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Locating Hop Sets in a Graph

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Abstract. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The open hop neighborhood of vertex $v \in V(G)$ is the set $N_G(v,2) = \{w \in V(G) : d_G(v,w) = 2\}$, where $d_G(v,w)$ denotes the distance between v and w. A non-empty set $S \subseteq V(G)$ is a locating hop set of G if $N_G(u,2) \cap S \neq N_G(v,2) \cap S$ for every pair of distinct vertices $u, v \in V(G) \setminus S$. The smallest cardinality of a locating hop set of G, denoted by $lhn(G)$ is called the locating hop number of G. This study focuses mainly on the concept of locating hop set in graphs. Characterizations of locating hop sets in the join and corona of two graphs are given and bounds for the corresponding locating hop numbers of these graphs are determined.

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Key Words and Phrases: Locating hop set, strictly locating hop set, join, corona

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $v \in V(G)$. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex v in a graph G, denoted by $deg_G(v)$, is the number of edges incident with v in G and the minimum degree $\delta(G)$ of the vertices of G is the *minimum degree* of G. The *open hop neighborhood* of vertex v is the set $N_G(v,2) = \{w \in V(G) : d_G(v,w) = 2\}$, where $d_G(v,w)$ denotes the distance between v and w. The *closed hop neighborhood* of vertex v is the set $N_G[v,2] = N_G(v,2) \cup \{v\}$. The concept of hop neighborhood was used in [10] to define and investigate the concept of hop domination. Hop domination and some of its variants had been studied also in [6], [7], [9], [12], and [13].

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The concept of locating set was first introduced by Slater (for which a protection device can determine the distance to an intruder) in 1975 (see [16]). Omega and Canoy in [11] studied the locating sets in graphs and characterized the locating sets in the join and corona of graphs where they also determined the locating numbers of these graphs. A set $S \subseteq V(G)$ is a *locating set* if for every two distinct vertices $u, v \in V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. A set $S \subseteq V(G)$ is *strictly locating* if it is locating and $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. The minimum cardinality of a locating set in G, denoted by $ln(G)$, is called the *locating number* of G. The minimum cardinality of a strictly locating set in G, denoted by $sln(G)$, is the *strict locating number* of G. Any locating (resp. strictly locating) set with cardinality equal to $ln(G)$ (resp. $sln(G)$), is called a *minimum locating set* or ln-set (resp. *minimum strictly locating set* or sln-set).

In 1987, Slater in [17] further investigated locating set with another concept called domination. A set $D \subseteq V(G)$ is a *dominating set* of G if $\cup_{x \in D} N[x] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Eventually, the concept of locating dominating set was introduced and is one of the widely studied topics nowadays (see [14], [15]). A locating subset $S \subseteq V(G)$ which is also a dominating set is called *locating-dominating set* (LD-set) in a graph G. A strictly locating subset S of $V(G)$ which is also a dominating set is called *strictly locating-dominating set* (SLD-set) in a graph G. The *locating-domination number or L-domination number* of G, denoted by $\gamma_L(G)$, is the minimum cardinality of a locating-dominating set. The minimum cardinality of a strictly locating-dominating set of G, denoted by $\gamma_{SL}(G)$, is called the *SL-domination number* of G. A locating-dominating (resp. strictly locating-dominating) set with cardinality equal to $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$)is called a *minimum locating-dominating set* or γ_L-set (*minimum strictly locating-dominating set* or γ_{SL}-set). Canoy et al. [8] characterized the locating dominating sets in the corona and composition of graphs. They also determined the locating-domination number of these graphs. There are other studies involving the concept of locating set and locating dominating set (see [4], [5], [8], [10], and [11]).

A non-empty set $S \subseteq V(G)$ is a *locating hop set* of G if $N_G(u,2) \cap S \neq N_G(v,2) \cap S$ for every pair of distinct vertices $u, v \in V(G) \setminus S$. A locating hop set is a *strictly locating hop* set if $N_G(v,2) \cap S \neq S$ for every $v \in V(G) \setminus S$. The smallest cardinality of a locating hop set (resp. strictly locating hop set) of G, denoted by $lhn(G)$ (resp. $slhn(G)$) is called the *locating hop* (resp. strictly locating hop) number of G. Any locating hop set (resp. strictly locating hop set) with cardinality equal to $lhn(G)$ (resp. $slhn(G)$) is called a *minimum locating hop set* or lhn-set (resp. *minimum strictly locating hop set* or $shln$-set). In this paper, we investigate the concept of locating hop set in the join and corona of two graphs. Investigation of several parameters in graphs under some binary operations had been done in many studies (see [1], [2], [3]).

A *point determining graph* is defined in [18] as a graph in which distinct non-adjacent vertices have distinct neighborhoods.

2. Preliminary Results

Proposition 1. For any graph G of order $n \geq 2$, $1 \leq lhn(G) \leq n-1$.
Proof: Let G be a connected non-trivial graph. By the definition of the locating hop set, $lhn(G) \geq 1$. Let $v \in V(G)$ and set $S = V(G) \setminus \{v\}$. Then S is a locating hop set of G. Hence, $lhn(G) \leq |S| = n-1$. □

Lemma 1. Let G be a graph with n vertices. If S is a locating hop set of G, then $n \leq |S| + 2^{|S|}$. In particular, $n \leq lhn(G) + 2^{lhn(G)}$.
Proof: Let G be a graph of order n and S is a locating hop set in G. By definition of locating hop set, the collection $\{N_G(a,2) \cap S : a \in V(G) \setminus S\}$ contains exactly $|V(G) \setminus S|$ distinct subsets of S. Hence, $|V(G) \setminus S| = n - |S| \leq 2^{|S|}$, i.e., $n \leq |S| + 2^{|S|}$. In particular, if S is an lhn-set of G, then $n \leq lhn(G) + 2^{lhn(G)}$. □

Theorem 1. Let G be a non-trivial graph. Then $lhn(G) = n-1$ if and only if every component of G is complete.
Proof: Suppose that $lhn(G) = n-1$ and suppose further that G has a component H which is not complete. Then there exist $x, y \in V(H)$ such that $d_H(x,y) = d_G(x,y) = 2$. Let $z \in N_G(x) \cap N_G(y)$ and $S = V(G) \setminus \{x,z\}$. Since $y \in N_G(x,2) \setminus N_G(z,2)$, it follows that $N_G(x,2) \cap S \neq N_G(z,2) \cap S$. Thus, S is a locating hop set and $lhn(G) \leq |S| = n-2$, contrary to the assumption $lhn(G) = n-1$. Therefore, every component of G is complete.

For the converse, suppose that every component of G is complete. Let S be an lhn-set of G. Since $N_G(u,2) \cap S = \varnothing$ $\forall u \in V(G)$, $V(G) \setminus S$ cannot contain two distinct vertices. Consequently, $S = V(G) \setminus \{v\}$ for some vertex v of G. Thus, $lhn(G) = |S| = n-1$. □

Corollary 1. For any positive integer $n \geq 2$, $lhn(K_n) = lhn(\overline{K}_n) = n-1$.

Proposition 2. Let G be a graph on n vertices. Then $lhn(G) = 1$ if and only if $G \in \{K_1, \overline{K}_2, P_2, P_3\}$.
Proof: Suppose $lhn(G) = 1$. By Lemma 1, $n \leq 3$. Clearly, $G = K_1$ if $n = 1$ and $G = K_2 = P_2$ or $G = \overline{K}_2$ if $n = 2$. Suppose $n = 3$. By Theorem 1, $lhn(K_3) = lhn(K_1 \cup P_2) = lhn(\overline{K}_3) = 2$. It follows that $G = P_3$. Thus, $G \in \{K_1, \overline{K}_2, P_2, P_3\}$.

The converse is clear. □

Proposition 3. Let G be a connected graph of order n. If $lhn(G) = 2$, then $3 \leq |V(G)| \leq 6$.
Proof: Suppose that $lhn(G) = 2$. By Lemma 1, $n \leq lhn(G) + 2^{lhn(G)} = 2 + 2^2 = 6$. By Proposition 2, it follows that $3 \leq |V(G)| \leq 6$. □

Proposition 4. Let G be a connected graph of order $n = 4$. Then $lhn(G) = 2$ if and only if $G \neq K_4$.
Proof: Let $lhn(G) = 2$. Then by Corollary 1, $G \neq K_4$.

For the converse, suppose that $G \neq K_4$. Since $n = 4$, by Proposition 2, $lhn(G) \geq 2$. Choose any $u, v \in V(G)$ such that $d_G(u,v) = 2$. Let $w \in N_G(u) \cap N_G(v)$ and let $s \in V(G) \setminus \{u,v,w\}$. Since $u \in N_G(v,2) \setminus N_G(w,2)$, it follows that $S = \{u, s\}$ is a locating set of G. Consequently, $lhn(G) = |S| = 2$. □

Proposition 5. Let G be a connected graph of order $n = 5$. Then $lhn(G) = 2$ if and only if there exist distinct vertices x and y of G satisfying one of the following properties:

(i) $|N_G(x,2) \cap N_G(y,2)| = 0$ and $|N_G(x,2) \setminus \{y\}| = |N_G(y,2) \setminus \{x\}| = 1$.

(ii) $|N_G(x,2) \cap N_G(y,2)| = 1$ and $[(|N_G(x,2) \setminus \{y\}| = |N_G(y,2) \setminus \{x\}| = 2)$ or $(|N_G(x,2) \setminus \{y\}| = 2$ and $|N_G(y,2) \setminus \{x\}| = 1)$ or $(|N_G(x,2) \setminus \{y\}| = 1$ and $|N_G(y,2) \setminus \{x\}| = 2)]$.

Proof: Suppose that $lhn(G) = 2$. Then there exist distinct vertices $x, y \in V(G)$ such that $S = \{x, y\}$ is a minimum locating hop set of G. Hence, $|N_G(x,2) \cap N_G(y,2)| \leq 1$. Suppose $|N_G(x,2) \cap N_G(y,2)| = 0$. Since S is a locating hop set, $|N_G(x,2) \setminus \{y\}| \leq 1$. Suppose $|N_G(x,2) \setminus \{y\}| = 0$. Then $|N_G(y,2) \setminus \{x\}| = 1$ since S is a locating hop set. This implies that there exist at least two vertices say z and w such that $z, w \notin N_G(x,2) \cup N_G(y,2)$. Consequently, $N_G(z,2) = N_G(w,2) = \varnothing$, contrary to our assumption that S is a locating hop set. Thus, $|N_G(x,2) \setminus \{y\}| = 1$. Similarly, $|N_G(y,2) \setminus \{x\}| = 1$. Hence, (i) holds.

Suppose that $|N_G(x,2) \cap N_G(y,2)| = 1$. Let $a \in V(G)$ such that $d_G(x,a) = 2$ and $d_G(y,a) = 2$ and let $b, c \in V(G) \setminus \{x, y, a\}$. Then $b, c \notin N_G(x,2) \cap N_G(y,2)$. Since the subset of S are $\varnothing, \{x, y\}$, $\{x\}$, $\{y\}$ and since $N_G(a,2) \cap S$ is $\{x, y\}$, the remaining two sets $N_G(b,2) \cap S$ and $N_G(c,2) \cap S$ are $\{x\}$ and $\{y\}$ or $\{x\}$ and $\varnothing$ or $\{y\}$ and $\varnothing$, respectively. Thus, $|N_G(x,2) \setminus \{y\}| = |N_G(y,2) \setminus \{x\}| = 2$ or $|N_G(x,2) \setminus \{y\}| = 2$ and $|N_G(y,2) \setminus \{x\}| = 1$ or $|N_G(x,2) \setminus \{y\}| = 1$ and $|N_G(y,2) \setminus \{x\}| = 2$. Therefore, (ii) holds.

For the converse, suppose there exist distinct vertices $x, y \in V(G)$ satisfying (i) or (ii). Let $S = \{x, y\}$. Then S is a minimum locating hop set in G. Therefore, $lhn(G) = 2$. □

Proposition 6. Let G be a connected graph of order $n \geq 3$. If $lhn(G) < slhn(G)$, then $1 + lhn(G) = slhn(G)$.

Proof: Let S be a minimum locating hop set in G. Then S is not a strictly locating hop set in G. Hence, there exists a vertex $u \in V(G) \setminus S$ such that $N_G(u,2) \cap S = S$. Let $S^* = S \cup \{u\}$ and let $z \in V(G) \setminus S^*$. Then $z \neq u$. Since S is a locating hop set and $N_G(u,2) \cap S = S$, $N_G(z,2) \cap S \neq S$. This implies that there exists $w \in S$ such that $w \notin N_G(z,2)$. Since $u \notin S$, $w \neq u$. Thus, $N_G(z,2) \cap S^* \neq S^*$. This implies that S^* is a strictly locating hop set in G. Hence, $slhn(G) \leq 1 + lhn(G)$. Since $lhn(G) < slhn(G)$, $1 + lhn(G) \leq slhn(G)$. Hence, $1 + lhn(G) = slhn(G)$. □

3. Locating Hop Sets in the Join of Graphs

The join of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2. Let G and H be connected non-trivial graphs. A set $S \subseteq V(G + H)$ is a locating hop set in $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets in G and H, respectively, and S_1 or S_2 is a strictly locating set.

Proof: Suppose that S is a locating hop set in $G+H$. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. Suppose $S_1 = \varnothing$. Then for any two distinct vertices $x, y \in V(G)$, $N_{G+H}(x,2) \cap S = N_{G+H}(y,2) \cap S = \varnothing$, contrary to our assumption that S is a locating hop set. Thus, $S_1 \neq \varnothing$. Similarly, $S_2 \neq \varnothing$.

Next, suppose S_1 or S_2, say S_1 is not a locating set. Then there exist $u, v \in V(G)$ such that $N_G(u) \cap S_1 = N_G(v) \cap S_1$. Thus, $x \in [V(G) \setminus N_G(u)] \cap S_1$ if and only if $x \in [V(G) \setminus N_G(v)] \cap S_1$. This implies that $[V(G) \setminus N_G(u)] \cap S_1 = [V(G) \setminus N_G(v)] \cap S_1$. Since $S_2 \cap N_{G+H}(u,2) = \varnothing$ and $S_2 \cap N_{G+H}(v,2) = \varnothing$, it follows that

$$\begin{aligned} N_{G+H}(u,2) \cap S &= N_{G+H}(u,2) \cap S_1 \\ &= [V(G) \setminus N_G(u)] \cap S_1 = [V(G) \setminus N_G(v)] \cap S_1 \\ &= N_{G+H}(v,2) \cap S_1 = N_{G+H}(v,2) \cap S. \end{aligned}$$

Thus, S is not a locating hop set in $G+H$, contrary to our assumption. Therefore, S_1 and S_2 are locating sets in G and H, respectively. Now, suppose that both are not strictly locating sets. Then there exist $p \in V(G) \setminus S_1$ and $q \in V(H) \setminus S_2$ such that $N_G(p) \cap S_1 = S_1$ and $N_H(q) \cap S_2 = S_2$. Consequently, $N_G(p,2) \cap S_1 = \varnothing$ and $N_H(q,2) \cap S_2 = \varnothing$. This implies that $N_{G+H}(p,2) \cap S = N_{G+H}(q,2) \cap S = \varnothing$, contrary to our assumption that S is a locating hop set. Therefore, S_1 is a strictly locating set in G or S_2 is a strictly locating set in H.

For the converse, suppose that S_1 and S_2 are locating sets in G and H, respectively, and S_1 or S_2 is a strictly locating set. Let $x, y \in V(G+H) \setminus S$ with $x \neq y$. If $x, y \in V(G)$, then $N_G(x) \cap S_1 \neq N_G(y) \cap S_1$. Moreover, $N_{G+H}(x,2) \cap S = [V(G) \setminus N_G(x)] \cap S_1 \neq [V(G) \setminus N_G(y)] \cap S_1 = N_{G+H}(y,2) \cap S$. Similarly, if $x, y \in V(H)$, then $N_{G+H}(x,2) \cap S \neq N_{G+H}(y,2) \cap S$. Suppose that $x \in V(G)$ and $y \in V(H)$ and suppose that S_1 is a strictly locating set in G. Then $N_G(x) \cap S_1 \neq S_1$. It follows that $[V(G) \setminus N_G(x)] \cap S_1 = N_{G+H}(x) \cap S \neq \varnothing$. Since $S_1 \cap N_{G+H}(y,2) = \varnothing$, $N_{G+H}(x,2) \cap S \neq N_{G+H}(y,2) \cap S$. Therefore, S is a locating hop set in $G+H$. □

Corollary 2. Let G and H be connected non-trivial graphs. Then

$$lhn(G+H) = min\{sln(H) + ln(G), sln(G) + ln(H)\}.$$

Proof: Let S be a minimum locating hop set in $G+H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. By Theorem 2, S_1 and S_2 are locating sets in G and H, respectively, where S_1 or S_2 is a strictly locating set. If S_1 is strictly locating set, then $sln(G)+ln(H) \leq |S_1| + |S_2| \leq |S| = lhn(G+H)$. If S_2 is strictly locating set, then $sln(H) + ln(G) \leq |S_2|+|S_1| \leq |S| = lhn(G+H)$. Thus, $lhn(G+H) \geq min\{sln(H)+ln(G), sln(G)+ln(H)\}$. Next, suppose that $sln(G) + ln(H) \leq sln(H) + ln(G)$. Let S_1 be a minimum strictly locating set in G and S_2 be a minimum locating set in H. Then $S = S_1 \cup S_2$ is a locating hop set by Theorem 2. Hence, $lhn(G+H) \leq |S| = |S_1|+|S_2| = sln(G)+ln(H)$. Therefore, $lhn(G+H) = min\{sln(H) + ln(G), sln(G) + ln(H)\}$. □

Theorem 3. ([5],[11]) Let G be a connected graph of order $n \geq 2$. If $ln(G) < sln(G)$, then $1 + ln(G) = sln(G)$.

Corollary 3. Let G be a connected non-trivial graph and let K_n be a complete graph of order $n \geq 2$. Then $lhn(G + K_n) = sln(G) + n - 1$.
Proof: Note that $ln(K_n) = n - 1$ and $sln(K_n) = n$. By Corollary 2, $lhn(G + K_n) = min\{sln(G) + n - 1, ln(G) + n\}$ and by Theorem 3, $sln(G) - 1 \leq ln(G)$. Therefore, $lhn(G + K_n) = min\{sln(G) + n - 1, ln(G) + n\} = sln(G) + n - 1$. □

Theorem 4. Let G be a connected non-trivial graph and let $K_1 = \langle v \rangle$. Then $S \subseteq V(G + K_1)$ is a locating hop set in $G + K_1$ if and only if $v \notin S$ and S is a strictly locating set in G or $S = \{v\} \cup S_1$, where S_1 is a locating set in G.
Proof: Let $S \subseteq V(G + K_1)$ be a locating hop set in $G + K_1$. If $v \notin S$, then $S \subseteq V(G)$. Let $u, s \in V(G) \setminus S$. Then $N_{G+K_1}(u, 2) \cap S \neq N_{G+K_1}(s, 2) \cap S$. It follows that $[V(G) \setminus N_G(u)] \cap S \neq [V(G) \setminus N_G(s)] \cap S$. Therefore,

$$\begin{aligned} N_G(u) \cap S &= [V(G) \setminus N_{G+K_1}(u, 2)] \cap S \\ &\neq [V(G) \setminus N_{G+K_1}(v, 2)] \cap S = N_G(v) \cap S, \end{aligned}$$

showing that S is a locating set in G. Suppose S is not a strictly locating set in G. Then there exists $z \in V(G) \setminus S$ such that $N_G(z) \cap S = S$. This implies that $N_G(z, 2) \cap S = \varnothing = N_G(v, 2) \cap S$, contrary to our assumption that S is a locating hop set. Hence, S is a strictly locating set in G. Next, suppose that $S = \{v\} \cup S_1$, where $S_1 = V(G) \cap S$. Then $S_1 \neq \varnothing$ and is a locating set in G. For the converse, suppose $v \notin S$ and S is a strictly locating set in G. Let $x, y \in V(G + K_1) \setminus S$. If $x, y \in V(G)$, then

$$\begin{aligned} N_{G+K_1}(x, 2) \cap S &= [V(G) \setminus N_G(x)] \cap S \\ &\neq [V(G) \setminus N_G(y)] \cap S = N_{G+K_1}(y, 2) \cap S. \end{aligned}$$

Suppose $x \in V(G)$ and $y = v$. Then $N_{G+K_1}(v, 2) \cap S = \varnothing$. Since S is a strictly locating set in G, $N_G(x) \cap S \neq S$. Then

$$\begin{aligned} N_{G+K_1}(x, 2) \cap S &= [V(G) \setminus N_G(x)] \cap S \\ &\neq [V(G) \setminus N_G(v)] \cap S = N_{G+K_1}(v, 2) \cap S. \end{aligned}$$

Therefore, S is a locating hop set in $G + K_1$. Next, suppose that $S = \{v\} \cup S_1$, where S_1 is a locating set of G. Let $x, y \in V(G + K_1) \setminus S$ with $x \neq y$. Then $x, y \in V(G) \setminus S_1$ and $N_G(x) \cap S_1 \neq N_G(y) \cap S_1$. Thus,

$$\begin{aligned} N_{G+K_1}(x, 2) \cap S &= [V(G) \setminus N_G(x)] \cap S_1 \\ &\neq [V(G) \setminus N_G(y)] \cap S_1 = N_{G+K_1}(y, 2) \cap S. \end{aligned}$$

Hence, S is a locating hop set in $G + K_1$. □

Corollary 4. Let G be a connected non-trivial graph. Then $lhn(G + K_1) = sln(G)$.
Proof: By Theorem 4, $lhn(G+K_1) = min\{sln(G), ln(G)+1\}$. By Theorem 3, $sln(G)-1 \leq ln(G)$. Hence, $sln(G) \leq ln(G) + 1$. Therefore, $lhn(G + K_1) = sln(G)$. □

4. Locating Hop Sets in the Corona of Graphs

The corona of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the vertex v_i of G to every vertex of the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are joined or attached to the vertex v. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v$.

Theorem 5. Let G be a non-trivial connected graph and let H be any non-trivial graph. Then $S \subseteq V(G \circ H)$ is a locating hop set of $G \circ H$ if and only if $S = A \cup [\cup_{v \in V(G)} D_v]$ and

- (i) $A \subseteq V(G)$ such that for any two distinct vertices $v, w \in V(G) \setminus A$, $N_G(v) \neq N_G(w)$ or $N_G(v, 2) \cap A \neq N_G(w, 2) \cap A$;
- (ii) D_v is a locating set in H^v for each $v \in V(G)$;
- (iii) D_w is a dominating set of H^w for each $w \in V(G)$ such that $N_G(v) = \{w\}$ for some $v \in V(G) \setminus A$; and
- (iv) D_v or D_w is a strictly locating set for each pair of distinct vertices v and w of G with $N_G(v) \cap A = N_G(w) \cap A$.

Proof: Suppose S is a locating hop set in $G \circ H$. Let $A = S \cap V(G)$ and let $D_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup [\cup_{v \in V(G)} D_v]$. Let $v, w \in V(G) \setminus A$ with $v \neq w$. Since S is a locating hop set in $G \circ H$,

$$\begin{aligned}
[N_G(v,2) \cap A] \cup [\cup_{x \in N_G(v)} D_x] &= N_{G \circ H}(v, 2) \cap S \\
&\neq N_{G \circ H}(w, 2) \cap S \\
&= [N_G(w, 2) \cap A] \cup [\cup_{y \in N_G(w)} D_y].
\end{aligned}$$

This implies that $N_G(v, 2) \cap A \neq N_G(w, 2) \cap A$ or $N_G(v) \neq N_G(w)$, showing that (i) holds.

Next, let $v \in V(G)$ and let $a, b \in V(H^v) \setminus D_v$ with $a \neq b$. Since S is a locating hop set in $G \circ H$,

$$\begin{aligned}
([V(H^v) \setminus N_{H^v}(a)] \cap D_v) \cup [N_G(v) \cap A] &= N_{G \circ H}(a, 2) \cap S \\
&\neq N_{G \circ H}(b, 2) \cap S \\
&= ([V(H^v) \setminus N_{H^v}(b)] \cap D_v) \cup [N_G(w) \cap A].
\end{aligned}$$

Hence, $[V(H^v) \setminus N_{H^v}(a)] \cap D_v \neq [V(H^v) \setminus N_{H^v}(b)] \cap D_v$. This implies that $N_{H^v}(a) \cap D_v \neq N_{H^v}(b) \cap D_v$, showing D_v is a locating set of H^v. Hence, (ii) holds. To show that (iii) holds, suppose there exists $w \in V(G)$ such that $N_G(v) = \{w\}$ for some $v \in V(G) \setminus A$. If $D_w = V(H^w)$, then we are done. So suppose that $D_w \neq V(H^w)$ and let $q \in V(H^w) \setminus D_w$. Then by assumption and the fact that S is a locating hop set in $G \circ H$,

$$D_w \cup (N_G(w) \cap A) = N_{G \circ H}(v, 2) \cap S$$

$$\neq N_{G\circ H}(q,2)\cap S$$
$$= ([V(H^w)\setminus N_{H^w}(q)]\cap D_w)\cup[N_G(w)\cap A].$$

This implies that $[(V(H^w)\setminus N_{H^w}(q))\cap D_w]\neq D_w$, that is, $N_{H^w}(q)\cap D_w\neq\varnothing$. This shows that D_w is a dominating set of H^w. Finally, let $v,w\in V(G)$ with $v\neq w$ and $N_G(w)\cap A=N_G(v)\cap A$. Suppose D_v and D_w are not strictly locating sets of H^v and H^w, respectively. Then there exist $x\in V(H^v)\setminus D_v$ and $y\in V(H^w)\setminus D_w$ such that $N_{H^v}(x)\cap D_v=D_v$ and $N_{H^w}(y)\cap D_w=D_w$. It follows that $[V(H^v)\setminus N_{H^v}(x)]\cap D_v=\varnothing$ and $[V(H^w)\setminus N_{H^w}(y)]\cap D_w=\varnothing$. This would imply that

$$\begin{aligned}N_{G\circ H}(x,2)\cap S&=[(V(H^v)\setminus N_{H^v}(x))\cap D_v]\cup(N_G(v)\cap A)\\&=N_G(v)\cap A=N_G(w)\cap A\\&=[(V(H^w)\setminus N_{H^w}(y))\cap D_w]\cup(N_G(w)\cap A)\\&=N_{G\circ H}(y,2)\cap S,\end{aligned}$$

contrary to the assumption that S is a locating hop set of $G\circ H$. Thus, (iv) holds.

For the converse, suppose that S is as described and satisfies properties (i)-(iv). Let $a,b\in V(G\circ H)\setminus S$ with $a\neq b$ and let $v,w\in V(G)$ such that $a\in V(v+H^v)$ and $b\in V(w+H^w)$. Consider the following cases:

Case 1: $v=w$.

Suppose $a,b\in V(H^v)\setminus D_v$. By (ii), $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose $a=v$ and $b\in V(H^v)\setminus D_v$. Pick any $z\in N_G(v)$. Since $D_z\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$, it follows that $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$.

Case 2: $v\neq w$.

Suppose $a=v$ and $b=w$. Then $v,w\in V(G)\setminus A$. By property (i), $N_G(v)\neq N_G(w)$ or $N_G(v,2)\cap A\neq N_G(w,2)\cap A$. If $N_G(v,2)\cap A\neq N_G(w,2)\cap A$, then $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose $N_G(v)\neq N_G(w)$. We may assume that there exists $p\in N_G(v)\setminus N_G(w)$. Then $D_p\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$. Hence, $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$.

Next, suppose that $a=v$ and $b\in V(H^w)\setminus D_w$ (or $b=w$ and $a\in V(H^v)\setminus D_v$). If $|N_G(v)|>1$ or $vw\notin E(G)$, pick any $z\in N_G(v)\setminus\{w\}$. Then $D_z\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$. It follows that $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose that $N_G(v)=\{w\}$. Then D_w is a dominating set by (iii). Hence, $[(V(H^w)\setminus N_{H^w}(b))\cap D_w]\neq D_w$. This implies that

$$\begin{aligned}N_{G\circ H}(a,2)\cap S&=D_w\cup(N_G(w)\cap A)\\&\neq[(V(H^w)\setminus N_{H^w}(b))\cap D_w]\cup(N_G(w)\cap A)\\&=N_{G\circ H}(b,2)\cap S.\end{aligned}$$

Finally, suppose that $a\in V(H^v)\setminus D_v$ and $b\in V(H^w)\setminus D_w$. If $[V(H^v)\setminus N_{H^v}(a)]\cap D_v\neq\varnothing$ and $[V(H^w)\setminus N_{H^w}(b)]\cap D_w\neq\varnothing$, then $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose one, say $[V(H^v)\setminus N_{H^w}(a)]\cap D_v=\varnothing$. If $N_G(v)\cap A\neq N_G(w)\cap A$, then $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. If $N_G(v)\cap A=N_G(w)\cap A$, then $[V(H^w)\setminus N_{H^w}(b)]\cap D_w\neq\varnothing$ by (iv). Thus, $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$.

Accordingly, S is a locating hop set of $G\circ H$. □

The next result is an immediate consequence of Theorem 5.

Corollary 5. Let G be a non-trivial connected graph of order m and let H be any graph. Then the following statements hold:

(i) $m \cdot ln(H) \leq lhn(G \circ H) \leq lhn(G) + m \cdot \gamma_{SL}(H)$.

(ii) If $\delta(G) \geq 2$, then $lhn(G \circ H) \leq lhn(G) + m \cdot sln(H)$.

(iii) If G is point determining, then $lhn(G \circ H) \leq m \cdot \gamma_{SL}(H)$.

(iv) If G is point determining and $\delta(G) \geq 2$, then

$$lhn(G \circ H) \leq m \cdot sln(H).$$

Moreover, if in addition, $ln(H) = sln(H)$, then

$$lhn(G \circ H) = m \cdot ln(H) = m \cdot sln(H).$$

Proof: Let S be a minimum locating hop set (lhn-set) in G. Then $S = A \cup [\cup_{v \in V(G)} D_v]$ and satisfies the conditions in Theorem 5. In particular, D_v is a (minimum) locating set in H^v for each $v \in V(G)$. Hence, $m \cdot ln(H) \leq |A| + \sum_{v \in V(G)} |D_v| = |S| = lhn(G \circ H)$.

Now, let A_1 be a locating hop set in G and let L_v be a strictly locating-dominating set (γ_{SL}-set) in H^v for each $v \in V(G)$. Then $S = A_1 \cup [\cup_{v \in V(G)} L_v]$ is a locating hop set in $G \circ H$ by Theorem 5. This implies that $lhn(G \circ H) \leq |S| = lhn(G) + m \cdot \gamma_{SL}(H)$, showing that (i) holds. If $\delta(G) \geq 2$ and each L_v is a minimum strictly locating set (sln-set) in H^v, then S is a locating hop set in $G \circ H$ by Theorem 5. Thus, (ii) holds, that is, $lhn(G \circ H) \leq |S| = lhn(G) + m \cdot sln(H)$.

Suppose G is a point determining graph. For each $v \in V(G)$, let T_v be a minimum strictly locating-dominating set (γ_{SL}-set) in H^v. Then $S_1 = \cup_{v \in V(G)} T_v$ is a locating hop set in $G \circ H$ by Theorem 5. This implies that $lhn(G \circ H) \leq |S| = m \cdot \gamma_{SL}(H)$, showing that (iii) holds. Moreover, if we impose that $\delta(G) \geq 2$, then each set T_v can be taken as a strictly locating set of H^v. Now, S_1 is still a locating hop set in $G \circ H$ by Theorem 5. Thus, $lhn(G \circ H) \leq m \cdot sln(H)$. Suppose now that, in addition, $ln(H) = sln(H)$. Then $lhn(G \circ H) \leq m \cdot sln(H) = m \cdot ln(H)$. Combining this with an inequality in (i), it follows that $lhn(G \circ H) = m \cdot ln(H) = m \cdot sln(H)$. □

Corollary 6. Let G be a cycle of order $m = 4$ and H be a non-trivial graph. Then

$$lhn(G \circ H) = \begin{cases} m \cdot sln(H) + 2 & \text{if } ln(H) = sln(H) \\ 2sln(H) + 2ln(H) + 2 & \text{if } ln(H) < sln(H). \end{cases}$$

Proof: Let $C_4 = [a, b, c, d, a]$ and let S be a minimum locating hop set in $C_4 \circ H$. Put $D_v = S \cap V(H^v)$ for each $v \in C_4$. By Theorem 5(i), $A = S \cap V(C_4) \neq \varnothing$. Without loss of generality, we suppose that $a \in A$. Suppose further that $b, d \notin A$. Since $N_{C_4}(b) = N_{C_4}(d)$ and $N_{C_4}(b, 2) \cap A = N_{C_4}(d, 2) \cap A = \varnothing$, A does not satisfy Theorem 5(i), contrary to our assumption that S is a locating hop set in $C_4 \circ H$. Hence, either $b \in A$ or $d \in A$, say $b \in A$. Since $N_{C_4}(a) \cap A = N_{C_4}(c) \cap A$, D_a or D_c, say D_a must be a minimum strictly locating set in H^a. It follows that D_c is a minimum locatng set in H^c by Theorem $5(iv)$ and the fact that S is an lhn-set. Similarly, one of D_b and D_d is a minimum strictly locating set and the other a minimum locating set. Since S is a minimum locating hop set in $C_4 \circ H$, $|A| = 2$ (increasing the number of elements of A will not change the above requirement for the sets D_v), and two subsets of S in copies of H are strictly locating sets. Therefore,

$$lhn(C_4 \circ H) = |A| + 2ln(H) + 2sln(H) = 2ln(H) + 2sln(H) + 2.$$

If $ln(H) = sln(H)$, then $lhn(C_4 \circ H) = 4ln(H) + 2 = 4sln(H) + 2$. □

5. Conclusion

As the concept of locating set plays an important role in the study of locating domination in a graph, the concept of locating hop set plays a similar important part in the study of locating hop domination. Locating hop sets in the join and the corona of two graphs have been characterized. These type of sets may be studied also in other graphs including those graphs which can be obtained by applying other binary operations of graphs. Furthermore, it may be interesting to study the relationship between this new parameter and other related known graph-theoretic parameters.

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