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Generalized Laguerre-Apostol-Frobenius-Type Poly-Genocchi Polynomials of Higher Order with Parameters a, b and c

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Abstract. In this paper, the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c are defined using the concept of polylogarithm, Laguerre, Apostol and Frobenius polynomials. These polynomials possess numerous properties including recurrence relations, explicit formulas and certain differential identity. Moreover, some connections of these higher order generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials to Stirling numbers of the second kind and different variations of higher order Euler and Bernoulli-type polynomials are obtained.

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1. Introduction

The Genocchi numbers G_n are defined by means of the following generating function

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t + 1}, \quad |t| < \pi.$$

These numbers have been generalized in different ways [2, 3, 8, 9, 11, 14, 19, 21–23, 26–28]. Most of the generalizations are done by mixing the Genocchi numbers with the concept of some known polynomials. For instance, mixing with exponential polynomials yields the Genocchi polynomials and Genocchi polynomials of higher order, which are given as follows:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}, \quad |t| < \pi, \tag{1}$$

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$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}. \tag{2}$$

These polynomials are well-studied and two of the most recent studies are the works of Corcino-Corcino [5, 7] on asymptotic approximations.

Now, mixing with the Apostol polynomials yields the Apostol-Genocchi polynomials, and Apostol-Genocchi polynomials of higher order, which are respectively defined as follows:

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1}e^{xt}, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{4}$$

where $|t| < \pi$ when $\lambda = 1$ and $|t| < \log(-\lambda)$ when $\lambda \neq 1, \lambda \in \mathbb{C}$. These polynomials were given Fourier series expansion in [6]. Also, mixing with Frobenius polynomials yields the so-called Frobenius-Genocchi polynomials, which are given by

$$\sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u}e^{xt}, \tag{5}$$

(see [3, 11–13, 22, 25, 27, 28]). Moreover, mixing the Genocchi numbers with the concept of polylogarithm $\mathrm{Li}_k(z)$

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^k}, k \in \mathbb{Z}, \tag{6}$$

yields the poly-Genocchi polynomials, which are defined as follows

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1}e^{xt}. \tag{7}$$

Furthermore, with a slight modification of the generating function, another generalization, denoted by $G_{n,2}^{(k)}(x)$, was defined by Kim et al. [26] as follows

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt}. \tag{8}$$

These polynomials are called modified poly-Genocchi polynomials. Note that, when $k = 1$, equations (7) and (8) give the Genocchi polynomials in (1). That is,

$$G_n^{(1)}(x) = G_{n,2}^{(1)}(x) = G_n(x).$$

Kim et. al [26] obtained several properties of these polynomials.

Kurt [14] defined two forms of generalized poly-Genocchi polynomials with parameters a, b, and c as follows

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{x^n}{n!} \tag{9}$$

$$\frac{2Li_k(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x;a,b,c)\frac{x^n}{n!}. \tag{10}$$

These were motivated by the generalizations introduced in (7) and (8), respectively. Note that, when $x = 0$, (7) reduces to

$$\frac{2Li_k(1-e^t)}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)}\frac{x^n}{n!}, \tag{11}$$

where $G_n^{(k)}$ are called the poly-Genocchi numbers. Recently, a new variation of poly-Genocchi polynomials with parameters a, b and c was defined in [8] by mixing the definitions of Apostol and Frobenius polynomials, namely, the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c. More precisely, the said polynomials, denoted by

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} c^{xt}. \tag{12}$$

It is worth-mentioning that, using multi-polylogarithm, the generalized poly-Genocchi polynomials in (9) and (10) have been extended further in [19].

On the other hand, a generalization of Laguerre polynomials, denoted by $L_n(x, y)$, was defined in [10] by means of the following generating function

$$e^{yt}C_0(xt) = \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}, \tag{13}$$

where $C_0(x)$ is the $0-th$ order Tricomi function [20]

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}, \quad C_0(0) := 1. \tag{14}$$

This 2-variable generalization of Laguerre polynomials possessed the following explicit formula

$$L_n(x,y) = \sum_{s=0}^{n} \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2}.$$

Also, the 2-variable generalization of Hermite polynomials were defined by Kampe de Feriet [1] as follows

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}, \tag{15}$$

which reduces to the ordinary Hermite polynomials $H_n(x)$ when taking $y = -1$ and x is replaced by $2x$. These generalized Hermite polynomials possessed the following explicit formula

$$H_n(x, y) = n! \sum_{r=0}^{n} \frac{y^r x^{n-2r}}{r!(n-2r)!}.$$

This can further be generalized through the following polynomials, denoted by $H_{n,L}(x; u, v)$ as follows

$$e^{vt+wt^2} C_0(xt) = \sum_{n=0}^{\infty} H_{n,L}(x; u, v) \frac{t^n}{n!}. \tag{16}$$

We call these polynomials as generalized Laguerre-Hermite polynomials.

In this paper, a new variation of poly-Genocchi polynomials with parameters a, b and c is constructed by mixing the concepts of Laguerre, Apostol and Frobenius polynomials. These polynomials are called the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c. Some special cases of these polynomials are enumerated and some identities that contain a number of relations of this new variation with some Genocchi-type polynomials are provided. One section of the paper devotes its discussion on some identities that link the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c to Appell polynomials. Finally, some connections of these higher order generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials to Stirling numbers of the second kind and different variations of higher order Euler and Bernoulli-type polynomials are discussed.

2. Definition and Some Preliminary Results

Let us formally define the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c.

Definition 2.1. The Generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c, denoted by $\mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, v, w, a, b, c)$, are defined as coefficients of the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x; \lambda, u, v, w, a, b, c) \frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - u a^{-t}} \right)^{\alpha} c^{vt+wt^2} C_0(xt), \tag{17}$$

where $|t| < \frac{\sqrt{(\ln(\frac{\lambda}{u}))^2 + 4\pi^2}}{|\ln a + \ln b|}$.

Now, let us consider some preliminary results of this paper. It is important to note that, using (15),

$$c^{xt+yt^2} = c^{xt} c^{yt^2} = e^{xt \ln c} e^{yt^2 \ln c} = e^{(x \ln c)t} e^{(y \ln c)t^2}$$

$$= \sum_{n=0}^{\infty} H_n(x \ln c, y \ln c)\frac{t^n}{n!}.$$

By making use of (17), we can easily establish the following relation.

Theorem 2.2. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation*

$$\begin{aligned}&\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v+y,w+z,a,b,c)\\&\quad=\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)H_m(y\ln c,z\ln c).\end{aligned}\tag{18}$$

Proof. We can write (17) as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v+y,w+z,a,b,c)\frac{t^n}{n!}\\&=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{vt+wt^2}C_0(xt)c^{yt+zt^2}\\&=\left(\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}H_n(y\ln c,z\ln c)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)H_m(y\ln c,z\ln c)\right)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

The next result is a kind of addition formula for $\mathcal{G}_{n}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)$.

Theorem 2.3. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation*

$$\begin{aligned}&\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v+y,w,a,b,c)\\&\quad=\sum_{m=0}^{n}\binom{n}{m}(y\ln c)^m\mathcal{G}_{n-m,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c).\end{aligned}\tag{19}$$

Proof. We can write (17) as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v+y,w,a,b,c)\frac{t^n}{n!}\\&=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{vt+wt^2}C_0(xt)e^{yt\ln c}\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(yt\ln c)^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)(y\ln c)^m\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

By giving special values to the parameters involved, the polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)$ reduce to some interesting Genocchi-type polynomials.

(i) When $c = e$, equation (17) reduces to

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,e)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} e^{vt+wt^2}C_0(xt). \quad (20)$$

For convenience, we use $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)$ to denote $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,e)$. That is,

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} e^{vt+wt^2}C_0(xt). \quad (21)$$

(ii) When $a = 1, b = e$, (21) will reduce to

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,1,e)\frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-(1-u)t})}{\lambda e^t - u}\right)^{\alpha} e^{vt+wt^2}C_0(xt). \quad (22)$$

We may use the notations

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w) = \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,1,e)$$
$$\mathcal{G}_{n,L}^{(k)}(x;\lambda,u,v,w) = \mathcal{G}_{n,L}^{(k)}(x;\lambda,u,v,w,1,e)$$

and call them generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order and generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials, respectively.

(iii) When $\lambda = 1$, (22) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;u,v,w,1,e)\frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-(1-u)t})}{e^t - u}\right)^{\alpha} e^{vt+wt^2}C_0(xt). \quad (23)$$

which is the higher order version of equation (8) and are called the higher order Laguerre-poly-Genocchi polynomials. We may use $\mathcal{G}_{n,L}^{(k,\alpha)}(x;u,v,w)$ to denote $\mathcal{G}_{n,L}^{(k,\alpha)}(x;u,v,w,1,e)$.

(iv) When $k = 1$, (22) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(1,\alpha)}(x;\lambda,u,v,w)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{vt+wt^2} C_0(xt), \tag{24}$$

and when $\lambda = 1$, (24) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(1,\alpha)}(x;1,u,v,w)\frac{t^n}{n!} = \left(\frac{(1-u)t}{e^t - u}\right)^{\alpha} e^{vt+wt^2} C_0(xt),$$

where $\mathcal{G}_{n,L}^{(1,\alpha)}(x;\lambda,u,v,w) = \mathcal{G}_{n,L}^{(\alpha)}(x;\lambda,u,v,w)$ and $\mathcal{G}_{n,L}^{(1,\alpha)}(x;1,u,v,w) = \mathcal{G}_{n,L}^{(\alpha)}(x;u,v,w)$ are called the Generalized Laguerre-Apostol-Frobenius-type Genocchi polynomials and Laguerre-Frobenius-Genocchi polynomials of higher order in (4) and (2), respectively. Furthermore, when $\alpha = 1$, we have

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}(x;\lambda,u,v,w)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u} e^{vt+wt^2} C_0(xt), \tag{25}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}(x;u,v,w)\frac{t^n}{n!} = \frac{(1-u)t}{e^t - u} e^{vt+wt^2} C_0(xt),$$

where $\mathcal{G}_{n,L}(x;\lambda,u,v,w)$ and $\mathcal{G}_{n,L}(x;u,v,w)$ are called the Generalized Laguerre-Apostol-Frobenius-type Genocchi polynomials and Laguerre-Frobenius-Genocchi polynomials in (4) and (2), respectively.

(v) When $v = w = 0$, (17) reduces to

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = C_0(xt)\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha}. \tag{26}$$

Consider a special case of (21) by taking $x = 0$. This gives

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(0;\lambda,u,v,w,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} c^{vt+wt^2}. \tag{27}$$

We use the notation $\mathcal{G}_{n,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c) = \mathcal{G}_{n,L}^{(k,\alpha)}(0;\lambda,u,v,w,a,b,c)$ and call them the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi numbers of higher order with parameters a, b and c. The following theorem contains an identity that expresses $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)$ as polynomial in x with $\mathcal{G}_{n,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c)$ as coefficients.

Theorem 2.4. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation,*

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c) = \sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i}{i!}\mathcal{G}_{n-i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c)x^i. \tag{28}$$

Proof. Equation (17) can be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{vt+wt^2}C_0(xt)$$
$$=\left(\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n(xt)^n}{(n!)^2}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(-xt)^{n-i}}{[(n-i)!]^2}\mathcal{G}_{i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c)\frac{t^i}{i!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{n-i}}{(n-i)!}\mathcal{G}_{i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b,c)x^{n-i}\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

The next identity gives the relation between

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c))\ \text{ and }\ \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w).$$

Theorem 2.5. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a,b,c satisfy the relation,*

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)=(\ln ab)^n\mathcal{G}_{n,L}^{(k,\alpha)}\left(\frac{x}{\ln ab};\lambda,u,\frac{v\ln c+\alpha\ln a}{\ln ab},\frac{w\ln c}{(\ln ab)^2}\right). \tag{29}$$

Proof. Using (17), we have

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}$$
$$=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{a^{-t}(\lambda(ab)^t-u)}\right)^{\alpha}e^{\frac{v\ln c}{\ln ab}t\ln ab+\frac{w\ln c}{(\ln ab)^2}(t\ln ab)^2}C_0\left(\frac{x}{\ln ab}t\ln ab\right)$$
$$=\left(\frac{Li_k(1-e^{-(1-u)t\ln ab})}{\lambda e^{t\ln ab}-u}\right)^{\alpha}e^{\frac{v\ln c+\alpha\ln a}{\ln ab}t\ln ab+\frac{w\ln c}{(\ln ab)^2}(t\ln ab)^2}C_0\left(\frac{x}{\ln ab}t\ln ab\right)$$
$$=\sum_{n=0}^{\infty}(\ln ab)^n\mathcal{G}_{n,L}^{(k,\alpha)}\left(\frac{x}{\ln ab};\lambda,u,\frac{v\ln c+\alpha\ln a}{\ln ab},\frac{w\ln c}{(\ln ab)^2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

3. A Differential Identity and Its Consequences

In this section, we consider $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)$ as polynomial in v. Now, applying the first derivative to equation (17) with respect to v yields

$$\sum_{n=0}^{\infty}\frac{d}{dv}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}=t(\ln c)\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{(\lambda b^t-ua^{-t})}\right)^{\alpha}e^{vt\ln c+wt^2\ln c}C_0(xt)$$

$$\sum_{n=0}^{\infty} \frac{d}{dv} \mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty}(\ln c)\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\frac{d}{dv} \mathcal{G}_{n+1,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^{n}}{n!} = \sum_{n=0}^{\infty}(\ln c)\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the following differential identity, which can be used to classify generalized Laguerre-Apostol-type poly-Genocchi polynomials of higher order as Appell polynomials [15, 16, 24].

Theorem 3.1. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b, c satisfy the relation,*

$$\frac{d}{dv}\mathcal{G}_{n+1,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c) = (n+1)(\ln c)\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c). \tag{30}$$

Remark 3.1. When $c = e$, equation (30) reduces to

$$\frac{d}{dv}\mathcal{G}_{n+1,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b) = (n+1)\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b), \tag{31}$$

where $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)$ is the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials in (21). Consequently, this makes $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)$ an Appell polynomial.

Being classified as Appell polynomials, the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)$ must possess the following properties

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b) = \sum_{i=0}^{n}\binom{n}{i}c_i x^{n-i}$$
$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b) = \left(\sum_{i=0}^{\infty}\frac{c_i}{i!}D^i\right)x^n,$$

for some scalar $c_i \neq 0$. It is then necessary to find the sequence $\{c_n\}$. However, by using (28) with $c = e$, $c_i = \mathcal{G}_{i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)$. This implies the following corollary.

Corollary 3.2. *The generalized Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b, c satisfy the formula,*

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b) = \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_{i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{i!}D^i\right)x^n. \tag{32}$$

In particular, when $n=3$, (32) gives

$$\begin{aligned}
\mathcal{G}_{3,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b) &= \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_{i,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{i!}D^i\right)x^3\\
&= \frac{\mathcal{G}_{0,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{0!}x^3+\frac{\mathcal{G}_{1,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{1!}D^1x^3\\
&\quad+\frac{\mathcal{G}_{2,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{2!}D^2x^3+\frac{\mathcal{G}_{3,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)}{3!}D^3x^3
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{G}_{0,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)x^3+3\mathcal{G}_{1,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)x^2+3\mathcal{G}_{2,L}^{(k,\alpha)}(\lambda,u,v,w,a,b)x\\
&\quad+\mathcal{G}_{3,L}^{(k,\alpha)}(\lambda,u,v,w,a,b).
\end{aligned}$$

The next corollary immediately follows from equation (31) and the characterization of Appell polynomials [15, 16, 24].

Corollary 3.3. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials with parameters a,b,c satisfy the addition formula*

$$\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v+y,w,a,b)=\sum_{i=0}^{\infty}\binom{n}{i}\mathcal{G}_{i,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b)y^{n-i}. \tag{33}$$

Remark 3.2. Corollary 3.3 can also be deduced immediately from Theorem 2.3 by taking $c=e$.

4. Connections with Some Special Numbers and Polynomials

In this section, some connections of the higher order generalized Laguerre-Apostol-type poly-Genocchi polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ with other well-known special numbers and polynomials will be established.

Recently, Pathan [17, 18] defined the generalized Hermite-Bernoulli polynomials of two variables, denoted by $B_{n,H}^{(s)}(v,w)$, as follows:

$$\left(\frac{t}{e^t-1}\right)^s e^{vt+wt^2}=\sum_{n=0}^{\infty}B_{n,H}^{(s)}(v,w)\frac{t^n}{n!}. \tag{34}$$

When $w=0$, these polynomials simply reduce to Bernoulli polynomials of order s. Here, we define the generalized Hermite-Apostol-type Frobenius-Euler polynomials, denoted by $E_{n,H}^{(s)}(\mu,v,w,\lambda)$, as follows

$$\left(\frac{1-\mu}{\lambda e^t-\mu}\right)^s e^{vt+wt^2}=\sum_{n=0}^{\infty}E_{n,H}^{(s)}(\mu,v,w,\lambda)\frac{t^n}{n!}. \tag{35}$$

When $s = 1, w = 0$, (35) gives $E^{(s)}_{n,H}(\mu, v, 0, \lambda)$, the Apostol-type Frobenius-Euler polynomials in [23]. Now, if $\lambda = 0$, we can define the generalized Hermite-Frobenius-Euler polynomials, denoted by $E^{(s)}_{n,H}(\mu, v, w)$, as follows:

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{vt+wt^2} = \sum_{n=0}^{\infty} E^{(s)}_{n,H}(\mu, v, w)\frac{t^n}{n!}. \tag{36}$$

The following theorem contains an identity that relates the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c to Stirling numbers of the second kind $\left\{ n \atop m \right\}$ defined in [4] by

$$\sum_{n=m}^{\infty} \left\{ n \atop m \right\} \frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{37}$$

Here, it is important to note that if $(c_0, c_1, \ldots, c_j, \ldots)$ is any sequence of numbers and l is a positive integer, then

$$\begin{aligned}\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^l &= \prod_{i=1}^{l}\left(\sum_{n_i=0}^{\infty} \frac{c_{n_i}}{n_i!} t^{n_i}\right)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{n_1+n_2+\ldots+n_\alpha=n} \prod_{i=1}^{l} c_{n_i}\binom{n}{n_1, n_2, \ldots, n_\alpha}\right\}\frac{t^n}{n!}. \end{aligned}\tag{38}$$

(see [4]). Now, we are ready to introduce the following theorem.

Theorem 4.1. *The generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfies the relation,*

$$\begin{aligned}&\mathcal{G}^{(k,\alpha)}_{n,L}(x; \lambda, u, v, w, a, b, c)\\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{\alpha}(\ln ab)^{n-j}\mathcal{G}^{(1,\alpha)}_{n-j,L}\left(\frac{x}{\ln ab}; \lambda, u, \frac{v\ln c + \alpha \ln a}{\ln ab}, \frac{w \ln c}{(\ln ab)^2}\right) d_j\end{aligned} \tag{39}$$

where

$$d_j = \sum_{n_1+n_2+\ldots+n_\alpha=j} \prod_{i=1}^{\alpha} c_{n_i}\binom{j}{n_1, n_2, \ldots, n_\alpha}$$

$$c_j = \sum_{m=0}^{j}(-1)^{m+j+1}\frac{((1-u)\ln ab)^j m! \left\{ j+1 \atop m+1 \right\}}{(j+1)(m+1)^{k-1}}.$$

Proof. Now, (17) can be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}=\frac{c^{vt+wt^2}C_0(xt)}{(\lambda b^t-ua^{-t})^\alpha}\left(\sum_{m=1}^{\infty}\frac{(1-e^{-(1-u)t\ln ab})^m}{m^k}\right)^\alpha$$

$$=\frac{c^{vt+wt^2}C_0(xt)}{(\lambda b^t-ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)^k}\right)^\alpha$$
$$=\frac{c^{vt+wt^2}C_0(xt)}{(\lambda b^t-ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{m!}{(m+1)^{k-1}}\frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)!}\right)^\alpha$$
$$=\frac{c^{vt+wt^2}C_0(xt)}{(\lambda b^t-ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}\left\{{j \atop m+1}\right\}\frac{(-(1-u)t\ln ab)^j}{j!}\right)^\alpha$$
$$=(-1)^\alpha c^{vt+wt^2}C_0(xt)\left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^\alpha\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha,$$

where

$$c_j=\sum_{m=0}^{j}(-1)^{m+j+1}\frac{((1-u)\ln ab)^j m!\left\{{j+1 \atop m+1}\right\}}{(j+1)(m+1)^{k-1}}.$$

Using the fact that $\mathrm{Li}_1(z)=-\ln(1-z)$, we get

$$Li_1(1-(ab)^{-(1-u)t})=-\ln(1-(1-(ab)^{-(1-u)t}))=(1-u)t\ln ab.$$

Hence,

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}(-1)^\alpha\mathcal{G}_{n,L}^{(1,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha.$$

Note that, using (38), $\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha$ can be expressed as

$$\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha=\sum_{n=0}^{\infty}d_n\frac{t^n}{n!},$$

where

$$d_n=\sum_{n_1+n_2+\ldots+n_\alpha=n}\prod_{i=1}^{\alpha}c_{n_i}\binom{n}{n_1,n_2,\ldots,n_\alpha}.$$

It follows that

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}(-1)^\alpha\mathcal{G}_{n-j,L}^{(1,\alpha)}(x;\lambda,u,v,w,a,b,c)d_j\right\}\frac{t^n}{n!}.$$

Comparing the coefficients and using equation (29) complete the proof of the theorem.

Remark 4.1. When $\alpha = 1$, $d_j = c_j$.

The identities in the following theorem are derived using the fact that the polynomials $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,)$ with parameters a and b satisfy the relation in (20).

Theorem 4.2. *The generalized Laguerre-Apostol-type poly-Genocchi polynomials of higher order* $\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)$ *with parameters* a,b,c *satisfy the following explicit formulas:*

$$\begin{aligned}&\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\\&=\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{{l+s \atop s}\right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\mathcal{G}_{n-l-m,L}^{(k,\alpha)}(x;\lambda,u,a,b)B_{m,H}^{(s)}(v\ln c,w\ln c), \qquad (40)\\&\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\\&=\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m,L}^{(k,\alpha)}(x;\lambda,u,j,a,b)E_{m,H}^{(s)}(\mu,v\ln c,w\ln c). \qquad (41)\end{aligned}$$

Proof. Using (34), (17) may be expressed as

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\\&=\left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{vt\ln c+wt^2\ln c}}{(e^t-1)^s}\right)C_0(xt)\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}\frac{s!}{t^s}\\&=\left(\sum_{n=0}^{\infty}\left\{{n+s \atop s}\right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_{m,H}^{(s)}(v\ln c,w\ln c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}\\&=\left(\sum_{n=0}^{\infty}\left\{{n+s \atop s}\right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B_{m,H}^{(s)}(v\ln c,w\ln c)\mathcal{G}_{n-m,L}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}\\&=\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left\{{l+s \atop s}\right\}\frac{t^{l+s}}{(l+s)!}\sum_{m=0}^{n-l}\binom{n-l}{m}B_{m,H}^{(s)}(v\ln c,w\ln c)\mathcal{G}_{n-l-m,L}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^{n-l}}{(n-l)!}\right)\frac{s!}{t^s}.\end{aligned}$$

This can further be written as

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_{n,L}^{(k,\alpha)}(x;\lambda,u,v,w,a,b,c)\frac{t^n}{n!}\\&=\left(\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=0}^{n-l}\left\{{l+s \atop s}\right\}\frac{l!s!}{(l+s)!}\binom{n-l}{m}B_{m,H}^{(s)}(v\ln c,w\ln c)\mathcal{G}_{n-l-m,L}^{(k,\alpha)}(x;\lambda,u,a,b)\frac{n!}{(n-l)!l!}\frac{t^n}{n!}\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \left\{ {l+s \atop s} \right\} \frac{\binom{n-l}{m}}{\binom{l+s}{s}} B_{m,H}^{(s)}(v \ln c, w \ln c) \mathcal{G}_{n-l-m,L}^{(k,\alpha)}(x; \lambda, u, a, b) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (40).

Now, to prove relation (41), (17) may be expressed as

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,L}^{(k,\alpha)}(x; \lambda, u, v, w, a, b, c) \frac{t^n}{n!}$$
$$= \left(\frac{(1-\mu)^s}{(e^t-\mu)^s} e^{vt \ln c + wt^2 \ln c} \right) \left(\frac{(e^t-\mu)^s}{(1-\mu)^s} \right) C_0(xt) \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}} \right)^{\alpha}$$
$$= \frac{1}{(1-\mu)^s} \left(\sum_{n=0}^{\infty} E_{n,H}^{(s)}(\mu, v \ln c, w \ln c) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \times \right.$$
$$\left. \times C_0(xt) \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}} \right)^{\alpha} e^{jt} \right)$$
$$= \frac{1}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \left(\sum_{n=0}^{\infty} E_{n,H}^{(s)}(\mu, v \ln c, w \ln c) \frac{t^n}{n!} \right) \times$$
$$\times \left(\sum_{n=0}^{\infty} \mathcal{G}_{n}^{(k,\alpha)}(x; \lambda, u, j, a, b) \frac{t^n}{n!} \right)$$
$$= \frac{1}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m,L}^{(k,\alpha)}(x; \lambda, u, j, a, b) \times \right.$$
$$\left. \times E_{m,H}^{(s)}(\mu, v \ln c, w \ln c) \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\binom{n}{m}}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \mathcal{G}_{n-m,L}^{(k,\alpha)}(x; \lambda, u, j, a, b) \times \right.$$
$$\left. \times E_{m,H}^{(s)}(\mu, v \ln c, w \ln c) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (41).

5. Conclusion and Recommendations

In this paper, a certain variation of poly-Genocchi polynomials, called the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order has been introduced using the concept of polylogarithm, Laguerre, Apostol and Frobenius polynomials. Some interesting properties and identities of these polynomials were explored parallel to those of the poly-Euler polynomials and poly-Bernoulli polynomials. Using their differential identities, the generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials in (21) were classified as Appell polynomials, which, consequently, gave some interesting relations. The paper was concluded by expressing these generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials of higher order in terms of Stirling numbers of the second kind, generalized Hermite-Frobenius-Bernoulli polynomials and generalized Hermite-Frobenius-Euler polynomials of higher order.

For future research work, one may try to define other variation of Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b and c by mixing these polynomials with the degenerate exponential polynomials. Moreover, it is also interesting to construct a q-analogue of these generalized Laguerre-Apostol-Frobenius-type poly-Genocchi polynomials using the method employed in [30]. Parallel to the construction of certain mixed type special polynomials in [29], it would also be interesting to construct another variation of poly-Genocchi polynomials by mixing these polynomials with Appell polynomials.

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