, \mu_{\mathcal{M}}, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ *which given by* $f_{\psi_{|\mathcal{M}}}(\delta_o^m) = f_\psi(\delta_o^m)$ *for every* $\delta_o^m \in \widetilde{\mathcal{M}}$ *is called a restriction S-map of* f_ψ *on* $\mathcal{M}$.

Lemma 1. *[23, 41] Let* $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ *be an IS-homeomorphism map. Then for any* $(\mathcal{H}, \mathcal{O}_1)$ *we have:*

(i) $f_\psi(int(\mathcal{H}, \mathcal{O}_1)) = int(f_\psi(\mathcal{H}, \mathcal{O}_1))$.

(ii) $f_\psi(cl(\mathcal{H}, \mathcal{O}_1)) = cl(f_\psi(\mathcal{H}, \mathcal{O}_1))$.

3. Main properties of infra soft b-open sets

Definition 14. *A S-set* $(\mathcal{H}, \mathcal{O})$ *in an ISTS* $(X, \mu, \mathcal{O})$ *is said to be IS-b-open if* $(\mathcal{H}, \mathcal{O})\widetilde{\subseteq} int(cl(\mathcal{H}, \mathcal{O}))\widetilde{\cup}cl(int(\mathcal{H}, \mathcal{O}))$*. Its complement is said to be an IS-b-closed set.*

Proposition 4. *Every IS-semi-open (IS-pre-open) set is IS-b-open.*

Proof. Let $(\mathcal{H}, \mathcal{O})$ be an IS-semi-open (resp. IS-pre-open) set. Then, $(\mathcal{H}, \mathcal{O})\widetilde{\subseteq}cl(int(\mathcal{H}, \mathcal{O}))$ (resp. $(\mathcal{H}, \mathcal{O})\widetilde{\subseteq}int(cl(\mathcal{H}, \mathcal{O}))$). Automatically, we obtain $(\mathcal{H}, \mathcal{O})\widetilde{\subseteq}int(cl(\mathcal{H}, \mathcal{O}))\widetilde{\cup}cl(int(\mathcal{H}, \mathcal{O}))$, which means that $(\mathcal{H}, \mathcal{O})$ is IS-b-open.

The converse of the above proposition fails as the next example shows.

Example 1. *Let* $X = \{x_1, x_2, x_3\}$ *and* $\mathcal{O} = \{o_1, o_2\}$*. Then* $\mu = \{\Phi, \widetilde{X}, (\mathcal{H}_1, \mathcal{O}), (\mathcal{H}_2, \mathcal{O})\}$ *is an IST on* X*, where*

$(\mathcal{H}_1, \mathcal{O}) = \{(o_1, \{x_1\}), (o_2, \{x_2, x_3\})\}$ *and*
$(\mathcal{H}_2, \mathcal{O}) = \{(o_1, \{x_3\}), (o_2, \{x_1\})\}$.

Let $(\mathcal{H}_5, \mathcal{O}) = \{(o_1, \{x_3\}), (o_2, \{x_3\})\}$ *and* $(\mathcal{H}_6, \mathcal{O}) = \{(o_1, \{x_1, x_2\}), (o_2, \{x_2, x_3\})\}$*. Then* $(\mathcal{H}_5, \mathcal{O})$ *and* $(\mathcal{H}_6, \mathcal{O})$ *are IS-b-open sets because* $cl(\mathcal{H}_5, \mathcal{O}) = \widetilde{X}$ *and* $cl(int(\mathcal{H}_6, \mathcal{O})) = (\mathcal{H}_6, \mathcal{O})$*. But* $(\mathcal{H}_5, \mathcal{O})$ *is not IS-semi-open because* $int(\mathcal{H}_5, \mathcal{O}) = \Phi$*, and* $(\mathcal{H}_6, \mathcal{O})$ *is not IS-pre-open because* $int(cl(\mathcal{H}_5, \mathcal{O})) = \{(o_1, \{x_1\}), (o_2, \{x_2, x_3\})\} \widetilde{\not\supseteq} (\mathcal{H}_6, \mathcal{O})$.

Proposition 5. *The unions of IS-b-open sets is IS-b-open.*

Proof. Consider $\{(\mathcal{H}_j, \mathcal{O}) : j \in J\}$ as a family of IS-b-open sets. Suppose $J \neq \emptyset$. Then $(\mathcal{H}_j, \mathcal{O}) \widetilde{\subseteq} int(cl(\mathcal{H}, \mathcal{O})) \widetilde{\cup} cl(int(\mathcal{H}, \mathcal{O}))$ for each $j \in J$. Thus, $\widetilde{\cup}_{j\in J}(\mathcal{H}_j, \mathcal{O}) \widetilde{\subseteq} \widetilde{\cup}_{j\in J}[int(cl(\mathcal{H}, \mathcal{O})) \widetilde{\cup} cl(int(\mathcal{H}, \mathcal{O}))] \widetilde{\subseteq} int(cl(\widetilde{\cup}_{j\in J}(\mathcal{H}_j, \mathcal{O}))) \widetilde{\cup}\ cl(int(\widetilde{\cup}_{j\in J}(\mathcal{H}_j, \mathcal{O})))$. Hence, $\widetilde{\cup}_{j\in J}(\mathcal{H}_j, \mathcal{O})$ is IS-b-open.

Corollary 1. *The intersections of IS-b-closed sets is IS-b-closed.*

Proposition 6. *If* $(\mathcal{H}_1, \mathcal{O})$ *is IS-open and* $(\mathcal{H}_2, \mathcal{O})$ *is IS-b-open, then* $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O})$ *is IS-b-open.*

Proof. Let $(\mathcal{H}_1, \mathcal{O})$ and $(\mathcal{H}_2, \mathcal{O})$ be as given in the proposition. Then $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O}) \widetilde{\subseteq} (\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} [int(cl(\mathcal{H}_2, \mathcal{O})) \widetilde{\cup} cl(int(\mathcal{H}_2, \mathcal{O}))] = [(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} int(cl(\mathcal{H}_2, \mathcal{O}))] \widetilde{\cup} [(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap}\ cl(int(\mathcal{H}_2, \mathcal{O}))]$. It follows from Proposition 2 that $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap}\ int(cl(\mathcal{H}_2, \mathcal{O})) \widetilde{\subseteq} int(cl[(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O})]$ and $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap}\ cl(int(\mathcal{H}_2, \mathcal{O})) \widetilde{\subseteq} cl(int[(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O})]$ Hence, $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O})$ is an IS-b-open set.

Corollary 2. *If* $(\mathcal{H}_1, \mathcal{O})$ *is IS-closed and* $(\mathcal{H}_2, \mathcal{O})$ *is IS-b-closed, then* $(\mathcal{H}_1, \mathcal{O}) \widetilde{\cap} (\mathcal{H}_2, \mathcal{O})$ *is IS-b-closed.*

Proposition 7. *The image of an IS-b-open set under an IS-homeomorphism is IS-b-open.*

Proof. Consider $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ as an IS-continuous map and let $(\mathcal{H}, \mathcal{O}_1)$ be an IS-b-open subset of $(X_1, \mu_1, \mathcal{O}_1)$. Then $f_\psi(\mathcal{H}, \mathcal{O}_1) \widetilde{\subseteq} f_\psi[cl(int(\mathcal{H}, \mathcal{O}_1)) \widetilde{\cup} int(cl(\mathcal{H}, \mathcal{O}_1))]$. It follows from Lemma 1 that $f_\psi(\mathcal{H}, \mathcal{O}_1) \widetilde{\subseteq} cl(int(f_\psi(\mathcal{H}, \mathcal{O}_1))) \widetilde{\cup} int(cl(f_\psi(\mathcal{H}, \mathcal{O}_1)))$. Hence, $f_\psi(\mathcal{H}, \mathcal{O}_1)$ is an IS-b-open subset of $(X_2, \mu_2, \mathcal{O}_2)$, as required.

4. Infra b-interior, infra b-closure, infra b-limit and infra b-boundary soft points of a soft set

Definition 15. *Let* $(\mathcal{H}, \mathcal{O})$ *be an S-set in* $(X, \mu, \mathcal{O})$*. Then:*

(i) *the IS-b-interior of* $(\mathcal{H}, \mathcal{O})$*, denoted by* $bint(\mathcal{H}, \mathcal{O})$*, is the union of all IS-b-open sets that are contained in* $(\mathcal{H}, \mathcal{O})$.

(ii) *the IS-b-closure of* $(\mathcal{H},\mathcal{O})$*, denoted by* $bcl(\mathcal{H},\mathcal{O})$*, is the intersection of all IS-b-closed sets containing* $(\mathcal{H},\mathcal{O})$.

Proposition 8. *We have the following properties.*

(i) $(\mathcal{H},\mathcal{O})$ *is an IS-b-open subset of* $(X,\mu,\mathcal{O})$ *iff* $bint(\mathcal{H},\mathcal{O}) = (\mathcal{H},\mathcal{O})$.

(ii) $(\mathcal{H},\mathcal{O})$ *is an IS-b-closed subset of* $(X,\mu,\mathcal{O})$ *iff* $bcl(\mathcal{H},\mathcal{O}) = (\mathcal{H},\mathcal{O})$.

Proof. It comes from Proposition 5 and Corollary 1.

The two characterizations given in the the above proposition are generally false for IS-open and IS-closed sets.

Proposition 9. *Let* $(\mathcal{H},\mathcal{O})$ *be a subset of* $(X,\mu,\mathcal{O})$.

(i) $\delta_o^x \in bint(\mathcal{H},\mathcal{O})$ *iff there exists an IS-b-open set* $(\mathcal{F},\mathcal{O})$ *such that* $\delta_o^x \in (\mathcal{F},\mathcal{O})\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$.

(ii) $\delta_o^x \in bcl(\mathcal{H},\mathcal{O})$ *iff the intersection of any IS-b-open set* $(\mathcal{F},\mathcal{O})$ *containing* δ_o^x *and* $(\mathcal{H},\mathcal{O})$ *is non-null.*

Proof. The proof of (i) is obvious, so we prove (ii).

Let $\delta_o^x \in bcl(\mathcal{H},\mathcal{O})$. Then every IS-$b$-closed set contains $(\mathcal{H},\mathcal{O})$ contains δ_o^x as well. Suppose that there exists an IS-b-open set $(\mathcal{F},\mathcal{O})$ containing δ_o^x such that $(\mathcal{H},\mathcal{O})\widetilde{\cap}(\mathcal{F},\mathcal{O}) = \Phi$. Therefore, $(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{F}^c,\mathcal{O})$ which means that $\delta_o^x \notin bcl(\mathcal{H},\mathcal{O})$. This is a contradiction. Conversely, suppose that there exists an IS-b-open set $(\mathcal{F},\mathcal{O})$ containing δ_o^x such that $(\mathcal{H},\mathcal{O})\widetilde{\cap}(\mathcal{F},\mathcal{O}) = \Phi$. Therefore, $bcl(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{F}^c,\mathcal{O})$ which means that $\delta_o^x \notin bcl(\mathcal{H},\mathcal{O})$. Hence, the result holds.

Proposition 10. *Let* $(\mathcal{H},\mathcal{O})$ *be a subset of* $(X,\mu,\mathcal{O})$*. Then:*

(i) $(bint(\mathcal{H},\mathcal{O}))^c = bcl(\mathcal{H}^c,\mathcal{O})$.

(ii) $(bcl(\mathcal{H},\mathcal{O}))^c = bint(\mathcal{H}^c,\mathcal{O})$.

Proof. (i): $(bint(\mathcal{H},\mathcal{O}))^c = \{\widetilde{\bigcup}_{j\in J}(\mathcal{F}_j,\mathcal{O}) : (\mathcal{F}_j,\mathcal{O})$ is an IS-b-open set contained in $(\mathcal{H},\mathcal{O})\}^c = \widetilde{\bigcap}_{j\in J}\{(\mathcal{F}_j^c,\mathcal{O}) : (\mathcal{F}_j^c,\mathcal{O})$ is an IS-b-closed set containing $(\mathcal{H}^c,\mathcal{O})\} = bcl(\mathcal{H}^c,\mathcal{O})$.

The proof of (ii) is similar to (i).

Proposition 11. *Let* $(\mathcal{F},\mathcal{O})$ *be an IS-open set and* $(\Lambda,\mathcal{O})$ *be an IS-closed set in* $(X,\mu,\mathcal{O})$*. Then:*

(i) $(\mathcal{F},\mathcal{O})\widetilde{\cap}bcl(\mathcal{H},\mathcal{O})\widetilde{\subseteq}bcl((\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}))$.

(ii) $bint((\Lambda,\mathcal{O})\widetilde{\cup}(\mathcal{H},\mathcal{O}))\widetilde{\subseteq}(\Lambda,\mathcal{O})\widetilde{\cup}bint(\mathcal{H},\mathcal{O})$.

Proof. (i): Let $\delta_o^x \in (\mathcal{F},\mathcal{O})\widetilde{\cap} bcl(\mathcal{H},\mathcal{O})$. Then $\delta_o^x \in (\mathcal{F},\mathcal{O})$ and $\delta_o^x \in bcl(\mathcal{H},\mathcal{O})$. This implies $(\mathcal{U},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}) \neq \Phi$ for every IS-b-open set $(\mathcal{U},\mathcal{O})$ containing δ_o^x. It follows from Proposition 6 that $(\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{U},\mathcal{O})$ is an IS-b-open set containing δ_o^x. Therefore, $[(\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{U},\mathcal{O})]\widetilde{\cap}(\mathcal{H},\mathcal{O}) \neq \Phi$. Now, $(\mathcal{U},\mathcal{O})\widetilde{\cap}[(\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O})] \neq \Phi$ which means that $\delta_o^x \in bcl((\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}))$. Hence, $(\mathcal{F},\mathcal{O})\widetilde{\cap} bcl(\mathcal{H},\mathcal{O})\widetilde{\subseteq} bcl((\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}))$.

One can prove (ii) following similar arguments.

Theorem 1. *Let $(\mathcal{H},\mathcal{O})$ and $(\mathcal{F},\mathcal{O})$ are in $(X,\mu,\mathcal{O})$. Then we have:*

(i) $bint(\widetilde{X}) = \widetilde{X}$.

(ii) $bint(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$.

(iii) *If* $(\mathcal{F},\mathcal{O})\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$, *then* $bint(\mathcal{F},\mathcal{O})\widetilde{\subseteq} bint(\mathcal{H},\mathcal{O})$.

(iv) $bint(bint(\mathcal{H},\mathcal{O})) = bint(\mathcal{H},\mathcal{O})$.

(v) $bint(\mathcal{F},\mathcal{O})\widetilde{\cap} bint(\mathcal{H},\mathcal{O})\widetilde{\subseteq} bint((\mathcal{F},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}))$.

Proof. (i): Since $\widetilde{X}$ is IS-b-open, $bint(\widetilde{X}) = \widetilde{X}$.
(ii) and (iii) are obvious.
(iv): Clearly $bint(bint(\mathcal{H},\mathcal{O}))$ is the largest IS-b-open set contained in $bint(\mathcal{H},\mathcal{O})$; however, $bint(\mathcal{H},\mathcal{O})$ is an IS-b-open set; hence, $bint(bint(\mathcal{H},\mathcal{O})) = bint(\mathcal{H},\mathcal{O})$.
(v): It comes from (iii).

Theorem 2. *Let $(\mathcal{H},\mathcal{O})$ and $(\mathcal{F},\mathcal{O})$ be subsets of $(X,\mu,\mathcal{O})$. Then we have:*

(i) $bcl(\Phi) = \Phi$.

(ii) $(\mathcal{H},\mathcal{O})\widetilde{\subseteq} bcl(\mathcal{H},\mathcal{O})$.

(iii) *If* $(\mathcal{F},\mathcal{O})\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$, *then* $bcl(\mathcal{F},\mathcal{O})\widetilde{\subseteq} bcl(\mathcal{H},\mathcal{O})$.

(iv) $bcl(bcl(\mathcal{H},\mathcal{O}))\widetilde{\subseteq} bcl(\mathcal{H},\mathcal{O})$.

(v) $bcl((\mathcal{F},\mathcal{O})\widetilde{\cup}(\mathcal{H},\mathcal{O})) = bcl(\mathcal{F},\mathcal{O})\widetilde{\cup} bcl(\mathcal{H},\mathcal{O})$.

Proof. It can be proved following similar arguments given in the proof of Theorem 1.

Definition 16. *A S-point δ_o^x is called an IS-b-limit point of a subset $(\mathcal{H},\mathcal{O})$ of $(X,\mu,\mathcal{O})$ provided that $[(\mathcal{F},\mathcal{O})\backslash\delta_o^x]\widetilde{\cap}(\mathcal{H},\mathcal{O}) \neq \Phi$ for any IS-b-open set $(\mathcal{F},\mathcal{O})$ containing δ_o^x.*

The S-set of all IS-b-limit points of $(\mathcal{H},\mathcal{O})$ is called an infra b-derived S-set. It is denoted by $(\mathcal{H},\mathcal{O})^{bs\prime}$.

Proposition 12. *Consider $(\mathcal{F},\mathcal{O})$ and $(\mathcal{H},\mathcal{O})$ as S-sets in $(X,\mu,\mathcal{O})$. Then*

(i) $\Phi^{bs\prime} = \Phi$ *and* $\widetilde{X}^{bs\prime}\widetilde{\subseteq}\widetilde{X}$.

(ii) *If* $(\mathcal{F}, \mathcal{O})\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})$, *then* $(\mathcal{F}, \mathcal{O})^{bs\prime}\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})^{bs\prime}$.

(iii) *If* $\delta_o^x \in (\mathcal{H}, \mathcal{O})^{bs\prime}$, *then* $\delta_o^x \in ((\mathcal{H}, \mathcal{O}) \setminus \delta_o^x)^{bs\prime}$.

(iv) $(\mathcal{F}, \mathcal{O})^{bs\prime}\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}\widetilde{\subseteq}((\mathcal{F}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O}))^{bs\prime}$.

Proof. Straightforward.

Theorem 3. *Let* $(\mathcal{H}, \mathcal{O})$ *be an S-set in* $(X, \mu, \mathcal{O})$. *Then*

(i) *If* $(\mathcal{H}, \mathcal{O})$ *is an IS-b-closed set, then* $(\mathcal{H}, \mathcal{O})^{bs\prime} \subseteq (\mathcal{H}, \mathcal{O})$.

(ii) $((\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime})^{bs\prime}\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}$.

(iii) $bcl(\mathcal{H}, \mathcal{O}) = (\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}$.

Proof.

(i) Consider $(\mathcal{H}, \mathcal{O})$ as an IS-b-closed set such that $\delta_o^x \notin (\mathcal{H}, \mathcal{O})$. Then $\delta_o^x \in (\mathcal{H}^c, \mathcal{O})$. Now, $(\mathcal{H}^c, \mathcal{O})$ is an IS-b-open set such that $(\mathcal{H}^c, \mathcal{O})\widetilde{\cap}(\mathcal{H}, \mathcal{O}) = \Phi$ which means that $\delta_o^x \notin (\mathcal{H}, \mathcal{O})^{bs\prime}$. Thus, $(\mathcal{H}, \mathcal{O})^{bs\prime}\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})$.

(ii) Consider $\delta_o^x \notin (\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}$. Then $\delta_o^x \notin (\mathcal{H}, \mathcal{O})$ and $\delta_o^x \notin (\mathcal{H}, \mathcal{O})^{bs\prime}$. Therefore, there exists an IS-b-open set $(\mathcal{F}, \mathcal{O})$ such that

$$(\mathcal{F}, \mathcal{O})\widetilde{\cap}(\mathcal{H}, \mathcal{O}) = \Phi \tag{1}$$

This implies that

$$(\mathcal{F}, \mathcal{O})\widetilde{\cap}(\mathcal{H}, \mathcal{O})^{bs\prime} = \Phi \tag{2}$$

It follows from (1) and (2) that $(\mathcal{F}, \mathcal{O})\widetilde{\cap}((\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}) = \Phi$. Thus, $\delta_o^x \notin ((\mathcal{H}, \mathcal{O})\widetilde{\cup}\,(\mathcal{H}, \mathcal{O})^{bs\prime})^{bs\prime}$. Hence, $((\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime})^{bs\prime}\widetilde{\subseteq}((\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime})$, as required.

(iii) It is clear that $(\mathcal{H}, \mathcal{O})\widetilde{\cup}(\mathcal{H}, \mathcal{O})^{bs\prime}\widetilde{\subseteq}bcl(\mathcal{H}, \mathcal{O})$. Conversely, let $\delta_o^x \in bcl(\mathcal{H}, \mathcal{O})$. Then for every IS-$b$-open set containing δ_o^x we have $(\mathcal{H}, \mathcal{O})\widetilde{\cap}(\mathcal{F}, \mathcal{O}) \neq \Phi$. Without loss of generality, let $\delta_o^x \notin (\mathcal{H}, \mathcal{O})$. Then $[(\mathcal{H}, \mathcal{O})\setminus\delta_o^x]\widetilde{\cap}(\mathcal{F}, \mathcal{O}) \neq \Phi$. Consequentially, $\delta_o^x \in (\mathcal{H}, \mathcal{O})^{bs\prime}$. Hence, the proof is complete.

Definition 17. *The IS-b-boundary points of a subset* $(\mathcal{H}, \mathcal{O})$ *of* $(X, \mu, \mathcal{O})$, *denoted by* $bB(\mathcal{H}, \mathcal{O})$, *are all the S-points which belong to the complement of* $bint(\mathcal{H}, \mathcal{O})\widetilde{\cup}bint(\mathcal{H}^c, \mathcal{O})$.

Proposition 13. *Let* $(\mathcal{H}, \mathcal{O})$ *be an S-set in* $(X, \mu, \mathcal{O})$. *Then:*

(i) $bB(\mathcal{H}, \mathcal{O}) = bcl(\mathcal{H}, \mathcal{O})\widetilde{\cap}bcl((\mathcal{H}^c, \mathcal{O}))$.

(ii) $bB(\mathcal{H}, \mathcal{O}) = bcl(\mathcal{H}, \mathcal{O}) \setminus bint(\mathcal{H}, \mathcal{O})$.

Proof.

(i) $bB(\mathcal{H},\mathcal{O}) = \{\delta_o^x \in \widetilde{X} : \delta_o^x \notin bint(\mathcal{H},\mathcal{O}) \text{ and } \delta_o^x \notin bint((\mathcal{H}^c,\mathcal{O}))\}$
$= \{\delta_o^x \in \widetilde{X} : \delta_o^x \notin (bcl(\mathcal{H}^c,\mathcal{O}))^c \text{ and } \delta_o^x \notin (bcl(\mathcal{H},\mathcal{O}))^c\}$
$= \{\delta_o^x \in \widetilde{X} : \delta_o^x \in bcl(\mathcal{H}^c,\mathcal{O}) \text{ and } \delta_o^x \in bcl(\mathcal{H},\mathcal{O})\}$
$= bcl(\mathcal{H},\mathcal{O})\widetilde{\cap} bcl(\mathcal{H}^c,\mathcal{O})$

(ii) $bB(\mathcal{H},\mathcal{O}) = bcl(\mathcal{H},\mathcal{O})\widetilde{\cap} bcl(\mathcal{H}^c,\mathcal{O})$
$= bcl(\mathcal{H},\mathcal{O})\widetilde{\cap}(bint(\mathcal{H},\mathcal{O}))^c$
$= bcl(\mathcal{H},\mathcal{O}) \setminus bint(\mathcal{H},\mathcal{O})$

Corollary 3. *Let $(\mathcal{H},\mathcal{O})$ be a subset of $(X,\mu,\mathcal{O})$. Then*

(i) $bB(\mathcal{H},\mathcal{O}) = bB(\mathcal{H}^c,\mathcal{O})$

(ii) $bcl(\mathcal{H},\mathcal{O}) = bint(\mathcal{H},\mathcal{O})\widetilde{\cup} bB(\mathcal{H},\mathcal{O})$

Proposition 14. *Let $(\mathcal{H},\mathcal{O})$ be a subset of $(X,\mu,\mathcal{O})$. Then*

(i) *$(\mathcal{H},\mathcal{O})$ is IS-b-open iff $bB(\mathcal{H},\mathcal{O})\widetilde{\cap}(\mathcal{H},\mathcal{O}) = \Phi$.*

(ii) *$(\mathcal{H},\mathcal{O})$ is IS-b-closed iff $bB(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{H},\mathcal{O})$.*

Proof.

(i) $bB(\mathcal{H},\mathcal{O}) \cap (\mathcal{H},\mathcal{O}) = bB(\mathcal{H},\mathcal{O}) \cap bint(\mathcal{H},\mathcal{O}) = \Phi$. Conversely, let $\delta_o^x \in (\mathcal{H},\mathcal{O})$. Then $\delta_o^x \in bint(\mathcal{H},\mathcal{O})$ or $\delta_o^x \in bB(\mathcal{H},\mathcal{O})$. Since $bB(\mathcal{H},\mathcal{O}) \cap (\mathcal{H},\mathcal{O}) = \Phi$, $\delta_o^x \in bint(\mathcal{H},\mathcal{O})$. Thus, $(\mathcal{H},\mathcal{O}) \subseteq bint(\mathcal{H},\mathcal{O})$ which means that $(\mathcal{H},\mathcal{O}) = bint(\mathcal{H},\mathcal{O})$. Hence, $(\mathcal{H},\mathcal{O})$ is IS-b-open.

(ii) $(\mathcal{H},\mathcal{O})$ is IS-b-closed $\Leftrightarrow$ $(\mathcal{H}^c,\mathcal{O})$ is IS-b-open $\Leftrightarrow bB(\mathcal{H}^c,\mathcal{O})\cap(\mathcal{H}^c,\mathcal{O}) = \Phi \Leftrightarrow bB(\mathcal{H},\mathcal{O})\cap (\mathcal{H}^c,\mathcal{O}) = \Phi \Leftrightarrow bB(\mathcal{H},\mathcal{O}) \subseteq (\mathcal{H},\mathcal{O})$.

Corollary 4. *A subset $(\mathcal{H},\mathcal{O})$ of $(X,\mu,\mathcal{O})$ is IS-b-open and IS-b-closed iff $bB(\mathcal{H},\mathcal{O}) = \Phi$.*

5. Infra soft b-homeomorphism maps

Definition 18. *$f_\psi : (X,\mu,\mathcal{O}) \to (\mathcal{S},\nu,\Delta)$ is said to be IS-b-continuous at $\delta_o^x \in \widetilde{X}$ if for any IS-b-open set $(\mathcal{F},\Delta)$ containing $f_\psi(\delta_o^x)$, there is an IS-b-open set $(\mathcal{H},\mathcal{O})$ containing δ_o^x such that $f_\psi(\mathcal{H},\mathcal{O})\widetilde{\subseteq}(\mathcal{F},\Delta)$.*

f_ψ is called IS-b-continuous if it is IS-b-continuous at all $\delta_o^x \in \widetilde{X}$.

Theorem 4. *If $f_\psi : (X,\mu,\mathcal{O}) \to (\mathcal{S},\nu,\Delta)$ is IS-b-continuous, then the next properties are equivalent.*

(i) *f_ψ is an IS-b-continuous map;*

(ii) *The inverse image of each IS-b-closed set is IS-b-closed;*

(iii) $bcl(f_\psi^{-1}(\mathcal{H},\Delta))\widetilde{\subseteq}f_\psi^{-1}(bcl(\mathcal{H},\Delta))$ *for each* $(\mathcal{H},\Delta)\widetilde{\subseteq}\widetilde{\mathcal{S}}$*;*

(iv) $f_\psi(bcl(\mathcal{F},\mathcal{O}))\widetilde{\subseteq}bcl(f_\psi(\mathcal{F},\mathcal{O}))$ *for each* $(\mathcal{F},\mathcal{O})\widetilde{\subseteq}\widetilde{X}$*;*

(v) $f_\psi^{-1}(bint(\mathcal{H},\Delta))\widetilde{\subseteq}bint(f_\psi^{-1}(\mathcal{H},\Delta))$ *for each* $(\mathcal{H},\Delta)\widetilde{\subseteq}\widetilde{\mathcal{S}}$.

Proof. $(i)\Rightarrow(ii)$: Let $(\mathcal{H},\Delta)$ be an IS-b-closed set in $(\mathcal{S},\nu,\Delta)$. Then $f_\psi^{-1}(\mathcal{H}^c,\Delta)$ is an IS-b-open subset of $\widetilde{X}$. Obviously, $f_\psi^{-1}(\mathcal{H}^c,\Delta)=\widetilde{X}-f_\psi^{-1}(\mathcal{H},\Delta)$; hence, $f_\psi^{-1}(\mathcal{H},\Delta)$ is an IS-b-closed subset of $\widetilde{X}$.
$(ii)\Rightarrow(iii)$: According to (ii), $f_\psi^{-1}(bcl(\mathcal{H},\Delta))$ is an IS-b-closed subset of $\widetilde{X}$. Then $bcl(f_\psi^{-1}(\mathcal{H},\Delta))\widetilde{\subseteq}bcl(f_\psi^{-1}(bcl(\mathcal{H},\Delta)))=f_\psi^{-1}(bcl(\mathcal{H},\Delta))$.
$(iii)\Rightarrow(vi)$: According to (iii), $bcl(f_\psi^{-1}(f_\psi(\mathcal{F},\mathcal{O})))\widetilde{\subseteq}f_\psi^{-1}(bcl(f_\psi(\mathcal{F},\mathcal{O})))$. Then $f_\psi(bcl(\mathcal{F},\mathcal{O}))\widetilde{\subseteq}f_\psi(f_\psi^{-1}(bcl(f_\psi(\mathcal{F},\mathcal{O}))))\widetilde{\subseteq}bcl(f_\psi(\mathcal{F},\mathcal{O}))$.
$(iv)\Rightarrow(v)$: According to (iv), $f_\psi(bcl(\widetilde{X}-f_\psi^{-1}(\mathcal{H},\Delta)))\widetilde{\subseteq}bcl(f_\psi(\widetilde{X}-f_\psi^{-1}(\mathcal{H},\Delta)))$. Therefore, $f_\psi(\widetilde{X}-bint(f_\psi^{-1}(\mathcal{H},\Delta)))=f_\psi(bcl(\widetilde{X}-f_\psi^{-1}(\mathcal{H},\Delta)))\subseteq bcl(\widetilde{\mathcal{S}}-(\mathcal{H},\Delta))=\widetilde{\mathcal{S}}-bint(\mathcal{H},\Delta)$. Thus $\widetilde{X}-bint(f_\psi^{-1}(\mathcal{H},\Delta))\widetilde{\subseteq}f_\psi^{-1}(\widetilde{\mathcal{S}}-bint(\mathcal{H},\Delta))=f_\psi^{-1}(\widetilde{\mathcal{S}})-f_\psi^{-1}(bint(\mathcal{H},\Delta))$. Hence $f_\psi^{-1}(bint(\mathcal{H},\Delta))\widetilde{\subseteq}bint(f_\psi^{-1}(\mathcal{H},\Delta))$.
$(v)\Rightarrow(i)$: Let $(\mathcal{H},\Delta)$ be an IS-b-open subset of $\widetilde{\mathcal{S}}$. According to (v), $f_\psi^{-1}(\mathcal{H},\Delta)\widetilde{\subseteq}bint(f_\psi^{-1}(\mathcal{H},\Delta))$. This implies $f_\psi^{-1}(\mathcal{H},\Delta)=bint(f_\psi^{-1}(\mathcal{H},\Delta))$. Hence, f_ψ is IS-b-continuous.

Theorem 5. *If* $f_\psi:(X,\mu,\mathcal{O})\to(\mathcal{S},\nu,\Delta)$ *is IS-b-continuous, then the restriction S-map* $f_{\psi_{|\mathcal{M}}}:(\mathcal{M},\mu_{\mathcal{M}},\mathcal{O})\to(\mathcal{S},\nu,\Delta)$ *is IS-b-continuous provided that* $\widetilde{\mathcal{M}}$ *is an IS-open set.*

Proof. Consider $(\mathcal{H},\Delta)$ is an IS-b-open set in $(\mathcal{S},\nu,\Delta)$. By hypothesis, $f_\psi^{-1}(\mathcal{H},\Delta)$ is IS-b-open. Now, $f_{\psi_{|\mathcal{M}}}^{-1}(\mathcal{H},\Delta)=f_\psi^{-1}(\mathcal{H},\Delta)\widetilde{\cap}\widetilde{\mathcal{M}}$. Since $\widetilde{\mathcal{M}}$ is an IS-open set, it follows from Proposition 6 that $f_{\psi_{|\mathcal{M}}}^{-1}(\mathcal{H},\Delta)$ is IS-b-open. Hence, $f_{\psi_{|\mathcal{M}}}$ is an IS-b-continuous map.

Definition 19. *If the image of each IS-b-open (resp., IS-b-closed) set under an S-map* $f_\psi:(X,\mu,\mathcal{O})\to(\mathcal{S},\nu,\Delta)$ *is IS-b-open (resp., IS-b-closed), then* f_ψ *is called IS-b-open (resp., IS-b-closed).*

Proposition 15. $f_\psi:(X,\mu,\mathcal{O})\to(\mathcal{S},\nu,\Delta)$ *is an IS-b-open map iff* $f_\psi(bint(\mathcal{H},\mathcal{O}))\widetilde{\subseteq}bint(f_\psi(\mathcal{H},\mathcal{O}))$ *for each subset of* $(\mathcal{H},\mathcal{O})$ *of* $\widetilde{X}$.

Proof. $\Rightarrow$: Let $(\mathcal{H},\mathcal{O})$ be a subset of $\widetilde{X}$. Now, $f_\psi(bint(\mathcal{H},\mathcal{O}))\widetilde{\subseteq}f_\psi(\mathcal{H},\mathcal{O})$ and $bint(\mathcal{H},\mathcal{O})$ is an IS-b-open set. By hypothesis, $f_\psi(bint(\mathcal{H},\mathcal{O}))$ is IS-b-open. Therefore, $f_\psi(bint(\mathcal{H},\mathcal{O}))\widetilde{\subseteq}bint(f_\psi(\mathcal{H},\mathcal{O}))$.
$\Leftarrow$: Let $(\Lambda,\mathcal{O})$ be an IS-open subset of $\widetilde{X}$. Then $f_\psi(\mathcal{H},\mathcal{O})\widetilde{\subseteq}bint(f_\psi(\mathcal{H},\mathcal{O}))$. Therefore, $f_\psi(\mathcal{H},\mathcal{O})=bint(f_\psi(\mathcal{H},\mathcal{O}))$ which means that f_ψ is an IS-b-open map.

Proposition 16. *$f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is an IS-b-closed map iff $bcl(f_\psi(\mathcal{H}, \mathcal{O})) \widetilde{\subseteq} f_\psi(bcl(\mathcal{H}, \mathcal{O}))$ for each subset $(\mathcal{H}, \mathcal{O})$ of $\widetilde{X}$.*

Proof. $\Rightarrow$: Let f_ψ be an IS-b-closed map and $(\mathcal{H}, \mathcal{O})$ be an S-set of $\widetilde{X}$. By hypothesis, $f_\psi(bcl(\mathcal{H}, \mathcal{O}))$ is IS-b-closed. Since $f_\psi(\mathcal{H}, \mathcal{O}) \widetilde{\subseteq} f_\psi(bcl(\mathcal{H}, \mathcal{O}))$, $bcl(f_\psi(\mathcal{H}, \mathcal{O})) \widetilde{\subseteq} f_\psi(bcl(\mathcal{H}, \mathcal{O}))$.
$\Leftarrow$: Suppose that $(\mathcal{H}, \mathcal{O})$ is an IS-b-closed subset of $\widetilde{X}$. By hypothesis, $f_\psi(\mathcal{H}, \mathcal{O}) \widetilde{\subseteq} bcl(f_\psi(\mathcal{H}, \mathcal{O})) \widetilde{\subseteq} f_\psi(bcl(\mathcal{H}, \mathcal{O})) = f_\psi(\mathcal{H}, \mathcal{O})$. Therefore, $f_\psi(\mathcal{H}, \mathcal{O})$ is IS-b-closed. Hence, f_ψ is an IS-b-closed map.

Proposition 17. *The concepts of IS-b-open and IS-b-closed maps are equivalent under bijectiveness.*

Proof. It comes from the fact that a bijective soft map $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ implies $f_\psi(\mathcal{H}^c, \mathcal{O}) = (f_\psi(\mathcal{H}, \mathcal{O}))^c$.

Proposition 18. *Let $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ and $F_\nu : (\mathcal{S}, \nu, \Delta) \to (\mathcal{V}, \sigma, \mathcal{U})$ be two S-maps. Then:*

(i) *If f_ψ and F_ν are IS-b-open maps, then $F_\nu \circ f_\psi$ is an IS-b-open map.*

(ii) *If $F_\nu \circ f_\psi$ is an IS-b-open map and f_ψ is a surjective IS-b-continuous map, then F_ν is an IS-b-open map.*

(iii) *If $F_\nu \circ f_\psi$ is an IS-b-open map and F_ν is an injective IS-b-continuous map, then f_ψ is an IS-b-open map.*

Proof.

(i) Straightforward.

(ii) Consider $(\mathcal{H}, \Delta)$ as an IS-b-open set in $(\mathcal{S}, \nu, \Delta)$. By hypothesis, $f_\psi^{-1}(\mathcal{H}, \Delta)$ is an IS-b-open subset of $(X, \mu, \mathcal{O})$. Again, by hypothesis, $(F_\nu \circ f_\psi)(f_\psi^{-1}(\mathcal{H}, \Delta))$ is an IS-b-open subset of $(\mathcal{V}, \sigma, \mathcal{U})$. Since f_ψ is surjective, then $(F_\nu \circ f_\psi)(f_\psi^{-1}(\mathcal{H}, \Delta)) = F_\nu(f_\psi(f_\psi^{-1}(\mathcal{H}, \Delta))) = F_\nu(\mathcal{H}, \Delta)$. Hence, F_ν is an IS-b-open map.

(iii) Consider $(\mathcal{H}, \mathcal{O})$ as an IS-b-open subset of $(X, \mu, \mathcal{O})$. By hypothesis, $(F_\nu \circ f_\psi)(\mathcal{H}, \mathcal{O})$ is an IS-b-open subset of $(\mathcal{V}, \sigma, \mathcal{U})$. Again, by hypothesis, $F_\nu^{-1}(F_\nu \circ f_\psi(\mathcal{H}, \mathcal{O}))$ is an IS-b-open subset of $(\mathcal{S}, \nu, \Delta)$. Since F_ν is injective, then $F_\nu^{-1}(F_\nu \circ f_\psi(\mathcal{H}, \mathcal{O})) = (F_\nu^{-1} F_\nu)(f_\psi(\mathcal{H}, \mathcal{O})) = f_\psi(\mathcal{H}, \mathcal{O})$. Hence, f_ψ is an IS-b-open map.

Definition 20. *A bijective S-map $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is said to be an IS-b-homeomorphism if it is IS-b-continuous and IS-b-open.*

The proofs of the following two results are easy and so is omitted.

Proposition 19. *Let $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ and $F_\nu : (\mathcal{S}, \nu, \Delta) \to (\mathcal{V}, \sigma, \mathcal{U})$ be IS-b-homeomorphism maps. Then $F_\nu \circ f_\psi$ is an IS-b-homeomorphism map.*

Proposition 20. *If $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is a bijective S-map, then the following items are equivalent.*

(i) *f_ψ is an IS-b-homeomorphism.*

(ii) *f_ψ and f_ψ^{-1} is IS-b-continuous.*

(iii) *f_ψ is IS-b-closed and IS-b-continuous.*

Proposition 21. *If $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is an IS-b-homeomorphism map, then the following items hold for each $(\mathcal{H}, \mathcal{O}) \in S(X)_A$.*

(i) *$f_\psi(bint(\mathcal{H}, \mathcal{O})) = bint(f_\psi(\mathcal{H}, \mathcal{O}))$.*

(ii) *$f_\psi(bcl(\mathcal{H}, \mathcal{O})) = bcl(f_\psi(\mathcal{H}, \mathcal{O}))$.*

Proof. (i): According to Proposition 15 (i), we obtain $f_\psi(bint(\mathcal{H}, \mathcal{O}))\widetilde{\subseteq} bint(f_\psi(\mathcal{H}, \mathcal{O}))$. Conversely, let $\delta_\kappa^s \in bint(f_\psi(\mathcal{H}, \mathcal{O})$. Then there is an IS-$b$-open set $(\mathcal{F}, \Delta)$ such that $\delta_\kappa^s \in (\mathcal{F}, \Delta)\widetilde{\subseteq} f_\psi(\mathcal{H}, \mathcal{O})$. By hypothesis, $\delta_o^x = f_\psi^{-1}(\delta_\kappa^s) \in f_\psi^{-1}(\mathcal{F}, \Delta)\widetilde{\subseteq}(\mathcal{H}, \mathcal{O})$ such that $f_\psi^{-1}(\mathcal{F}, \Delta)$ is an infra soft b-open set. So that, $\delta_o^x \in bint(\mathcal{H}, \mathcal{O})$ which means that $\delta_\kappa^s \in f_\psi(bint(\mathcal{H}, \mathcal{O}))$. One can achieve item (ii) following similar arguments.

Theorem 6. *The property of an IS-b-dense set is an IS-topological invariant.*

Proof. Let $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ be an IS-b-homeomorphism map and consider $(\mathcal{H}, \mathcal{O})$ as an IS- b-dense subset of $(X, \mu, \mathcal{O})$, i.e. $bcl(\mathcal{H}, \mathcal{O}) = \widetilde{X}$. It comes from Proposition 21 (ii) that $bcl(f_\psi(\mathcal{H}, \mathcal{O})) = f_\psi(bcl(\mathcal{H}, \mathcal{O})) = f_\psi(\widetilde{X}) = bcl(\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}$. Thus, $f_\psi(\mathcal{H}, \mathcal{O})$ is an IS-b-dense set in $(\mathcal{S}, \nu, \Delta)$, as required.

We complete this section by studying the concept of fixed soft points with respect to IS-b-open sets.

Definition 21. *We say that $(X, \mu, \mathcal{O})$ has a b-fixed S-point property provided that for every IS-b-continuous map $f_\psi : (X, \mu, \mathcal{O}) \to (X, \mu, \mathcal{O})$ there exists $\delta_o^s \in X$ such that $f_\psi(\delta_o^s) = \delta_o^s$.*

Proposition 22. *The property of being a b-fixed S-point is preserved under an IS-b-homeomorphism.*

Proof. Consider $(X_1, \mu_1, \mathcal{O}_1)$ and $(X_2, \mu_2, \mathcal{O}_2)$ as two IS-b-homeomorphism. This means that there exists a bijective S-map $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ such that f_ψ and f_ψ^{-1} are IS-b-continuous. Suppose that $(X_1, \mu_1, \mathcal{O}_1)$ has the property of b-fixed soft point. That is any IS-b-continuous map $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_1, \mu_1, \mathcal{O}_1)$ has a b-fixed S-point. Now, consider $C_\psi : (X_2, \mu_2, \mathcal{O}_2) \to (X_2, \mu_2, \mathcal{O}_2)$ is IS-b-continuous. It is clear that $C_\psi \circ f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ is IS-b-continuous. Therefore, $f_\psi^{-1} \circ C_\psi \circ f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_1, \mu_1, \mathcal{O}_1)$ is IS-b-continuous. Since $(X_1, \mu_1, \mathcal{O}_1)$ has a b-fixed S-point property, $f_\psi^{-1}(h_\psi(f_\psi(\delta_o^s))) = \delta_o^s$ for some $\delta_o^s \in \widetilde{X}$. Thus, $f_\psi(f_\psi^{-1}(h_\psi(f_\psi(\delta_o^s)))) = f_\psi(\delta_o^s)$. This implies that $h_\psi(f_\psi(\delta_o^s)) = f_\psi(\delta_o^s)$. Hence, $f_\psi(\delta_o^s)$ is a b-fixed soft point of C_ψ which means that $(X_2, \mu_2, \mathcal{O}_2)$ has a b-fixed S-point property.

6. Concluding remark and further work

In this paper, we have formulated the concept of IS-b-open sets and discussed its main properties. Then, we have defined novel operators and mappings between ISTSs depending on the classes of IS-b-open and IS-b-closed sets. We have revealed the relationships between these operators and mappings and investigated their basic features. As we have noted that several topological characterizations still have been valid via the structures of infra topologies, which confirms the importance of infra ST-structures.

Our future works will focus on studying further topological concepts and notions via infra ST-structures. Also, we will research the hybridizations structures obtained from ISTs and other structures such as rough soft and FS-structures.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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