EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 1237-1253
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Using the sum of triangular numbers as the most fundamental number of combinations found using heuristic methods

Yukio Kobayashi

Department of Information Systems Science, Faculty of Science and Engineering, Soka University 1-236 Tangi-machi, Hachioji-shi, Tokyo 192-8577, Japan

Abstract. The same mathematical meaning can be expressed in many different ways. Students should deduce this and thus perceive that the different expressions have the same meaning. In this work, a pedagogical approach to the determination of various equations related to the sum of triangular numbers is presented as an example to show how students can deduce these different expressions and their meanings. The sum of triangular numbers is closely related to the sums of natural numbers, square numbers, and two consecutive numbers. These relationships can be found using a heuristic method in which the equations representing the sum of natural numbers are simply listed. Guiding students to discover the formula by themselves is important, rather than giving them the formula from the beginning and asking them to prove it. Triangular numbers are the fundamental numbers of combinations. The sum of n consecutive triangular numbers is equal to ${}_{n+2}C_3$, as shown by combinatoric tree diagrams. The sum of the triangular numbers can be regarded as an extended version of the sum of the natural numbers. The sum of square numbers is also related to the sum of triangular numbers, which is also easy to understand in terms of combinatorics. The combinatoric expression for the sum of triangular numbers can be extended to $\sum_{\ell=1}^{k-1} {}_{n-\ell}C_{k-1} = {}_nC_k$, which can be expressed as the sum of triangular numbers. This study provides an example of teaching material that broadens students' view of a given equation by crossing different learning units such as combinatorics and algebra. The sum of triangular numbers discussed in this study is simply an example of developing the habit of considering various interpretations of the same formula, but even well-known formulas can be utilized for this type of subject matter.

2020 Mathematics Subject Classifications: 97H20

Key Words and Phrases: Sum of natural numbers, sum of triangular numbers, sum of square numbers, tree diagram, combinatorics, heuristic method

DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4498

Email address: koba@t.soka.ac.jp (Y. Kobayashi)

1. Introduction

In previous studies [3–5], we considered the geometrical meaning of the sums of natural numbers, square numbers, and cubic numbers in terms of elementary combinatorics or tree diagrams. This basic concept involves counting the number of points or edges. Adopting a different perspective and focusing on triangular numbers, it may be observed that the same number of points or edges can be expressed by different formulas, depending on how they are counted. Counting is an appropriate subject to help students realise that an equation is both a means of writing the process of a calculation and a composition for expressing numbers.

As students become familiar with various expressions of the sum of triangular numbers, they can discover the close relationships between the sums of natural numbers, square numbers, and triangular numbers. In contrast to passive learning, guiding students to independently discover the characteristics of numbers or equations using heuristic methods is essential [6]. Through these learning experiences, students should deduce and thus perceive that the different expressions have the same meaning. Proofs without words can provide a very useful support in this process to improve students' learning performance [1]. In this study, we consider the relationship between the sums of natural numbers, square numbers, and triangular numbers using various equations to represent the sum of the triangular numbers. In standard teaching methods, combinatorics and algebra are treated as separate subjects. However, in this study, the sum of triangular numbers is regarded as the basis of combinatorics and the relationships between series and combinatorics are shown by a simple approach.

This paper presents an example of how students can deduce mathematical expressions; thus, the pedagogical approach can be used to determine the mathematical meaning of various equations related to the sum of triangular numbers. Although they are beyond the scope of the present work, some advanced studies that extend this idea beyond mathematics education are also in progress, which consider approaches such as representation of integers as sums of triangular numbers [10].

2. Equations for the sum of triangular numbers

To begin, we consider the various ways of representing triangular numbers. The sum of triangular numbers can be expressed in four ways [7, 9], as given below.

$$\frac{1}{2}[1\cdot 2+2\cdot 3+3\cdot 4+\cdots+(k-1)\cdot k+\cdots+(n-1)\cdot n+n\cdot(n+1)] \tag{1}$$

$$= {}_2C_2 + {}_3C_2 + {}_4C_2 + \cdots + {}_kC_2 + \cdots + {}_nC_2 + {}_{n+1}C_2 \tag{2}$$

$$= 1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+n) \tag{3}$$

$$= 1\cdot n+2\cdot(n-1)+\cdots+n\cdot 1. \tag{4}$$

This is a useful pedagogical exercise for students, in which they consider why the values of these four equations are equal despite their different forms. Computational skills are important to help students cultivate the ability to observe expressions of equations. To this end, becoming sufficiently skilled such that one can clearly perceive the values of these equations as equal is an appropriate learning goal.

First, instructors should begin by having students explain how (1) and (2) are equivalent. If students notice that the denominator 2 is equal to 2!, then it is clear that (1) expresses the sum of combinations $\sum_{k=1}^{n} {}_{k+1}C_2$ [7, 9, 11].

Second, students should proceed to explain how (2) and (3) are equivalent. If students draw combinations ${}_{k+1}C_2$ in terms of tree diagrams, they can determine that the total number of edges is $1 + 2 + \cdots + k$. The equivalence of (1) and (3) can also be explained by rewriting (1) as the sum of the arithmetical series

$$\frac{1(1+1)}{2} + \frac{2(2+1)}{2} + \cdots + \frac{n(n+1)}{2}. \tag{5}$$

Third, (4) is derived by rewriting (3) as

$$\begin{aligned} & 1 + (1+2) + (1+2+3) + \cdots + [1+2+3+\cdots+(n-1)] + [1+2+3+\cdots+(n-1)+n] \\ = & \underbrace{(1+1+\cdots+1)}_{n \text{ terms}} + \underbrace{(2+2+\cdots+2)}_{(n-1) \text{ terms}} + \cdots + \underbrace{[(n-1)+(n-1)]}_{\text{two terms}} + \underbrace{n}_{\text{one term}} . \end{aligned} \tag{6}$$

Equations (1), (2), (3), and (4) can be also represented by a pictorial proof because the value of these equations is equal to the total number of tree diagrams in Figure 1. If the edges are replaced with filled circles and the points are rearranged into equilateral triangles, students should be convinced that the number of filled circles is triangular.

3. Relationship between the sums of natural numbers, triangular numbers, and the product of consecutive numbers

Having students first determine the characteristics of concrete numbers or equations by arranging them is essential, rather than providing formulas and having students prove them. Instructors can adopt the following heuristic method to implement this policy. Instructors should display Figure 2 and ask students to consider the problem of finding the relationship between the sums of natural numbers, triangular numbers, and the product of consecutive numbers, referring to the equations presented in Section 2. Considering the sum of the natural numbers from 1 to n, the following sequence of n equations can be provided (instead of the five equations in Figure 2).

Depending on how we interpret the sequence of equations in Figure 2, we can determine the features shown in Figures 3 and 4. The features in Figure 3 can be expressed as given below.

$$\begin{aligned} & (1+2+3+4+5)\cdot 5 \\ = & (1\cdot 5 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 + 5\cdot 1) \\ & +(2\cdot 1 + 3\cdot 2 + 4\cdot 3 + 5\cdot 4). \end{aligned} \tag{7}$$

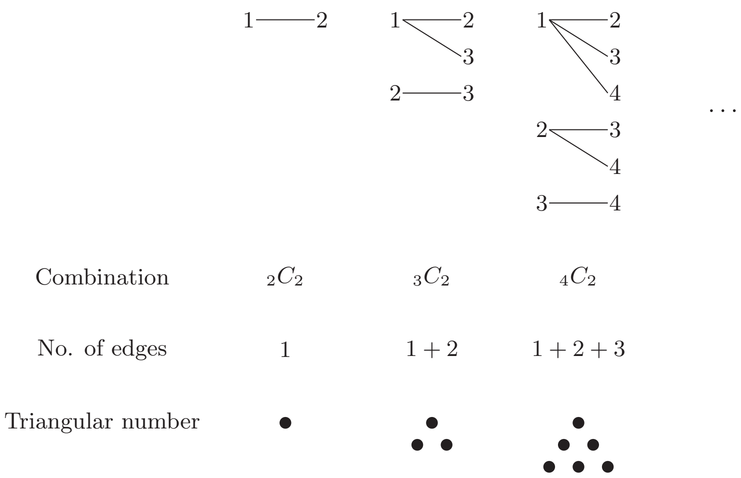

Figure 1: Pictorial proof of the equivalence of (1), (2), (3), and (4). The numerals in the tree diagrams indicate the number of cards.

$$\begin{array}{ccccccccc}
1 & + & 2 & + & 3 & + & 4 & + & 5 \\
1 & + & 2 & + & 3 & + & 4 & + & 5 \\
1 & + & 2 & + & 3 & + & 4 & + & 5 \\
1 & + & 2 & + & 3 & + & 4 & + & 5 \\
1 & + & 2 & + & 3 & + & 4 & + & 5
\end{array}$$

Figure 2: Sequence of five equations expressing the sum of natural numbers from 1 to 5.

The first summation $1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1$ and the second $2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4$ are the numbers in the dark grey triangle and in the light grey triangle in Figure 3, respectively. From Figure 4, we can express the feature as follows.

$$\begin{aligned}
& (1+2+3+4+5) \cdot 5 - (1+2+3+4+5) \\
= \; & (1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1) \\
& + (2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4). \qquad (8)
\end{aligned}$$

By generalising (7) and (8), we obtain

$$\begin{aligned}
& (1 + 2 + \cdots + n) \cdot n \\
= \; & [1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1] \\
& + [2 \cdot 1 + 3 \cdot 2 + \cdots + n \cdot (n-1)], \qquad (9)
\end{aligned}$$

and

$$(1 + 2 + \cdots + n) \cdot (n - 1)$$

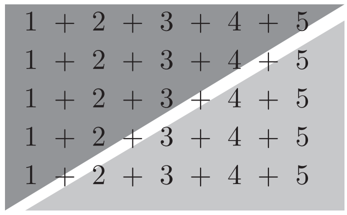

Figure 3: Relationship between the sums of natural numbers, triangular numbers, and the product of consecutive numbers.

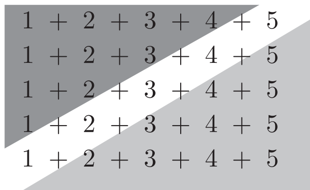

Figure 4: Relationship between the sums of natural numbers, triangular numbers, and the product of consecutive numbers.

$$\begin{aligned} &= [1 \cdot (n-1) + 2 \cdot (n-2) + \cdots + (n-1) \cdot 1] \\ &\quad +[2 \cdot 1 + 3 \cdot 2 + \cdots + n \cdot (n-1)]. \end{aligned} \tag{10}$$

Considering (4), Equations (9) and (10) indicate the relationship between the sums of natural numbers, triangular numbers, and the product of consecutive numbers.

Equations (1) and (9) can be used to obtain the sum of both triangular numbers and the product of consecutive numbers. From (1),

$$\begin{aligned} & 1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 \\ = \; & \frac{1}{2}[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n + n \cdot (n+1)]. \end{aligned} \tag{11}$$

Thus, (9) can be written as

$$\begin{aligned} & (1 + 2 + \cdots + n) \cdot n \\ = \; & \frac{3}{2}[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n] + \frac{1}{2} n \cdot (n+1). \end{aligned} \tag{12}$$

By performing slight rearrangements, we obtain

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n \\ = \; & \frac{1}{3}(n-1)n(n+1), \end{aligned} \tag{13}$$

using $1 + 2 + \cdots + n = n \cdot (n+1)/2$. Similarly, from Equation (11),

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n \\ = \; & 2\left[1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1\right] - n \cdot (n+1). \end{aligned} \tag{14}$$

Thus, (9) can be written as

$$\begin{aligned} & (1 + 2 + \cdots + n) \cdot n \\ = \; & 3\left[1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1\right] - n \cdot (n+1). \end{aligned} \tag{15}$$

By performing some slight rearrangements, we obtain

$$\begin{aligned} & 1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 \\ = \; & \frac{1}{6} n(n+1)(n+2). \end{aligned} \tag{16}$$

Note that Equations (13) and (16) can be obtained from (10) rather than (9).

Equation (16) can be rewritten in two ways. The relationship between the sum of the natural numbers and the sum of triangular numbers can be expressed as either

$$\begin{aligned} & 1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 \\ = \; & \frac{1}{3}(n+2) \cdot \frac{1}{2} n(n+1) \\ = \; & \frac{1}{3}(n+2) \sum_{k=1}^{n} k, \end{aligned} \tag{17}$$

or

$$\begin{aligned} & 1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 \\ = \; & \frac{1}{3} n \cdot \frac{1}{2}(n+1)(n+2) \\ = \; & \frac{1}{3} n \sum_{k=1}^{n+1} k. \end{aligned} \tag{18}$$

Equations (17) and (18) can also be explained in terms of elementary combinatorics, as shown by Section 6.

4. Relationship between the sums of natural numbers, triangular numbers, and square numbers

The sequence of equations expressing the sum of natural numbers shown in Figure 2 can also be used to determine the relationship between the sums of natural numbers, triangular numbers, and square numbers. The sum of natural numbers shown in Figure 2 can be expressed differently from that in Figures 3 and 4. From Figure 5, we can express the feature as

$$\begin{aligned} & (1+2+3+4+5)\cdot 5 \\ = & (1\cdot 5+2\cdot 4+3\cdot 3+4\cdot 2+5\cdot 1) \\ & +(1\cdot 1+2\cdot 2+3\cdot 3+4\cdot 4+5\cdot 5) \\ & -(1+2+3+4+5). \end{aligned} \tag{19}$$

By performing some slight rearrangements, we obtain

$$\begin{aligned} & 1\cdot 5+2\cdot 4+3\cdot 3+4\cdot 2+5\cdot 1 \\ = & (1+2+3+4+5)\cdot 6 \\ & -(1^2+2^2+3^2+4^2+5^2). \end{aligned} \tag{20}$$

Equation (20) defines the relationship between the sums of natural numbers, square numbers, and triangular numbers.

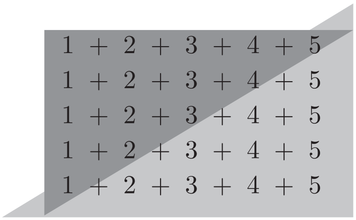

Figure 5: Relationship between the sums of natural numbers, square numbers, and triangular numbers.

Equations (20) and (17) can be used to find the sum of square numbers. By generalising (20), we obtain

$$\begin{aligned} & 1^2+2^2+\cdots+n^2 \\ = & (1+2+\cdots+n)\cdot(n+1) \\ & -[1\cdot n+2\cdot(n-1)+\cdots+n\cdot 1] \\ = & \left[\frac{1}{2}n(n+1)\right](n+1)-\frac{1}{3}(n+2)\cdot\frac{1}{2}n(n+1) \\ = & \frac{1}{6}n(n+1)(2n+1). \end{aligned} \tag{21}$$

Equation (21) can be rewritten to express the relationship between the sums of natural numbers and square numbers, as given below.

$$1^2+2^2+\cdots+n^2=\frac{1}{3}(2n+1)\sum_{k=1}^{n}k. \tag{22}$$

5. View of integral representations

The sum of triangular numbers can be obtained using a heuristic method, as shown in Section 3. The sum of triangular numbers may appear to have been found easily using a character expression $\sum_{k=1}^{n} k(n+1-k)$, instead of a numerical expression; however, this is not always the case. The following expression indicates the relationship between the sums of triangular numbers and square numbers.

$$\sum_{k=1}^{n} k(n+1-k) = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2. \tag{23}$$

If students are unfamiliar with (21) or (22), they should consider how to find the sum of triangular numbers (23) without these expression. To calculate $\sum_{k=1}^{n} k(n+1-k)$ in a way that does not rely on (21) or (22), it is convenient to note the integral representations [5]. Instructors should encourage students to realise

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \int_0^k dx, \tag{24}$$

and

$$\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n} \int_0^k x dx. \tag{25}$$

Using (24) and (25),

$$\begin{aligned}
& \sum_{k=1}^{n} k(n+1-k) \\
= \; & \sum_{k=1}^{n} \int_0^k (n+1-2x)dx \\
= \; & n\int_0^1 (n+1-2x)dx + (n-1)\int_1^2 (n+1-2x)dx + \cdots \\
& +[n-(k-1)]\int_{k-1}^k (n+1-2x)dx + \cdots + \int_{n-1}^n (n+1-2x)dx \\
= \; & \sum_{k=1}^{n} (n+1-k)\int_{k-1}^k (n+1-2x)dx \\
= \; & \sum_{k=1}^{n} (n+1-k)\Big[(n+1)x - x^2\Big]_{k-1}^k
\end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{n}(n+1-k)[(n+1)k-k^2-(n+1)(k-1)+(k-1)^2] \\ &= \sum_{k=1}^{n}(n+1-k)(n-2k+2) \\ &= (n+2)\sum_{k=1}^{n}(n+1-k)-2\sum_{k=1}^{n}k(n+1-k), \end{aligned} \tag{26}$$

and by rearranging, we obtain

$$\begin{aligned} &3\sum_{k=1}^{n}k(n+1-k) \\ &= (n+2)\sum_{k=1}^{n}(n+1-k) \\ &= (n+2)(n+1)\sum_{k=1}^{n}1-(n+2)\sum_{k=1}^{n}k \\ &= (n+2)(n+1)n-(n+2)\cdot\frac{1}{2}(n+1)n \\ &= \frac{1}{2}n(n+1)(n+2). \end{aligned} \tag{27}$$

Thus, we obtain

$$\sum_{k=1}^{n}k(n+1-k)=\frac{1}{6}n(n+1)(n+2). \tag{28}$$

This approach uses the definite integral to calculate the series. From a pedagogical viewpoint, instructors can apply this material as a useful exercise towards integrated learning of integrals and series. We can also determine that $\sum_{k=1}^{n}k^{\ell}$ using integral representations, where ℓ is a positive integer, as shown in a previous study [5].

6. Combinatoric sums of triangular numbers and square numbers

6.1. Triangular numbers

The relationship between the sum of natural numbers and triangular numbers expressed by (17) and (18), respectively, indicates the number of combinations, ${}_{n+2}C_3$. Instructors should ask students to consider why the sum of triangular numbers is equal to ${}_{n+2}C_3$. Students can be convinced of this equivalence by drawing tree diagrams for ${}_{n+2}C_3$. Drawing diagrams in connection with the following expansion of a polynomial is preferable.

$$(a+b)^{n+2}$$

$$= \overbrace{(a+b)}^{\text{term 1}}\overbrace{(a+b)}^{\text{term 2}}\cdots\overbrace{(a+b)}^{\text{term } n+2}$$
$$= {}_{n+2}C_0 a^{n+2}b^0 + {}_{n+2}C_1 a^{n+1}b^1 + {}_{n+2}C_2 a^n b^2 + {}_{n+2}C_3 a^{n-1}b^3$$
$$+\cdots+ {}_{n+2}C_{n+2}a^0 b^{n+2}. \tag{29}$$

The combination of $(a+b)$s required to choose $(n-1)$ as and three bs can be represented by ${}_{n+2}C_3$ in (29).

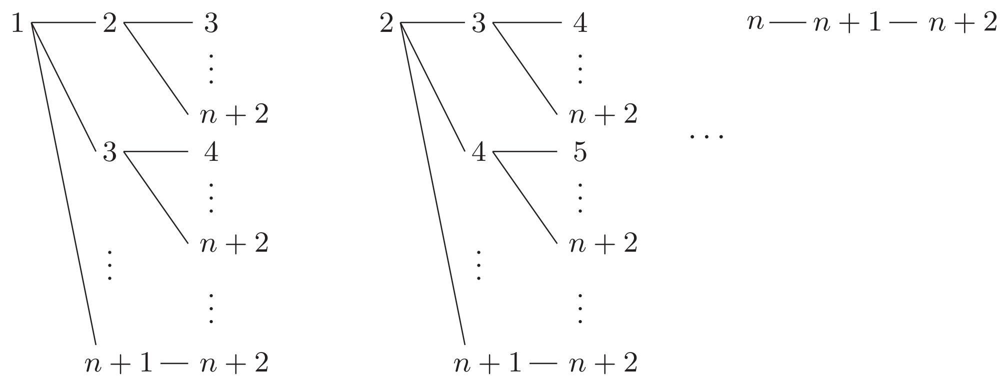

Figure 6: Tree diagrams for ${}_{n+2}C_3$. The numerals indicate the number of $(a+b)$ terms in Equation (29).

Among the four representations of triangular numbers, (1), (2), (3), and (4), (3) is appropriate to compare the sum of triangular numbers with ${}_{n+2}C_3$. The diagrams in Figure 6 indicate the sum of triangular numbers expressed in Equation (3). Counting the number of edges is another heuristic method. From (2) and (3), we can determine the relationship

$$\sum_{k=1}^{n} {}_{k+1}C_2 = {}_{n+2}C_3, \tag{30}$$

which indicates the total number of edges in the tree diagrams in Figures 1 and 6.

To summarise, the sum of triangular numbers can be expressed in various ways, as follows.

$$\sum_{\ell=1}^{n}\sum_{k=1}^{\ell} k = \begin{cases} \frac{1}{3}(n+2)_{n+1}C_2, \\ 1\cdot n + 2\cdot(n-1) + \cdots + 3\cdot(n-3) + \cdots + k\cdot[n-(k-1)] + \cdots + n\cdot 1, \\ \sum_{k=1}^{n} k(n+1-k), \\ (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2, \\ (n+1)_{n+1}C_2 - \frac{1}{3}(2n+1)_{n+1}C_2, \\ {}_{n+2}C_3, \\ \sum_{k=1}^{n} {}_{k+1}C_2. \end{cases}$$

These equations can also be interpreted in various ways to express ${}_{n+2}C_3$.

6.2. Square numbers

The sum of natural numbers and of triangular numbers can be explained in terms of combinatorics; thus, (20) suggests that the sum of square numbers can also be implied by combinatorics, and a diagram corresponding to Figure 1 can be drawn as shown in Figure 7. Instructors should begin by asking students about the relationship between triangular and square numbers.

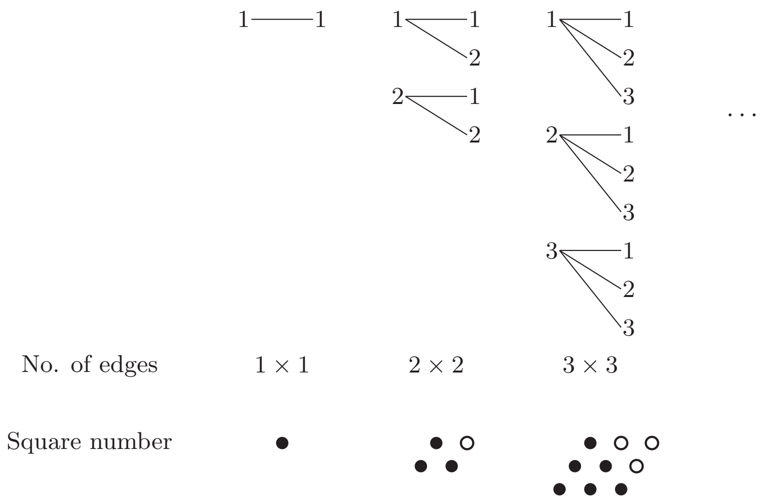

Figure 7: Pictorial proof of the relationship between triangular and square numbers. The numerals in the tree diagrams indicate the number of cards and the edges represent how the two numbered cards are assembled. To show that square numbers are the sum of two consecutive triangular numbers, square numbers are represented by open or filled circles.

Students should notice the following additional formulae from Figure 7 using a heuristic method. Equation (31) indicates that square numbers are the sum of two consecutive triangular numbers.

$$\left.\begin{array}{lcl} 1^2 &=& 1. \\ 2^2 &=& 1 + (1+2). \\ 3^2 &=& \quad (1+2) + (1+2+3). \\ && \cdots \end{array}\right\} \tag{31}$$

The sum of square numbers can be written as the sum of triangular numbers by counting the number of points drawn in Figure 7, as shown in Figure 8.

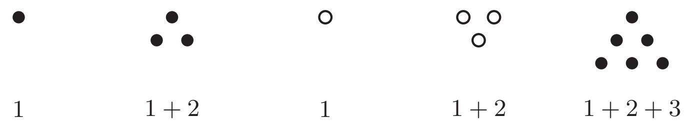

Figure 8: Visual interpretation of the sum of square numbers.

Generating

$$\begin{array}{rl} & 1^2 + 2^2 + 3^2 \\ = & 2 \cdot [1 + (1+2)] + (1+2+3), \end{array} \tag{32}$$

we obtain

$$\begin{array}{rl} & 1^2 + 2^2 + \cdots + n^2 \\ = & 2 \cdot \{1 + (1+2) + \cdots + [1 + 2 + \cdots + (n-1)]\} \\ & +(1 + 2 + \cdots + n) \\ = & 2 {}_{n+2}C_3 + {}_{n+1}C_2. \end{array} \tag{33}$$

Using the expression for the sum of triangular numbers in Section 6.1, instructors should have students verify that (33) agrees with $\frac{1}{6}n(n+1)(2n+1)$.

Considering Table 1, the sum of triangular numbers can be regarded as an extended version of the sum of natural numbers, and the sum of square numbers is related to the sum of both triangular numbers and natural numbers.

Table 1. Comparison between the sums of triangular numbers, natural numbers, and square numbers.

$1 + 2 + \cdots + n = {}_{n+1}C_2$.
$1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 = {}_{n+2}C_3$.
$1^2 + 2^2 + \cdots + n^2 = 2\, {}_{n+1}C_3 + {}_{n+1}C_2$.
Note the difference between ${}_{n+2}C_3$ and ${}_{n+1}C_3$.

7. Related problems

Triangular numbers are fundamental numbers in the field of combinations. The combination ${}_nC_k$ can be expressed using triangular numbers, as shown in Table 2. The additional formulae for ${}_nC_k$ indicate the total number of edges in the tree diagrams for ${}_nC_k$. As shown in Sections 2 and 6.1, ${}_nC_2$s are triangular numbers and ${}_nC_3$s are the sums of triangular numbers. Instructors should invite students discover the relationship between ${}_nC_k$s and the sum of triangular numbers using a heuristic method, as shown in Table 2. For example, for ${}_nC_4$,

$$\begin{aligned} {}_6C_4 &= 1+4+10 \\ &= 1+(1+3)+(1+3+6) \\ &= 1+[1+(1+2)]+[1+(1+2)+(1+2+3)] \\ &= {}_3C_3+{}_4C_3+{}_5C_3, \end{aligned} \tag{34}$$

which is also the sum of the sums of triangular numbers. For example, for ${}_nC_5$,

$$\begin{aligned} {}_8C_5 &= 1+5+15+35 \\ &= 1+(1+4)+(1+4+10)+(1+4+10+20) \\ &= 1+[1+(1+3)]+[1+(1+3)+(1+3+6)]+[1+(1+3)+(1+3+6)+(1+3+6+10)] \\ &= {}_4C_4+{}_5C_4+{}_6C_4+{}_7C_4 \\ &= {}_3C_3+({}_3C_3+{}_4C_3)+({}_3C_3+{}_4C_3+{}_5C_3)+({}_3C_3+{}_4C_3+{}_5C_3+{}_6C_3), \end{aligned} \tag{35}$$

which is also the sum of the sums of triangular numbers. Corresponding to (30), the following relationship holds.

$$\sum_{\ell=1}^{k-1} {}_{n-\ell}C_{k-1} = {}_nC_k. \tag{36}$$

Table 2. Addition formulae for combinations.

${}_2C_2 = 1.$
${}_3C_2 = 1 + 2.$
${}_4C_2 = 1 + 2 + 3.$
${}_5C_2 = 1 + 2 + 3 + 4.$
${}_3C_3 = 1.$
${}_4C_3 = 1 + 3.$
${}_5C_3 = 1 + 3 + 6.$
${}_6C_3 = 1 + 3 + 6 + 10.$
${}_4C_4 = 1.$
${}_5C_4 = 1 + 4.$
${}_6C_4 = 1 + 4 + 10.$
${}_7C_4 = 1 + 4 + 10 + 20.$
${}_5C_5 = 1.$
${}_6C_5 = 1 + 5.$
${}_7C_5 = 1 + 5 + 15.$
${}_8C_5 = 1 + 5 + 15 + 35.$
See Figures 1 and 6.

To cultivate an understanding of the addition formulae, students should check the total number of edges in the tree diagrams. The number of edges is the sum of triangular numbers. The tree diagrams for ${}_nC_k$ are inductively extended to those for ${}_{n+1}C_k$. For example, the tree diagrams for ${}_6C_4$ include those for ${}_5C_4$, as shown in Figure 9. The sum of ${}_4C_3$ and ${}_3C_3$ indicate the method of choosing four cards from two, three, four, five, and six cards, with the exception of one; thus, ${}_4C_3 \ + \ {}_3C_3 = \ {}_5C_4$.

8. Concluding remarks

In this paper, we have proposed a pedagogical approach to the determination of various equations related to the sum of triangular numbers that include the following steps (in order); however, the sum of triangular numbers can be shown simply using a pictorial proof [8].

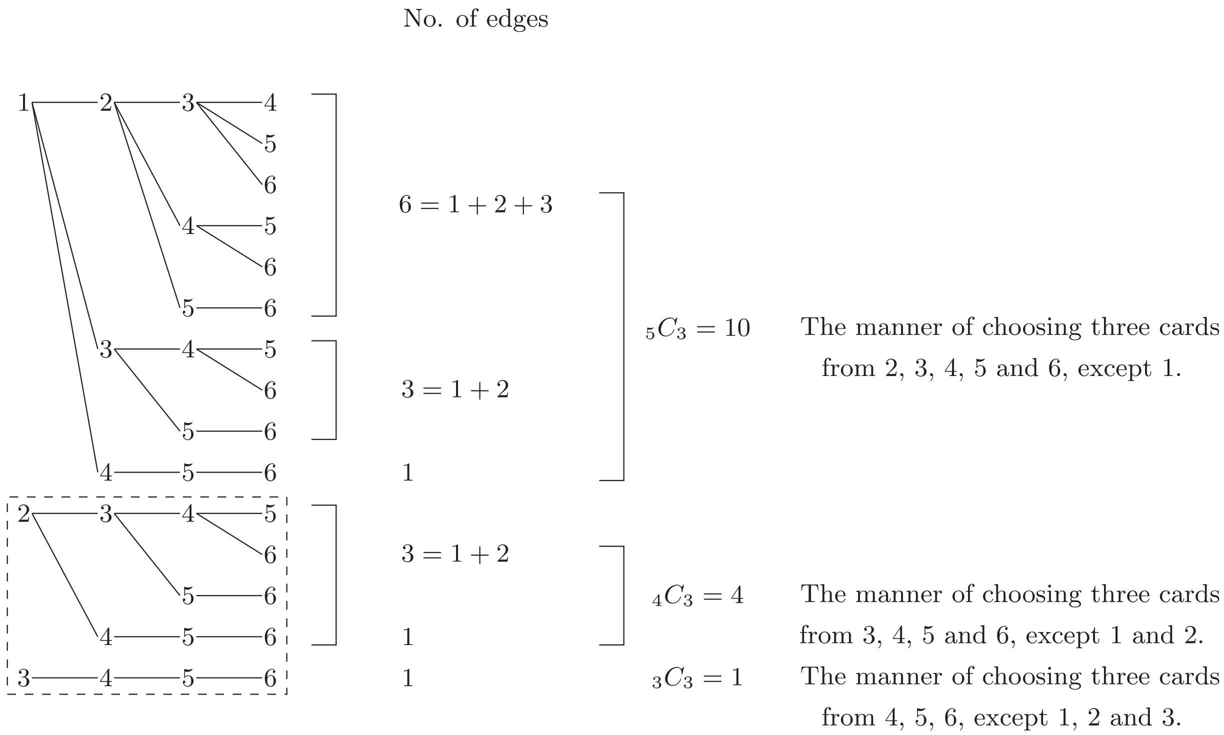

Figure 9: Tree diagrams for ${}_6C_4$. The numerals in the tree diagrams indicate the number of cards. The edges represent how the four two-numbered cards are assembled. The total number of edges in the area enclosed by the dashed line represents ${}_5C_4$s.

List of equations expressing the sum of natural numbers
↓
Tree diagrams
↓
Array of filled circles
↓
Combinatorics perspective

The same mathematical meaning can be expressed in many different ways, as shown in Equations (1), (2), (3), and (4). Students should deduce and thus perceive that the different expressions have the same meaning. Equations (24) and (25) also serve as good examples of this and are useful for calculating the series. Finding equations such as (34) and (35) can be relatively difficult if students consider the problem within the framework of combination. As we can understand through this example, the most fundamental number of combinations is the sum of triangular numbers. The sum of triangular numbers expressed by Equation (1) is the basis, and thus Equations (2), (3), and (4) are different expressions of (1). The utility of the sum of triangular numbers is significant.

Although indirectly related to this study, understanding the principles of statistical mechanics is difficult if we are unfamiliar with an equation of the form [2]

$$\frac{\eta(E_0 - E_r)}{\eta(E_0 - E_{r'})} = e^{\ln \eta(E_0 - E_r) - \ln \eta(E_0 - E_{r'})}, \tag{37}$$

in which the left- and right-hand sides appear completely different. The ability to calculate using memorised formulas is important; however, practice in expressing mathematical meanings with equations helps students become familiar with a variety of expressions using heuristic methods. By combining different topics that are normally learned separately such as combinatorics and algebra, students can understand that different notions are connected at the root of mathematics, and they can acquire the idea of reading the mathematical meanings of equations and expressing mathematical meanings by equations.

Slater and Frank [12] posited that the greatest difficulty that students have in mastering theoretical physics involves learning how to apply mathematics to a physical situation and mathematically formulate a problem, rather than solving the problem when formulated. Heuristic methods are effective in implementing learning, as noted by Slater and Frank.

Acknowledgements

The author thanks Editage (www.editage.jp) for English language editing.

References

[1] C. Alsina and R. B. Nelsen. An invitation to proofs without words. *European Journal of Pure and Applied Mathematics.*, 3:118–127, 2010.

[2] R. P. Feynman. *Statistical Mechanics: A Set of Lectures.* Benjamin, 1972.

[3] Y. Kobayashi. Geometrical meaning of arithmetic series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$, and $\sum_{k=1}^{n} k^3$ in terms of the elementary combinatorics. *Int. J. Math. Sci. Tech.*, Corrigendum 42:657–664, 2011.

[4] Y. Kobayashi. Integral representation of the pictorial proof of $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$. *Int. J. Math. Sci. Tech.*, 42:235–239, 2011.

[5] Y. Kobayashi. Recursion formulae of $1^{ell} + 2^{ell} + \cdots + n^{ell}$ and their combinatoric meaning in terms of the tree diagrams. *Int. J. Math. Sci. Tech.*, 44:132–142, 2013.

[6] Y. Kobayashi. Derivation of some formulae in combinatrics by heuristic method. *Int. J. Math. Sci. Tech.*, Erratum 46:469–476, 2015.

[7] Y. Kobayashi. Proof without words: relationship between combination ${}_nC_2$ and arithmetic series. *Far East J. Math. Educ.*, 19:57–58, 2019.

[8] Y. Kobayashi. Proof without words: relationship between the sum of triangular numbers, sum of natural numbers, and sum of square numbers. *Far East J. Math. Educ.*, 21:83–91, 2021.

[9] L. C. Larson. A discrete look at $1 + 2 + \cdots + n$. *College Math. J.*, 16:369–382, 1985.

[10] Zhi-Guo Liu. An identity of ramanujan and the representation of integers as sums of triangular numbers. *The Ramanujan Journal.*, 7:407–434, 2003.

[11] R. B. Nelsen, editor. *Proofs Without Words.* The Mathematical Association of America, 2000.

[12] J. C. Slater and N. H. Frank. *Introduction to Theoretical Physics.* McGrawhill, 1933.