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Path-Induced Closed Geodetic Domination of Some Common Graphs and Edge Corona of Graphs

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Abstract. Let G be a connected graph of order n and $S \subseteq V(G)$. A closed geodetic cover S of G is a path-induced closed geodetic dominating set of a graph G if a subgraph $\langle S \rangle$ has a Hamiltonian path and S is a dominating set of G. The minimum cardinality of a path-induced closed geodetic dominating set is called path-induced closed geodetic domination number of G. This study presents the characterization of the path-induced closed geodetic dominating sets of some common graphs and edge corona of two graphs. The path-induced closed geodetic domination numbers of these graphs are also determined.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Geodetic set, geodetic dominating set, path-induced closed geodetic set, path-induced closed geodetic dominating set, path-induced closed geodetic domination number

1. Introduction

Domination in graph is one of the most studied concepts in Graph Theory. It was first developed in the late 1950's and 1960's, beginning with C. Berge in 1958. On his study, he referred the domination number as the "coefficient of external stability". In 1962, O. Ore introduced the terms "dominating set" and "domination number". Years later, a new domination parameter called geodetic domination in graph was introduced by Escuadro et al. (2011) and defined that a vertex in a graph G dominates itself and its neighbors.

On the other hand, O. Cauntongan and I. Aniversario [3] introduced and studied the concept on path-induced closed geodetic number of some graphs. This concept follows from the definition of geodetic numbers of graphs introduced by Buckley and Harary in [2], closed geodetic numbers in [1], and path-induced geodetic numbers in [7]. In their studies, they were able to present some properties and characterized the path-induced closed geodetic set of some common graphs and determined the path-induced closed geodetic numbers of those graphs. The researchers believe that the concept of path-induced closed

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geodetic numbers of graphs can be applied in travel time saving, facility location, goods distribution, project crashing and other things in which this concept will be of great help.

These previous studies motivated the researchers to combine the concepts of path-induced closed geodesic and dominating sets in graphs. That is, a set $S \subseteq V(G)$ is both a path-induced closed geodetic set and a dominating set of G.

In this study, we only consider a connected simple nontrivial graph G. The *distance* between the vertices u and v, denoted by $d_G(u, v)$, is the shortest length of the u-v path in G. A u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. For every two vertices u and v of G, the interval $I_G[u, v]$ denotes the *set interval* containing u, v and all vertices lying in some u-v geodesic. The *geodetic closure* $I_G[S]$ is the union of intervals between all pairs of vertices from S, that is, $I_G[S] = \bigcup\{I_G[u, v] : u, v \in S\}$. A *geodetic set* of G is a set S with $I_G[S] = V(G)$. The *geodetic number*, $gn(G)$ of a graph G is the minimum cardinality among geodetic sets of G. A set S is a *closed geodetic cover* of G if $S = \{v_1, v_2, \cdots, v_k\}$ such that $v_1 \neq v_2$, $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$ and $I_G[S_k] = V(G)$, where $S_i = \{v_1, v_2, \cdots, v_i\}$ for $i = 1, 2, \cdots, k$. The *closed geodetic number* $cgn(G)$ of G is the minimum cardinality among closed geodetic covers of G [1].

2. Preliminary Concepts and Results

Definition 1. [2] The *removal of a vertex v* from a graph G results in the subgraph $\langle G \setminus \{v\}\rangle$ with $V(G \setminus \{v\}) = V(G) \setminus \{v\}$ and $E(\langle G \setminus \{v\}\rangle) = \{uw \in E(G) : u \neq v \textit{ and } w \neq v\}$. We may use the notation $G \setminus v$ for $G \setminus \{v\}$.

Definition 2. [2] A vertex x of a graph G is called a *cut-vertex* if the removal of x increases the number of components of the graph G. We will use $\omega(G)$ to describe the number of components a graph G has.

Definition 3. [2] In a graph G, the *neighborhood* $N_G(u)$ of a vertex $u \in V(G)$ is the set consisting of all vertices v which are adjacent to u, that is, $N_G(u) = \{v \in V(G) | uv \in E(G)\}$. A vertex $u \in V(G)$ is an *extreme vertex* if the neighborhood $N_G(u)$ of u induces a complete subgraph of G.

Definition 4. [2] A nontrivial connected graph without cut-vertices is called *non-separable* graph. Otherwise, such graphs are *separable*.

Definition 5. [2] Let G be a nontrivial connected graph. A *block B* of G is a subgraph of G that is itself non-separable and which is maximal with respect to this property.

Definition 6. [2] A *Hamiltonian path* of a graph G is a path that contains all vertices of G and passes through each vertex of G exactly once.

Definition 7. [6] The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H.

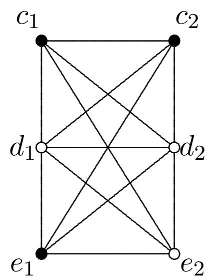
Definition 8. [4] Let G be a connected graph and $S \subseteq V(G)$. The 2-*path closure* $P_2[S]_G$ of set S is the set

$$P_2[S]_G = S \cup \{w \in V(G) : w \in I_G(u,v) \text{ for some } u, v \in S \text{ with } d_G(u,v) = 2\ \}.$$

A set S is called 2-*path closure absorbing* if $P_2[S]_G = V(G)$. The minimum cardinality of a 2-path closure absorbing set of G is denoted by $\varphi(G)$.

Definition 9. [3] Let G be a connected graph of order n and $S \subseteq V(G)$. A closed geodetic cover S of G is called a *path-induced closed geodetic set* of a graph G, denoted by *picg*-set, if $\langle S\rangle$ has a Hamiltonian path. The minimum cardinality of a path-induced closed geodetic set is called *path-induced closed geodetic number* of G, denoted by $picgn(G)$. A path-induced closed geodetic set S with $|S| = picgn(G)$ is called a *path-induced closed geodetic basis* of G, denoted by $picgb(G)$.

Definition 10. Let G be a connected graph of order n and $S \subseteq V(G)$. A closed geodetic cover S of G is called a *path-induced closed geodetic dominating set* of a graph G, denoted by *picgd*-set, if S is both a path-induced closed geodetic set and a dominating set of G. The minimum cardinality of a path- induced closed geodetic dominating set is called *path-induced closed geodetic domination number* of G, denoted by $\gamma_{picg}(G)$. A path-induced closed geodetic dominating set with $|S| = \gamma_{picg}(G)$ is said to be a γ_{picg}-*set* of G.



G

Figure 1: A graph G

Example 1. Consider the graph G in Figure 1. Let $S_1 = \{c_1\}$, $S_2 = \{c_1, c_2\}$ and $S_3 = \{c_1, c_2, e_1\}$. Then $I_G[S_2] = \{c_1, c_2\}$ and

$$\begin{aligned} I_G[S_3] &= I_G[c_1, c_2] \cup I_G[c_1, e_1] \cup I_G[c_2, e_1] \\ &= \{c_1, c_2\} \cup \{c_1, c_2, d_1, d_2, e_2, e_1\} \cup \{c_2, e_1\} \\ &= \{c_1, c_2, d_1, d_2, e_1, e_2\} = V(G). \end{aligned}$$

Let $S = \{c_1, c_2, e_1\} = S_3$. Then $I_G[S] = V(G)$. Note that $\langle S\rangle$ contains a Hamiltonian path $[c_1, c_2, e_1]$. Thus, S is a path-induced closed geodetic set of G. Observe that we cannot find a path-induced closed geodetic set S of cardinality less than 3. Thus, $picgn(G) = 3$.

Now, since each vertex of $V(G) \setminus S$ is adjacent to at least two vertices of S, S is a dominating set of G. Therefore, S is a path-induced closed geodetic dominating set and it can be verified that $\gamma_{picg}(G) = 3$.

Remark 1. [7]. Let G be a connected nontrivial graph of order n. If G admits a path-induced geodetic set, then $2 \leq gn(G) \leq pign(G) \leq n$.

Theorem 1. [3] Path-induced closed geodetic number of a few well-known graphs:

(i) For a complete graph K_n, $picgn\,(K_n) = n$.

(ii) For a path P_n on n vertices, $picgn\,(P_n) = n$.

(iii) For a cycle C_n of length n, $picgn(C_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even;} \\ \frac{n+1}{2} + 1, & \text{if } n \text{ is odd.} \end{cases}$

Theorem 2. [3] Let G be a connected graph with cut-vertices. If $S \subseteq V\,(G)$ is a path-induced closed geodetic basis of G and x is a cut-vertex of G, then every component of $G \setminus x$ contains a vertex in S.

Remark 2. [3] Every cut-vertex of a connected graph G belongs to every path-induced closed geodetic set of G.

Theorem 3. [3] Let G be a connected graph with cut-vertices. If G admits path-induced closed geodetic set, then $G \setminus x$ has exactly two components for each cut-vertex x of G.

Theorem 4. [3] Let G be a connected graph with cut-vertices. If G admits a path-induced closed geodetic set, then each block of G admits at most 2 cut-vertices.

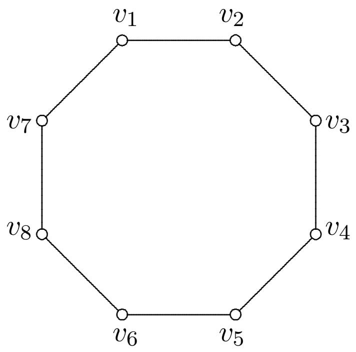
Theorem 5. [3] Let G be a connected graph of order m such that G admits a path-induced closed geodetic set. If every vertex of G is either an extreme vertex or a cut-vertex, then $picgn\,(G) = m$.

Theorem 6. [3] Let T be a tree. Then T admits a path-induced closed geodetic sets if and only if T is a path.

3. Path-induced Closed Geodetic Domination Numbers of Some Common Graphs

In view of Definition 9, a *picg*-set S may not be a *picgd*-set. Consider the cycle C_8 with vertex-set $\{v_1, v_2, \cdots, v_8\}$ in Figure 2. A $picgb(C_8)$ are $S = \{v_1, v_2, \cdots, v_5\}$ and $S^* = \{v_1, v_8, \cdots, v_5\}$ but both sets are not dominating sets of C_8. However, a *picgd*-set S of G is always a *picg*-set of G and a γ_{picg}-set of G is always a $picgb(G)$ of G. Hence, the next remarks follow.

In general, the graph C_n does not have a path-induced closed geodetic dominating set, for all $n \geq 8$.



C_8

Figure 2: The cycle C_8

Remark 3. Every *picgd*-set of a graph G is a *picg*-set of G.

Remark 4. Every γ_{picg}-set of a graph G is a $picgb(G)$ of G.

In view of Definition 10, not all connected graphs have path-induced closed geodetic dominating set. To illustrate this, let us have the following example.

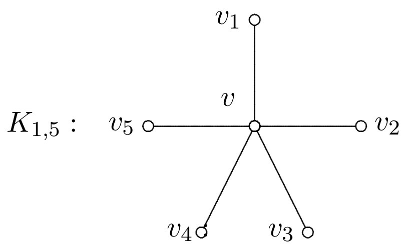


Figure 3: A graph without a *picg*-set

Example 2. Consider the star $K_{1,5}$ in Figure 3. Observe that the only geodetic covers of $K_{1,5}$ are sets $S = \{v_1, v_2, v_3, v_4, v_5\}$ and $S^* = V(K_{1,5})$ which are also dominating sets of $K_{1,5}$. But $\langle S \rangle$ and $\langle S^* \rangle$ do not contain a Hamiltonian path. Therefore, S and S^* are not path-induced closed geodetic dominating sets of $K_{1,5}$.

In general, the graph $K_{1,n}$ does not have a path-induced closed geodetic dominating set, for all $n \geq 3$.

To this extent, we will examine the properties of those graphs which admit path-induced closed geodetic dominating sets and provide some conditions that will help us determine whether a graph G admits a path-induced closed geodetic dominating set or not. Let us consider the following theorem.

Theorem 7. Let G be a connected graph with cut-vertices. If $S \subseteq V(G)$ is a γ_{picg}-set of G and x is a cut-vertex of G, then every component of $G \setminus x$ contains an element in S.

Proof: Let G be a connected graph and $x \in V(G)$ be a cut-vertex of G. Let $S \subseteq V(G)$ be a γ_{picg}-set of G. Then by Remark 4, S is a *picgb*(G). Hence, by Theorem 2, every component of $G \setminus x$ contains a vertex in S. ■

As a consequence of Remark 2, Theorem 3 and Theorem 4, the next results follow.

Remark 5. Every cut-vertex of a connected graph G belongs to every path-induced closed geodetic dominating set of G.

Theorem 8. Let G be a connected graph with cut-vertices. If G has a path-induced closed geodetic dominating set, then $\omega(G - x) = 2$ for every cut-vertex x of G.

The contrapositive of Theorem 8 says that if there exists a cut-vertex x of G with $\omega(G - x) \geq 3$, then G has no path-induced closed geodetic dominating set.

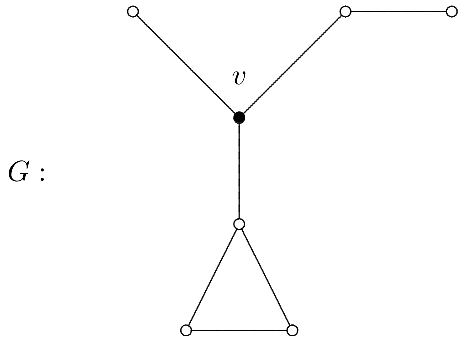


Figure 4: A graph with a cut-vertex and without a γ_{picg}-set

Figure 4 shows an illustration of the situation described in Theorem 8 with $\omega(G-x) = 3$ and therefore does not allow G to have a path-induced closed geodetic dominating set.

Theorem 9. Let G be a connected graph with cut-vertices. If G has a path-induced closed geodetic dominating set, then each block of G contains at most 2 cut-vertices.

The contrapositive of Theorem 9 says that if there exists a block of G with three or more cut-vertices, then G has no path-induced closed geodetic dominating set.

Figure 5 shows an illustration of the situation described in Theorem 9. Block B has three cut-vertices v_1, v_2, v_3 and hence G has no path-induced closed geodetic dominating set.

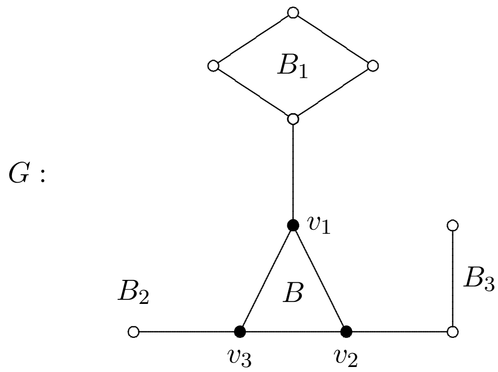


Figure 5: A graph with a cut-vertex and without a γ_{picg}-set

The next remark is a restatement of Remark 1.

Remark 6. Let G be a connected nontrivial graph of order n. If G admits a path-induced closed geodetic dominating set, then $2 \leq gn(G) \leq \gamma_{picg}(G) \leq n$.

Theorem 10. Let G be a connected nontrivial graph. Then $\gamma_{picg}(G) = 2$ if and only if $G = K_2$.

Proof: Let $S = \{x, y\}$ be a γ_{picg}-set of G. Then by Definition 10, S is both a path-induced closed geodetic set and a dominating set. Thus, $I_G[S] = V(G)$. But note that $\langle S \rangle = K_2$. Hence, $G = K_2$.

Conversely, suppose $G = K_2$. Then by Theorem 1 (i), $picgn\,(K_2) = 2$. Thus, $V(G)$ is a dominating set of G. Therefore, $\gamma_{picg}(G) = 2$. ■

Remark 7. Every vertex of a complete graph K_m is an extreme vertex.

In G. J. Changa et al. [5], it is shown that every geodetic basis of a graph contains its extreme vertices. Now, since every vertex of a complete graph K_m is an extreme vertex, then the next result follows.

Proposition 1. For any natural number m, $\gamma_{picg}(K_m) = m$.

The following theorem provides some of the necessary conditions for a graph G of order m to have $\gamma_{picg}(G) = m$.

Theorem 11. Let G be a connected graph of order m such that G has a path-induced closed geodetic dominating set. If every vertex of G is either an extreme vertex or a cut-vertex, then $\gamma_{picg}(G) = m$.

Proof: Let G be a connected graph of order m and G has a path-induced closed geodetic dominating set S. By Remark 3, S is a path-induced closed geodetic set of G. Let any $v \in V(G)$ be either an extreme vertex or a cut-vertex. Then by Theorem 5, $v \in S$ and $picgn(G) = m$. This means that $S = V(G)$. Hence, S is a dominating set of G. Therefore, $\gamma_{picg}(G) = m$. ■

Remark 8. Every end-vertex in a graph G is an extreme vertex.

Corollary 1. Let $G = P_m$. Then $\gamma_{picg}(G) = m$ for all $m \geq 1$.

Proof: Let $G = P_m$ and $V(G) = \{u_1, u_2, \cdots, u_m\}$. Thus, by Theorem 1(ii), $picgn(G) = m$. Since $V(G)$ is a dominating set of G, then $\gamma_{picg}(G) = m$. ■

The next theorem characterizes those trees which admit path-induced closed geodetic dominating sets.

Theorem 12. Let T be a tree. Then T admits a path-induced closed geodetic dominating set if and only if T is a path.

Proof: Suppose T is a tree that admits a path-induced closed geodetic dominating set. Then by Definition 10, T admits a path-induced closed geodetic set. Thus, by Theorem 6, T is a path.

Conversely, suppose T is a path. Then, by Corollary 1, T admits a path-induced closed geodetic dominating set. ■

4. Path-Induced Closed Geodetic Domination Numbers of the Edge Corona of Graphs

In this section, we discuss the path-induced closed geodetic domination number of a graph obtained from the edge corona of two graphs. We remark that not all edge corona of two graphs admit path-induced closed geodetic dominating sets. Consider the edge corona $G = C_6 \diamond P_2$ shown in Figure 6. Observe that the only connected geodetic sets in G are $V(G)$, $V(G) \setminus v_6$, and $V(G) \setminus v_2$. But these sets are not closed geodetic sets of G. Hence, G does not admit path-induced closed geodetic dominating set.

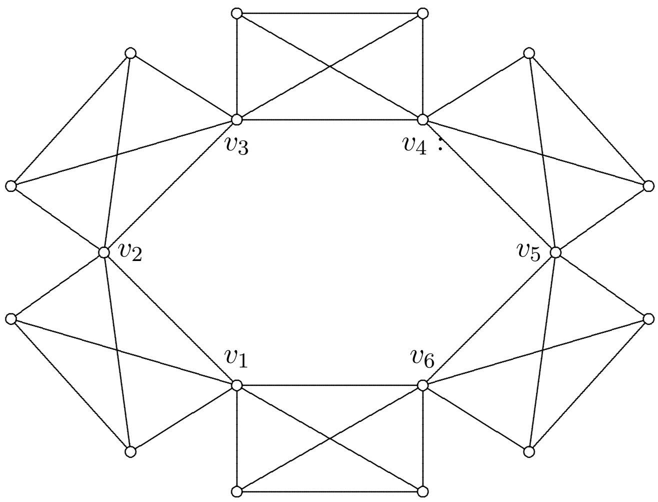


Figure 6: $G = C_6 \diamond P_2$

In general, the graph $C_n \diamond H$ does not admit path-induced closed geodetic dominating set, for all $n \geq 6$.

To this extent, we will give necessary conditions for those graphs whose edge corona admits a path-induced closed geodetic dominating set. First, let us consider the following theorem. Here, we let $G = K \diamond H$,

$$\Omega_G = \{S \subseteq V(G) : P_2[S]_G = V(G) \text{ and } \langle S \rangle \text{ has a Hamiltonian path}\}$$

and let H^{uv} be the copy of the graph H for each $uv \in E(K)$.

Remark 9. *If K is a connected graph of order 2 and H is any graph, then $G = K \diamond H = K + H$.*

Theorem 13. Let H be any graph and K be a connected noncomplete graph of order $n \geq 3$ and both K and H admit $S_K \in \Omega_K$ and $S_H \in \Omega_H$, respectively. Then $G = K \diamond H$ admits a path-induced closed geodetic dominating set S if and only if $S = \left(\bigcup_{uv \in E(K)} S_{uv} \right) \cup A$ where the following holds:

(*i*) $A \subseteq V(K)$ and $A \in \Omega_K$; and

(*ii*) For all $S_{uv} \subseteq V(H^{uv})$, $S_{uv} \in \Omega_{H^{uv}}$ for each $uv \in E(K)$.

Proof: Let $G = K \diamond H$ admits a path-induced closed geodetic dominating set S and let $A = S \cap V(K)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(K)$. Then $A \subseteq V(K)$ and $S_{uv} \subseteq V(H^{uv})$. Note that $I_G[S] = V(G)$ and so $S = \left(\bigcup_{uv \in E(K)} S_{uv} \right) \cup A$. By Definition 7, for each $uv \in E(K)$ there exists H^{uv} copy of H. Note that H admits $S_H \in \Omega_H$ and so for any H^{uv} copy of H, H^{uv} admits an element in $\Omega_{H^{uv}}$. Since S is a path-induced closed geodetic dominating set of G, A and S_{uv} must be elements of Ω_K and $\Omega_{H^{uv}}$, respectively. Otherwise, $I_G[S] \neq V(G)$ or $\langle S \rangle$ can not contain a Hamiltonian path or S is not a dominating set of G, which is a contradiction.

Conversely, suppose $S = \left(\bigcup_{uv \in E(K)} S_{uv} \right) \cup A$ and (i) and (ii) hold. Since A and S_{uv} are 2-path closure absorbing of K and H^{uv} for each $uv \in E(K)$, respectively, it follows that $I_G[S] = V(G)$. Note that each of $\langle S_{uv} \rangle$ and $\langle A \rangle$ contains a Hamiltonian path. Thus, by Definition 7, $\langle S \rangle$ contains a Hamiltonian path. Since A is a 2-path closure absorbing of K, for any $w_K \in V(K) \setminus A$, w_K is adjacent to at least two vertices of A. This means that A is a dominating set of K. Thus, by Definition 7, A is a dominating set of G and so S is a dominating set of G. Therefore, S is a path-induced closed geodetic dominating set of G. ∎

Theorem 14. Let H be any graph and K be a connected noncomplete graph with $|V(K)| = m \geq 3$, $|E(K)| = n$ and both K and H admit $S_K \in \Omega_K$ and $S_H \in \Omega_H$, respectively. Let $G = K \diamond H$. Then

$$\gamma_{picg}(G) = n \cdot \min\{|S_H| : S_H \in \Omega_H\} + \min\{|S_K| : S_K \in \Omega_H\}.$$

Proof: Let K and H admit $S_K \in \Omega_K$ and $S_H \in \Omega_H$, respectively. Let $G = K \diamond H$ where $|V(K)| = m \geq 3$ and $|E(K)| = n$. Then by Definition 7, there are n copies of H in G. Thus, by Theorem 13, the path-induced closed geodetic dominating sets of G are of the form $S = \left(\bigcup_{uv \in E(K)} S_{uv}\right) \cup A$ where $A \subseteq V(K)$ and $A \in \Omega_K$ and for all $S_{uv} \subseteq V(H^{uv})$, $S_{uv} \in \Omega_{H^{uv}}$ for each $uv \in E(K)$. Note that the minimum cardinality of S is obtained when each $|S_{uv}|$ and $|S_K|$ are minimum in $\Omega_{H^{uv}}$ and Ω_K, respectively. That is,

$$\gamma_{picg}(G) = \bigcup_{uv \in E(K)} \min\{|S_{uv}| : S_{uv} \in \Omega_{H^{uv}}\} + \min\{|S_K| : S_K \in \Omega_K\}.$$

Since all $\langle S_{uv} \rangle$ are just copies of the graph H, it follows that $\min\{|S_{uv}| : S_{uv} \in \Omega_{H^{uv}}\} = \min\{|S_H| : S_H \in \Omega_H\}$. Therefore,
$\gamma_{picg}(G) = n \cdot \min\{|S_H| : S_H \in \Omega_H\} + \min\{|S_K| : S_K \in \Omega_K\}$. ■

Corollary 2. Let $m, n \geq 3$ be natural numbers. Then

$$\gamma_{picg}(P_n \diamond K_m) = (n-1)m + n.$$

Proof: Note that $\Omega_{P_n} = \{V(P_n)\}$, $|E(P_n)| = n - 1$ and $\Omega_{K_m} = \{V(K_m)\}$ for all $m, n \geq 1$. Then by Theorem 14,

$$\begin{aligned}\gamma_{picg}(P_n \diamond K_m) &= |E(P_n)| \cdot \min\{|S_{K_m}| : S_{K_m} \in \Omega_{K_m}\} + \min\{|S_{P_n}| : S_{P_n} \in \Omega_{P_n}\} \\ &= (n-1) \cdot \min\{|S_{K_m}| : S \in \Omega_{K_m}\} + n \\ &= (n-1) \cdot m + n.\end{aligned}$$ ■

Theorem 15. Let $G = K_2 \diamond H$ where H is any graph that admits $S \in \Omega_H$ with $|V(H)| = n \geq 3$ and $diam(H) \geq 2$. Then

$$\gamma_{picg}(G) = \gamma_{picg}(H).$$

Proof: Let $G = K_2 \diamond H$ where H is any graph that admits $S \in \Omega_H$ with $|V(H)| = n \geq 3$ and $diam(H) \geq 2$. Then, by Remark 9, $G = K_2 \diamond H = K_2 + H$. Now, we let $u, v \in V(K_2)$. By Definition 7, all vertices of H are joined to u and v. Since $diam(H) \geq 2$, for any non adjacent $x, y \in S$, u and v lie in some x-y geodesic. Hence, $I_G[S] = V(G)$. Note that $S \in \Omega_H$ and so $\langle S \rangle$ contains a Hamiltonian path and S is a 2-path closure absorbing of H. Hence, by Definition 8, for any $a \in V(H) \setminus S$, a is adjacent to at least two vertices of S. It follows that S is a dominating set of H and so S is a dominating set of G. Therefore, $\gamma_{picg}(G) = S = \gamma_{picg}(H)$. ■

Corollary 3. For all integers $n \geq 3$,

$$\gamma_{picg}(K_2 \diamond P_n) = n.$$

Proof: Note that $\Omega_{P_n} = \{V(P_n)\}$, for all integers $n \geq 1$. Hence, by Theorem 15,

$$\begin{aligned}\gamma_{picg}(K_2 \diamond P_n) =& \gamma_{picg}(P_n)\\ =&|V(P_n)|\\ =&n.\end{aligned}$$ ■

Corollary 4. Let H be a graph that has a path-induced closed geodetic dominating set. If $diam(H) = 2$, then $\gamma_{picg}(K_2 \diamond H) = \gamma_{picg}(H)$.

Proof: Let S be a path-induced closed geodetic dominating set of H where $diam(H) = 2$. That is, for every $u, v \in V(H)$, $d_H(u, v) \leq 2$. Thus, $P_2[S]_H = V(H)$. This means that $S \in \Omega_H$ is with minimum cardinality. Therefore, by Theorem 15, $\gamma_{picg}(K_2 \diamond H) = |S| = \gamma_{picg}(H)$. ■

Example 3. Since $diam(F_n) = 2$ for all $n \geq 3$ and $diam(W_m) = 2$ for all $m \geq 4$, by Corollary 4, we have the following:

(i) $\gamma_{picg}(K_2 \diamond F_n) = n$, for all $n \geq 3$.

(ii) $\gamma_{picg}(K_2 \diamond W_m) = m$, for all $m \geq 4$.

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