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Fourier Series for Bernoulli-Type Polynomials, Euler-Type Polynomials and Genocchi-Type Polynomials of Integer Order

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Abstract. Parameters a, b, c, and α are introduced to form the Bernoulli-type, Euler-type and Genocchi-type polynomilas where α is the order of the polynomial and is a positive integer. Analytic methods are used here to obtain the Fourier series for these polynomials.

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Key Words and Phrases: Fourier Series, Bernoulli polynomials, Euler polynomials, Genocchi polynomials

1. Introduction

The polynomials that will be considered are given by the generating functions (1)-(3) where $B_n^{(\alpha)}(x; a, b, c)$ denotes the Bernoulli-type polynomials of order α, $E_n^{(\alpha)}(x; a, b, c)$ denotes the Euler-type polynomials of order α and $G_n^{(\alpha)}(x; a, b, c)$ denotes the Genocchi-type polynomials of order α with $\alpha \in \mathbb{Z}^+$, a, b, c are positive real numbers and $B = \ln b - \ln a > 0$.

$$\left(\frac{t}{b^t - a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; a, b, c)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{B} \tag{1}$$

$$\left(\frac{2}{b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; a, b, c)\frac{t^n}{n!}, \quad |t| < \frac{\pi}{B} \tag{2}$$

$$\left(\frac{2t}{b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; a, b, c)\frac{t^n}{n!}, \quad |t| < \frac{\pi}{B}. \tag{3}$$

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These polynomials are generalizations of the classical Bernoulli, Euler and Genocchi polynomials, respectively. The Apostol-type of these polynomials were mentioned in [9] in the introduction of the paper. Fourier series for the tangent type of these polynomials were obtained in [7] while the Fourier series for the Apostol-Tangent polynomials were obtained in [6]. Integral representation and explicit formula at rational arguments of tangent polynomials of higher order were derived in [8]. Properties of higher order Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b and c were studied in [10]. Other interesting polynomials related to Bernoulli, Euler and Genocchi were studied in [1–4].

In this paper, the Fourier series for $B_n^{(\alpha)}(x;a,b,c)$, $E_n^{(\alpha)}(x;a,b,c)$ and $G_n^{(\alpha)}(x;a,b,c)$ of positive integer order α will be derived. The method used here is analytic. In particular, there will be heavy use of contour integration and residue theory. For elaborate discussion of these topics see [5].

2. The case $\alpha = 1$

Lemma 2.1. *Let $n \geq 2$, $N > 1$ and C_N be the circle about zero of radius $R = (2N\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$, $b > a$. For*

$$0 < x < \left(\ln a - \frac{B}{2\pi - \varepsilon}\right) \Big/ \ln c, \qquad \ln c > 0$$

we have

$$\lim_{N \to +\infty} \int_{C_N} \frac{c^{xt}}{b^t - a^t} \frac{dt}{t^n} = 0.$$

Proof.

$$\left| \int_{C_N} \frac{c^{xt}}{(b^t - a^t)} \frac{dt}{t^n} \right| \leq \int_{C_N} \frac{|c^{xt}|}{|b^t - a^t|} \frac{|dt|}{|t^n|}.$$

We will show that under the conditions in the lemma, the function $\dfrac{c^{xt}}{(b^t - a^t)}$ is bounded on C_N.

Write $c^{xt} = e^{xt \ln c}$, $b^t = e^{t \ln b}$, $a^t = e^{t \ln a}$, where $t \in C_N$. Let $t = \gamma + i\rho$. Then

$$\gamma = \frac{2N\pi - \varepsilon}{B} \cos\theta, \qquad \rho = \frac{2N\pi - \varepsilon}{b} \sin\theta,$$

where $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned} \frac{|c^{xt}|}{|b^t - a^t|} &= \frac{e^{x\gamma \ln c}}{|e^{(\gamma + i\rho)\ln b} - e^{(\gamma + i\rho)\ln a}|} \\ &= \frac{e^{x\gamma \ln c}}{e^{\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{1}{2}}} \end{aligned}$$

$$= \frac{1}{e^{\gamma[\ln a - x\ln c]}[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{1}{2}}}.$$

With

$$x < \frac{\ln a}{\ln c} - \frac{B}{(2\pi - \varepsilon)\ln c}$$
$$\Longrightarrow x\ln c < \ln a - \frac{B}{2\pi - \varepsilon}$$
$$\Longrightarrow x\ln c - \ln a < -\frac{B}{2\pi - \varepsilon}$$
$$\Longrightarrow \ln a - x\ln c > \frac{B}{2\pi - \varepsilon} \geq \frac{B}{2\pi N - \varepsilon}, \quad \forall\, N \geq 1.$$

Thus,

$$\frac{1}{e^{\gamma[\ln a - x\ln c]}} \leq \frac{1}{e^{cos\theta}} \leq \frac{1}{e^{-1}} = e,$$

and

$$\frac{|c^{xt}|}{|b^t - a^t|} \leq \frac{e}{[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{1}{2}}}.$$

The denominator of the preceding expression must not be zero. With $0 \leq \theta \leq 2\pi$, we look at 3 cases:

Case 1: $\cos\theta < 0$
As $N \to +\infty, \gamma \to -\infty$ and $e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1 \longrightarrow 1$ provided $B > 0$.

Case 2: $\cos\theta > 0$
As $N \to +\infty, \gamma \to +\infty$ and $e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1 = e^{2\gamma B}\left(1 - \frac{2\cos\rho B}{e^{\gamma B}} + \frac{1}{e^{\gamma B}}\right) \longrightarrow +\infty$, provided $B > 0$.

Case 3: $\cos\theta = 0$
Then $\gamma = 0$ and $e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1 = 2 - 2\cos\rho B$, which is nonzero provided that $\cos\rho B \neq 1$. Because $\cos\theta = 0$, we have $\rho = \pm(2N\pi - \varepsilon)/B$. Thus,

$$\cos\rho B = \cos[(\pm 2N\pi - \varepsilon)] = 1 \text{ iff } 2N\pi - \varepsilon = 2k\pi, \text{ for some integer } k.$$

This gives

$$2(N - k)\pi = \varepsilon,$$

which is not possible because $0 < \varepsilon < 1$.

Thus, under the conditions in the lemma, in all 3 cases $c^{xt}/(b^t - a^t)$ is bounded $\forall t \in C_N$. Let M be a positive integer such that

$$\left|\frac{c^{xt}}{b^t - a^t}\right| < M.$$

Then

$$\begin{aligned}\left|\int_{C_N} \frac{c^{xt}}{b^t - a^t}\frac{dt}{t^n}\right| &< M\int_{C_N}\frac{|dt|}{|t^n|}\\ &= M \cdot \frac{(2N\pi - \varepsilon)2\pi}{\frac{(2N\pi-\varepsilon)^n}{B^{n-1}}}\\ &= \frac{2M\pi B^{n-1}}{(2N\pi - \varepsilon)^{n-1}} \longrightarrow 0 \text{ as } N \to +\infty \text{ for } n \geq 2.\end{aligned}$$

This completes the proof of the lemma.

Theorem 2.2. *Let a, b, c be positive real numbers. The Fourier series of the Bernoulli-type polynomials $B_n(x; a, b, c)$ is given by*

$$\frac{B_n(x; a, b, c)}{n!} = -\frac{1}{B}\sum_{k \in \mathbb{Z}^+}\frac{e^{t_k(x\ln c - \ln a)}}{t_k^n},$$

valid for

$$0 < x < \left(\ln a - \frac{B}{2\pi - \varepsilon}\right)\Big/\ln c, \qquad \ln c > 0$$

where $t_k = 2k\pi i/B$, $B = \ln b - \ln a > 0$.

Proof. When $\alpha = 1$, the generating function (1) reduces to

$$\frac{t}{b^t - a^t}c^{xt} = \sum_{n=0}^{\infty} B_n(x; a, b, c)\frac{t^n}{n!}, |t| < \frac{2\pi}{B}.$$

Applying the Cauchy Integral Formula yields

$$\frac{B_n(x; a, b, c)}{n!} = \frac{1}{2\pi i}\int_C \frac{c^{xt}}{b^t - a^t}\frac{dt}{t^n},$$

where C is a circle with center at 0 and radius less than $\frac{2\pi}{B}$. Let

$$f(t) = \frac{c^{xt}}{(b^t - a^t)t^n}.$$

The function $f(t)$ has simple poles at t such that $b^t - a^t = 0$ and a pole at $t = 0$ of order n. Let t_k be those values of t such that $b^t - a^t = 0$. These values are obtained as follows.

$$\begin{aligned} b^t - a^t &= 0 \\ e^{t\ln b} - e^{t\ln a} &= 0 \\ (e^{t\ln b} &= e^{t\ln a})e^{-t\ln a} \\ \log(e^{t(\ln b - \ln a)} &= 1) \\ t(\ln b - \ln a) &= \log 1 = i\ Arg\ 1 + 2k\pi i \\ t &= \frac{2k\pi i}{B}, \end{aligned}$$

where $B = \ln b - \ln a$.

Let $t_k = 2k\pi i/B,\ k \in \mathbb{Z}$. Now let C_N be the circle described in Lemma 2.1. Applying the Residue Theorem, we have

$$\lim_{N\to+\infty} \frac{1}{2\pi i} \int_{C_N} \frac{c^{xt}}{b^t - a^t} \frac{dt}{t^n} = Res(f(t), t = 0) + \sum_{k\in\mathbb{Z}, k\neq 0} Res(f(t), t = t_k).$$

By Lemma 2.1,

$$\begin{aligned} 0 =& Res(f(t), t = 0) + \sum_{k\in\mathbb{Z}, k\neq 0} Res(f(t), t = t_k) \\ 0 =& \frac{B_n(x; a, b, c)}{n!} + \sum_{k\in\mathbb{Z}, k\neq 0} Res(f(t), t = t_k) \\ \Longrightarrow& \frac{B_n(x; a, b, c)}{n!} = - \sum_{k\in\mathbb{Z}, k\neq 0} Res(f(t), t = t_k). \end{aligned}$$

Computing the residue at t_k:

$$\begin{aligned} Res(f(t), t = t_k) &= \lim_{t\to t_k} (t - t_k) \frac{2c^{xt}}{(b^t - a^t)t^n} \\ &= \frac{2c^{xt_k} t_k^{-n}}{\mu}, \end{aligned}$$

where

$$\begin{aligned} \mu &= \frac{d}{dt}(b^t - a^t)|_{t=t_k} \\ &= \frac{d}{dt}(e^{t\ln b} - e^{t\ln a})|_{t=t_k} \\ &= (\ln b\ e^{t_k \ln b} - \ln a\ e^{t_k \ln a}) \frac{e^{-t_k \ln a}}{e^{-t_k \ln a}} \end{aligned}$$

$$
\begin{aligned}
&= e^{t_k \ln a}(\ln b \; e^{t_k(\ln b - \ln a)} - \ln a) \\
&= e^{t_k \ln a}(\ln b - \ln a) \\
&= B \cdot e^{t_k \ln a}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Res(f(t), t = t_k) &= \frac{c^{xt_k} t_k^{-n}}{B \cdot e^{t_k \ln a}} \\
&= \frac{e^{t_k(x \ln c - \ln a)}}{B \cdot t_k^n}.
\end{aligned}
$$

Consequently,

$$
\frac{B_n(x; a, b, c)}{n!} = -\frac{1}{B} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{e^{t_k(x \ln c - \ln a)}}{t_k^n}.
$$

Lemma 2.3. *Let a, b, c be positive real numbers. Let $n \geq 1$, $N > 1$ and C_N be the circle about zero of radius $R = ((2N+1)\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$, $b > a$. For*

$$
0 < x < \left(\ln a - \frac{B}{\pi - \varepsilon} \right) \Big/ \ln c, \qquad \ln c > 0
$$

we have

$$
\lim_{N \to +\infty} \int_{C_N} \frac{c^{xt}}{b^t + a^t} \frac{dt}{t^{n+1}} = 0.
$$

Proof. We will show that the function $\dfrac{c^{xt}}{b^t + a^t}$ is bounded on C_N under the conditions in Lemma 2.3.

From the proof of Lemma 2.1,

$$
\frac{|c^{xt}|}{|b^t + a^t|} = \frac{e^{x\gamma \ln c}}{e^{\gamma \ln a}[e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1]^{\frac{1}{2}}},
$$

where here,

$$
\begin{aligned}
\gamma &= \frac{(2N+1)\pi - \varepsilon}{B} \cos \theta, \\
\rho &= \frac{(2N+1)\pi - \varepsilon}{B} \sin \theta,
\end{aligned}
$$

$0 \leq \theta \leq 2\pi$. With

$$
\begin{aligned}
& x < \frac{\ln a}{\ln c} - \frac{B}{(\pi - \varepsilon) \ln c} \\
\Longrightarrow \ln a - x \ln c > & \frac{B}{\pi - \varepsilon} \geq \frac{B}{(2N+1)\pi - \varepsilon}, \quad \forall \; N \; \geq \; 0.
\end{aligned}
$$

Then

$$\frac{1}{e^{\gamma(\ln a - x\ln c)}} \leq \frac{1}{e^{\cos\theta}} \leq \frac{1}{e^{-1}} = e.$$

Thus,

$$\frac{|c^{xt}|}{|b^t + a^t|} \leq \frac{e}{[e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1]^{\frac{1}{2}}}.$$

The expression $e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1$ must not be zero. The results for the cases $\cos\theta < 0$ and $\cos\theta > 0$ obtained in the proof of Lemma 2.1 still hold. We reconsider here the case $\cos\theta = 0$.

In the case $\theta = 0$, $\gamma = 0$ and

$$e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1 = 2 + 2\cos\rho B,$$

which is nonzero provided that $\cos\rho B \neq -1$. Since $\cos\theta = 0$, we have $\rho = (\pm 1)\dfrac{(2N+1)\pi - \varepsilon}{B}$.

Thus,

$$\cos\rho B = \cos(\pm(2N+1)\pi - \varepsilon) = -1 \quad \text{iff} \quad (2N+1)\pi - \varepsilon = (2k+1)\pi,$$

for some integer k. Equivalently,

$$(2N+1)\pi - (2k+1)\pi = \varepsilon$$

$$2(N-k)\pi = \varepsilon,$$

which is not possible because $0 < \varepsilon < 1$. Thus, under the conditions in the Lemma, the function $\dfrac{c^{xt}}{b^t + a^t}$ is bounded on C_N as $N \to +\infty$.

Let M^* be a positive integer such that

$$\frac{|c^{xt}|}{|b^t + a^t|} < M^*, \qquad \forall t \in C_N.$$

Then

$$\begin{aligned}\left|\int_{C_N} \frac{c^{xt}}{b^t + a^t} \cdot \frac{dt}{t^{n+1}}\right| &\leq \int_{C_N} \left|\frac{c^{xt}}{b^t + a^t}\right| \frac{|dt|}{|t^{n+1}|} \\ &< \frac{M^* \dfrac{(2N+1)\pi - \varepsilon}{B} \cdot 2\pi}{\left(\dfrac{(2N+1)\pi - \varepsilon}{B}\right)^{n+1}} \\ &< \frac{2M^*\pi B^n}{((2N+1)\pi - \varepsilon)^n},\end{aligned}$$

which goes to zero as $N \to +\infty$.

Theorem 2.4. *Let a, b, c be positive real numbers. The Fourier series of the Euler-type polynomials $E_n(x; a, b, c)$ is given by*

$$\frac{E_n(x;a,b,c)}{n!} = \frac{2}{B}\sum_{k\in\mathbb{Z}} \frac{e^{t_k(x\ln c - \ln a)}}{t_k^{n+1}} ,$$

valid for

$$0 < x < \left(\ln a - \frac{B}{\pi - \varepsilon}\right) \Big/ \ln c, \quad \ln c > 0$$

where $t_k = (2k+1)\pi i/B,\ B = \ln b - \ln a > 0$.

Proof. When $\alpha = 1$, the generating function (2) reduces to

$$\left(\frac{2}{b^t + a^t}\right) c^{xt} = \sum_{n=0}^{\infty} E_n(x;a,b,c)\frac{t^n}{n!}, \quad |t| < \frac{\pi}{B}.$$

Applying the Cauchy Integral Formula,

$$\frac{E_n(x;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C \frac{2c^{xt}}{(b^t + a^t)t^{n+1}} dt ,$$

where C is a circle about zero of radius $\frac{\pi}{B}$. Let

$$g(t) = \frac{2c^{xt}}{(b^t + a^t)t^{n+1}}.$$

The function $g(t)$ has a pole at $t = 0$ of order $n+1$ and simple poles at the values of t such that $b^t + a^t = 0$. These values are $t_k = (2k + 1)\pi i/B$, $k \in \mathbb{Z}$ which are obtained similarly as those in Theorem 2.2. Let C_N be the circle described in Lemma 2.3. From the Residue Theorem,

$$\lim_{N\to+\infty} \frac{1}{2\pi i}\int_{C_N} g(t)d(t) = Res(g(t), t = 0) + \sum_{k\in\mathbb{Z}} Res(g(t), t = t_k).$$

By Lemma 2.3, we have

$$\frac{E_n(x;a,b,c)}{n!} = -\sum_{k\in\mathbb{Z}} Res(g(t), t = t_k).$$

Computing the residues of $g(t)$ at t_k:

$$Res(g(t), t = t_k) = \lim_{t\to t_k} (t - t_k)\frac{2e^{xt\ln c}}{b^t + a^t} t^{-n-1}$$

$$= \frac{2e^{xt_k \ln c} t_k^{-n-1}}{\nu} ,$$

where

$$\begin{aligned}\nu &= \frac{d}{dt}(b^t + a^t)|_{t=t_k}\\ &= ((\ln b)e^{t_k \ln b} + (\ln a)e^{t_k \ln a})\frac{e^{-t_k \ln a}}{e^{-t_k \ln a}}\\ &= e^{t_k \ln a}[(\ln b)e^{t_k(\ln b - \ln a)} + \ln a]\\ &= e^{t_k \ln a}[-\ln b + \ln a]\\ &= -B \cdot e^{t_k \ln a}.\end{aligned}$$

Thus,

$$\begin{aligned}Res(g(t), t = t_k) &= \frac{2e^{t_k x \ln c} t_k^{-n-1}}{-B \cdot e^{t_k \ln a}}\\ &= \frac{2e^{t_k(x \ln c - \ln a)}}{-B \cdot t_k^{n+1}}.\end{aligned}$$

Consequently,

$$\frac{E_n(x; a, b, c)}{n!} = \frac{2}{B}\sum_{k\in\mathbb{Z}} \frac{e^{t_k(x \ln c - \ln a)}}{t_k^{n+1}}.$$

Theorem 2.5. *Let a, b, c be positive real numbers. The Fourier series of the Genocchi-type polynomials $G_n(x; a, b, c)$ is given by*

$$\frac{G_n(x; a, b, c)}{n!} = \frac{2}{B}\sum_{k\in\mathbb{Z}} \frac{e^{t_k(x \ln c - \ln a)}}{t_k^{n}},$$

valid for

$$0 < x < \left(\ln a - \frac{B}{\pi - \varepsilon}\right) \Big/ \ln c, \quad \ln c > 0$$

where $t_k = (2k + 1)\pi i/B$, $B = \ln b - \ln a > 0$.

Proof. The theorem follows from Theorem 2.4.

3. The case $\alpha \geq 2$

Lemma 3.1. *Let a, b, c be positive real numbers. Let $n \geq \alpha \geq 2$, $\alpha \in \mathbb{Z}^+$, $N > 1$ and C_N be the circle about zero of radius $R = (2N\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a > 0$. For*

$$0 < x < \left(\alpha \ln a - \frac{B}{2\pi - \varepsilon}\right) \Big/ \ln c, \quad \ln c > 0$$

we have

$$\lim_{N\to+\infty} \int_{C_N} \frac{c^{xt}}{(b^t - a^t)^\alpha} \frac{dt}{t^{n-\alpha+1}} = 0.$$

Proof. We will show that the function $\frac{c^{xt}}{(b^t - a^t)^\alpha}$ is bounded on C_N. From Lemma 2.1,

$$|b^t - a^t| = e^{\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{1}{2}},$$

where $t \in C_N$, $t = \gamma + i\rho$. That is,

$$\gamma = \frac{2N\pi - \varepsilon}{B}\cos\theta, \qquad \rho = \frac{2N\pi - \varepsilon}{B}\sin\theta,$$

$0 \le \theta \le 2\pi$. Then

$$|b^t - a^t|^\alpha = e^{\alpha\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{\alpha}{2}},$$

and

$$\left|\frac{c^{xt}}{(b^t - a^t)^\alpha}\right| = \frac{e^{x\gamma \ln c}}{e^{\alpha\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B} cos\rho B + 1]^{\frac{\alpha}{2}}}.$$

Impose that $\alpha \ln a - x \ln c > \frac{B}{2N\pi - \varepsilon}, \quad \forall N \ge 1.$
This is satisfied when

$$\alpha \ln a - x \ln c > \frac{B}{2\pi - \varepsilon}.$$

Equivalently, impose that

$$0 < x < \left(\alpha \ln a - \frac{B}{2\pi - \varepsilon}\right) \Big/ \ln c.$$

Then

$$\frac{1}{e^{\gamma(\alpha \ln a - x \ln c)}} < \frac{1}{e^{\cos\theta}} \le \frac{1}{e^{-1}} = e.$$

Consequently,

$$\left|\frac{c^{xt}}{(b^t - a^t)^\alpha}\right| < \frac{e}{[e^{2\gamma B} - 2e^{\gamma B}\cos\rho B + 1]^{\frac{\alpha}{2}}}.$$

It follows from Lemma 2.1 that the right hand side above is bounded on C_N as $N \to +\infty$.
That is, there is a constant M such that

$$\left|\frac{c^{xt}}{(b^t - a^t)^\alpha}\right| < M, \;\; t \in C_N \;\text{ and }\; 0 < x < \left(\alpha \ln a - \frac{B}{2\pi - \varepsilon}\right) \Big/ \ln c.$$

Thus,

$$\begin{aligned}\left|\int_{C_N} \frac{c^{xt}}{(b^t - a^t)^\alpha}\frac{dt}{t^{n-\alpha+1}}\right| &< M\int_{C_N}\frac{|dt|}{|t^{n-\alpha+1}|} \\ &< \frac{M \cdot \frac{2N\pi - \varepsilon}{B} \cdot 2\pi}{\left(\frac{2N\pi - \varepsilon}{B}\right)^{n-\alpha+1}}\end{aligned}$$

$$= \frac{2\pi M B^{\alpha-n}}{(2N\pi-\varepsilon)^{n-\alpha}}, \quad n \geq \alpha.$$

$$\longrightarrow 0 \quad as \quad N \to +\infty.$$

Lemma 3.2. *For $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\nu, \alpha \in \mathbb{Z}^+$ with fixed $\nu \geq \alpha$,*

$$B_\nu^{(\alpha)}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} B_l^{(\alpha)}(0; a, b, c)(x \ln c)^{\nu-l}.$$

Proof.

$$\left(\frac{t}{b^t - a^t}\right)^\alpha c^{xt} \cdot c^{yt} = \left(\sum_{n=0}^{\infty} B_n^{(\alpha)}(x; a, b, c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(yt \ln c)^n}{n!}\right)$$

$$\left(\frac{t}{b^t - a^t}\right)^\alpha c^{(x+y)t} = \sum_{n=0}^{\infty}\sum_{l=0}^{n} B_l^{(\alpha)}(x; a, b, c)\frac{t^l}{l!}\frac{(yt \ln c)^{n-l}}{(n-l)!} \cdot \frac{n!}{n!}$$

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x+y; a, b, c)\frac{t^y}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{n}{l} B_l^{(\alpha)}(x; a, b, c)(y \ln c)^{n-l}\frac{t^n}{n!}.$$

Thus,

$$B_n^{(\alpha)}(x+y; a, b, c) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(\alpha)}(x; a, b, c)(y \ln c)^{n-l}.$$

Take $y = z$, $x = 0$. Then

$$B_n^{\alpha}(z; a, b, c) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(\alpha)}(0; a, b, c)(z \ln c)^{n-l}$$

Now take $n = \nu$ and $z = x$, we have

$$B_\nu^{(\alpha)}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} B_l^{(\alpha)}(0; a, b, c)(x \ln c)^{v-l}.$$

Theorem 3.3. *Let a, b, c be positive real numbers, $N, n, \alpha \in \mathbb{Z}^+$ with $n \geq \alpha \geq 2$, $N > 1$ and C_N be the circle about zero of radius $R = (2N\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a > 0$. The Fourier series of the Bernoulli-type polynomials $B_n^{(\alpha)}(x; a, b, c)$ of order α is given by*

$$\frac{B_n^{(\alpha)}(x;a,b,c)}{n!} = -\sum_{k\in\mathbb{Z},k\neq 0}\left(\sum_{\nu=0}^{\alpha-1}\frac{(\alpha-n-1)_{\alpha-1-\nu}}{\nu!(\alpha-1-\nu)!}(2k\pi i)^{\nu}B_{\nu}^{(\alpha)}(x;a,b,c)\right)\frac{e^{2k\pi i(x\ln c-\alpha\ln b)}}{(2k\pi i)^n},$$

valid for $0 < x < \left(\alpha\ln a - \frac{B}{2\pi-\varepsilon}\right)\Big/\ln c$, $\ln c > 0$, where $B_{\nu}^{(\alpha)}(x;a,b,c)$ is given in Lemma 3.2.

Proof. Applying the Cauchy Integral Formula to (1),

$$\frac{B_n^{(\alpha)}(x;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C \frac{c^{xt}}{(b^t-a^t)^{\alpha}}\frac{dt}{t^{n+1-\alpha}},$$

where C is a circle about the origin with radius less than $\frac{2\pi}{B}$. Let

$$f_\alpha(t) = \frac{c^{xt}}{(b^t-a^t)^{\alpha}t^{n-\alpha+1}}, \quad n > \alpha.$$

The function $f_\alpha(t)$ has a pole of order $n-\alpha+1$ at $t=0$ and a pole of order α at the zeros of $b^t - a^t$ which are given by $t_k = \frac{2k\pi i}{B}$, $k \in \mathbb{Z}$. Now let C_N, $N > 1$ be the circle described in Lemma 3.1. Applying the Residue Theorem,

$$\lim_{N\to+\infty}\frac{1}{2\pi i}\int_{C_N}\frac{c^{xt}}{(b^t-a^t)^{\alpha}}\frac{dt}{t^{n-\alpha+1}} = Res(f_\alpha(t), t=0) + \sum_{k\in\mathbb{Z},k\neq 0} Res(f_\alpha(t), t=t_k).$$

By Lemma 3.1,

$$0 = Res(f_\alpha(t), t=0) + \sum_{k\in\mathbb{Z},k\neq 0} Res(f_\alpha(t), t=t_k)$$

$$0 = \frac{B_n^{(\alpha)}(x;a,b,c)}{n!} + \sum_{k\in\mathbb{Z},k\neq 0} Res(f_\alpha(t), t=t_k)$$

$\Longleftrightarrow$

$$\frac{B_n^{\alpha}(x;a,b,c)}{n!} = -\sum_{k\in\mathbb{Z},k\neq 0} Res(f_\alpha(t), t=t_k). \tag{4}$$

Computing the residues at t_k:

$$Res(f_\alpha(t), t=k) = \frac{1}{(\alpha-1)!}\lim_{t\to t_k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}(t-t_k)^{\alpha}\left(\frac{e^{xt\ln c}}{(b^t-a^t)^{\alpha}}\right)\frac{1}{t^{n-\alpha+1}}$$

$$= \frac{1}{(\alpha-1)!}\lim_{t\to t_k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left[\frac{(t-t_k)^{\alpha}}{(b^t-a^t)^{\alpha}}\frac{e^{xt\ln c}}{t^{n-\alpha+1}}\right]. \tag{5}$$

Taking $x = 0$ in (1) gives

$$\left(\frac{t}{b^t - a^t}\right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{t^n}{n!}.$$

Replacing $t \mapsto t - t_k$ and writing $b^t = e^{t \ln b}$, $a^t = e^{t \ln a}$,

$$\frac{(t - t_k)^\alpha}{(e^{(t-t_k)\ln b} - e^{(t-t_k)\ln a})^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{(t - t_k)^n}{n!}. \tag{6}$$

Multiplying and dividing the left hand side of (6) by $e^{\alpha t_k \ln b}$ gives

$$\frac{(t - t_k)^\alpha e^{\alpha t_k \ln b}}{(e^{t\ln b} - e^{t\ln a}e^{t_k B})^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{(t - t_k)^n}{n!}. \tag{7}$$

With $t_k = (2k\pi i)/B$, we have $e^{t_k B} = e^{2k\pi i} = 1$. Thus, (7) becomes

$$\frac{(t - t_k)^\alpha e^{\alpha t_k \ln b}}{(b^t - a^t)^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{(t - t_k)^n}{n!}$$

$$\frac{(t - t_k)^\alpha}{(b^t - a^t)^\alpha} = e^{-\alpha t_k \ln b}\sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{(t - t_k)^n}{n!}. \tag{8}$$

Substituting (8) to (5) gives,

$$Res(f_\alpha(t), t = t_k) = \frac{e^{-\alpha t_k \ln b}}{(\alpha - 1)!}\lim_{t\to t_k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left(\frac{e^{xt\ln c}}{t^{n-\alpha+1}}\sum_{n=0}^{\infty} B_n(0; a, b, c)\frac{(t - t_k)^n}{n!}\right).$$

The derivatives will be obtained using Leibniz Rule. This is done as follows. Recalling the Leibniz Rule for derivatives,

$$\frac{d^n}{dt^n}(fg) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^{n-k}}{dt^{n-k}}f\right)\left(\frac{d^k}{dt^k}g\right).$$

Let $f = t^{\alpha-n-1}$, $g = e^{xt\ln c}\sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c)\frac{(t - t_k)^n}{n!}$.
Then

$$\frac{d^{\alpha-1}}{dt^{\alpha-1}}(fg) = \sum_{\nu=0}^{\alpha-1}\binom{\alpha - 1}{\nu}\left(\frac{d^{d-1-\nu}}{dt^{d-1-\nu}}f\right)\left(\frac{d^\nu}{dt^\nu}g\right)$$

$$= \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha-n-1)_{\alpha-1-\nu} t^{\alpha-n-1-(\alpha-1-\nu)} \left(\frac{d^\nu}{dt^\nu} g \right)$$
$$= \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha-n-1)_{\alpha-1-\nu} t^{-n+\nu} \left(\frac{d^\nu}{dt^\nu} g \right), \tag{9}$$

where the notation $(n)_k$ is designed as

$$(n)_k = n(n-1)(n-2)...(n-k+1).$$

Also,

$$(\alpha-n-1)_{\alpha-1-\nu} = (-1)^{\alpha-1-\nu} (n-\alpha+1)(n-\alpha+2)(n-\alpha+3)\dots((n-\alpha)+\alpha-\nu-1)$$
$$= (-1)^{\alpha-1-\nu} \langle n-\alpha+1 \rangle_{\alpha-\nu-1}.$$

On the other hand,

$$\frac{d^\nu}{dt^\nu} g = \frac{d\nu}{dt^\nu} \left(\sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c) \frac{(t-t_k)^n}{n!} \cdot e^{xt \ln c} \right)$$
$$= \sum_{l=0}^{\nu} \binom{\nu}{l} \frac{d^{\nu-l}}{dt^{v-l}} e^{t(x \ln c)} \cdot \frac{d^l}{dt^l} \left(\sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c) \frac{(t-t_k)^n}{n!} \right)$$
$$= \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu-l} e^{xt \ln c} \sum_{n \ge l} B_n^{(\alpha)}(0;a,b,c)(n)_l \frac{(t-t_k)^{n-l}}{n!}.$$

Now take the limit as $t \to t_k$. Then

$$\lim_{t \to t_k} \frac{d^\nu}{dt^\nu} g = \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{v-l} e^{t_k x \ln c} B_l^{(\alpha)}(0;a,b,c).$$

Substituting to (9) and taking the limit as $t \to t_k$ will yield

$$\lim_{t \to k} \frac{d^{\alpha-1}}{dt^{\alpha-1}}(fg) = \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha-n-1)_{\alpha-1-\nu} \, t_k^{-n+\nu}$$
$$\sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu-l} e^{t_k \ln c} B_l^{(\alpha)}(0;a,b,c)$$
$$= \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha-n-1)_{\alpha-1-\nu} \, t_k^{-n+\nu} e^{t_k \ln c} \left(\sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{v-l} B_l^{(\alpha)}(0;a,b,c) \right). \tag{10}$$

Applying Lemma 3.2 to (10),

$$\lim_{t\to k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}(fg)=\sum_{\nu=0}^{\alpha-1}\binom{\alpha-1}{\nu}(\alpha-n-1)_{\alpha-1-\nu}t_k^{-n+\nu}e^{t_k\ln c}B_\nu^{(\alpha)}(x;a,b,c).$$

Thus,

$$Res(f_\alpha(t),t=t_k)=\frac{e^{t_k(x\ln c-\alpha\ln b)}}{t_k^n}\sum_{\nu=0}^{\alpha-1}\frac{(\alpha-n-1)_{\alpha-1-\nu}}{\nu!(\alpha-1-\nu)!}t_k^\nu B_\nu^{(\alpha)}(x;a,b,c). \tag{11}$$

The desired Fourier series is obtained by substituting (11) to (4).

Taking $\alpha=1$, the Fourier series in Theorem 3.3 reduces to that in Theorem 2.2. For $\alpha=2$, Theorem 3.3 gives the Fourier series of the Bernoulli-type polynomials of order 2.

This is given by

$$\frac{B_n^{(2)}(x;a,b,c)}{n!}=\frac{-1}{B^2}\sum_{k\in\mathbb{Z},k\neq0}(-n+1+x\ln c)\frac{e^{2k\pi i(x\ln c-2\ln b)}}{(2k\pi i)^n},$$

valid under the conditions in Theorem 3.3.

Lemma 3.4. *Let a,b,c be positive real numbers with $b>a$, $n,\alpha\in\mathbb{Z}^+$ with $n\geq\alpha$, $N>1$ and C_N be the circle about zero of radius $R=((2N+1)\pi-\varepsilon)/B$, where $0<\varepsilon<1$ and $B=\ln b-\ln a$. For $\ln c>0$ and*

$$0<x<\left(\alpha\ln a-\frac{B}{\pi-\varepsilon}\right)\Big/\ln c \tag{12}$$

we have

$$\lim_{N\to+\infty}\int_{C_N}\frac{c^{xt}}{(b^t+a^t)^\alpha}\frac{dt}{t^{n+1}}=0.$$

Proof. From the proof of Lemma 3.2,

$$|b^t+a^t|^\alpha=e^{\alpha\gamma\ln a}[e^{2\gamma B}+2e^{\gamma B}\cos\rho B+1]^{\frac{\alpha}{2}},\quad t\in C_N$$

where $t=\gamma+i\rho=\dfrac{(2N+1)\pi-\varepsilon}{B}(\cos\theta+i\sin\theta)$, $0\leq\theta\leq2\pi$.
Thus,

$$\gamma=\frac{(2N+1)\pi-\varepsilon}{B}\cos\theta,\qquad\rho=\frac{(2N+1)\pi-\varepsilon}{B}\sin\theta.$$

For x satisfying (12), it follows that

$$\alpha \ln a - x \ln c > \frac{B}{\pi - \varepsilon} \geq \frac{B}{(2N+1)\pi - \varepsilon}, \qquad \forall N \geq 1.$$

Then

$$\frac{1}{e^{\gamma[\alpha \ln a - x \ln c]}} = \frac{1}{e^{\frac{(2N+1)-\varepsilon}{B} \cos\theta[\alpha \ln a - x \ln c]}} < \frac{1}{e^{\cos\theta}} < e.$$

Consequently,

$$\begin{aligned} \left| \frac{c^{xt}}{(b^t + a^t)^\alpha} \right| = \frac{|c^{xt}|}{|b^t + a^t|^\alpha} &= \frac{1}{e^{\gamma[\alpha \ln a - x \ln c]}[e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1]^{\frac{\alpha}{2}}} \\ &< \frac{e}{(e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1)^{\frac{\alpha}{2}}}. \end{aligned}$$

The expression $e^{2\gamma B} + 2e^{\gamma B}\cos\rho B + 1 \neq 0\ \forall t \in C_N$ as discussed in Lemma 2.3. Thus, $\exists$ an integer M $s.t.$

$$\left| \frac{c^{xt}}{(b^t + a^t)^\alpha} \right| < M, \quad \forall t \in C_N.$$

Hence,

$$\begin{aligned} \left| \int_{C_N} \frac{c^{xt}}{(b^t + a^t)^\alpha} \frac{dt}{t^{n+1}} \right| &\leq M \int_{C_N} \frac{|dt|}{|t^{n+1}|} \\ &= M \cdot \frac{\frac{(2N+1)\pi - \varepsilon}{B} \cdot 2\pi}{\left(\frac{(2N+1)\pi - \varepsilon}{B} \right)^{n+1}} \\ &= \frac{2\pi M B^n}{((2N+1)\pi - \varepsilon)^{n+1}}, \quad n > 1. \\ &\longrightarrow 0 \text{ as } N \to +\infty. \end{aligned}$$

Lemma 3.5. *For $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\nu, \alpha \in \mathbb{Z}^+$ with fixed $\nu \geq \alpha \geq 2$,*

$$E_\nu^{(\alpha)}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} E_l^{(\alpha)}(0; a, b, c)(x \ln c)^{v-l}.$$

Proof. The proof is done similarly as that of Lemma 3.2.

Theorem 3.6. *Let a, b, c be positive real numbers with $b > a$, N, n, $\alpha \in \mathbb{Z}^+$, $n \geq \alpha \geq 2$, $N > 1$ and C_N be the circle about zero of radius $R = ((2N+1)\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$*

and $B = \ln b - \ln a$*. The Fourier series of the Euler-type polynomials* $E_n^{(\alpha)}(x;a,b,c)$ *of order* α *is given by*

$$\frac{E_n^{(\alpha)}(x;a,b,c)}{n!} = \frac{-2^{\alpha}}{(\alpha-1)!}\sum_{k\in\mathbb{Z}}\sum_{\nu=0}^{\alpha-1}\binom{\alpha-1}{\nu}(-n-1)_{\alpha-1-\nu}B_{\nu}^{(\alpha)}(x;a,b,c)\frac{e^{t_k(x\ln c-\alpha\ln b)}}{t_k^{n+\alpha-\nu}},$$

valid for

$$0 < x < \left(\alpha\ln a - \frac{B}{\pi-\varepsilon}\right)\Big/\ln c, \quad \ln c > 0.$$

Proof. Applying the Cauchy-Integral Formula to (2),

$$\frac{E_n^{(\alpha)}(x;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C \frac{2^{\alpha}c^{xt}}{(b^t+a^t)^{\alpha}}\frac{dt}{t^{n+1}},$$

where C is a circle about zero of radius less than $\frac{\pi}{B}$. Let

$$g_{\alpha}(t) = \frac{c^{xt}}{(b^t+a^t)^{\alpha}t^{n+1}}.$$

Then

$$\frac{E_n^{(\alpha)}(x;a,b,c)}{2^{\alpha}(n!)} = \frac{1}{2\pi i}\int_C g_{\alpha}(t)dt.$$

The function $g_{\alpha}(t)$ has a pole of order $n+1$ at $t=0$ and a pole of order α at the zeros of $b^t + a^t$ which are given by $t_k = ((2k+1)\pi i)/B$, $k \in Z$. Applying the Residue Theorem and taking the limit as $N \to +\infty$,

$$\lim_{N\to+\infty}\frac{1}{2\pi i}\int_C g_{\alpha}(t)dt = Res(g_{\alpha}(t), t=0) + \sum_{k\in\mathbb{Z}} Res(g_{\alpha}(t), t=t_k).$$

It follows from Lemma 3.4 that

$$\frac{E_n^{(\alpha)}(x;a,b,c)}{2^{\alpha}(n!)} = -\sum_{k\in\mathbb{Z}} Res(g_{\alpha}(t), t=t_k).$$

Computing the residues at t_k:

$$Res(g_{\alpha}(t), t=t_k) = \frac{1}{(\alpha-1)!}\lim_{t\to t_k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left((t-t_k)^{\alpha}\frac{c^{xt}}{(b^t+a^t)^{\alpha}}\cdot\frac{1}{t^{n+1}}\right). \tag{13}$$

Now use (7). With $t_k = (2k+1)\pi i/B$, $e^{t_kB} = e^{(2k+1)\pi i} = -1$. Thus, (7) becomes,

$$\frac{(t-t_k)^{\alpha}e^{\alpha t_k\ln b}}{(e^{t\ln b}+e^{t\ln a})^{\alpha}} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c)\frac{(t-t_k)^n}{n!}$$

$$\frac{(t-t_k)^\alpha}{(b^t+a^t)^\alpha} = e^{-\alpha t_k \ln b}\sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c)\frac{(t-t_k)^n}{n!}. \tag{14}$$

Substituting (14) to (13),

$$Res(g_\alpha(t), t=t_k) = \frac{e^{-\alpha t_k \ln b}}{(\alpha-1)!}\lim_{t\to t_k}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left(c^{xt}t^{-n-1}\sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c)\frac{(t-t_k)^n}{n!}\right).$$

Applying the Leibniz Rule for differentiation,

$$Res(g_\alpha(t), t=t_k) = \frac{e^{t_k(x\ln c-\alpha\ln b)}}{(\alpha-1)!}\sum_{\nu=0}^{\alpha-1}\binom{\alpha-1}{\nu}(-n-1)_{\alpha-1-\nu}\, t_k^{-n-\alpha+\nu}B_\nu^{(\alpha)}(x;a,b,c).$$

Thus,

$$\begin{aligned}\frac{E_n^{(\alpha)}(x;a,b,c)}{n!} &= -\frac{2^\alpha}{(\alpha-1)!}\sum_{k\in\mathbb{Z}} e^{t_k(x\ln c-\alpha\ln b)}\sum_{\nu=0}^{\alpha-1}\binom{\alpha-1}{\nu}(-n-1)_{\alpha-1-\nu}\, t_k^{-n-\alpha+\nu}\, B_\nu^{(\alpha)}(x;a,b,c)\\ &= -\frac{2^\alpha}{(\alpha-1)!}\sum_{k\in\mathbb{Z}}\sum_{\nu=0}^{\alpha-1}\binom{\alpha-1}{\nu}(-n-1)_{\alpha-1-\nu}\, B_\nu^{(\alpha)}(x;a,b,c)\frac{e^{t_k(x\ln c-\alpha\ln b)}}{t_k^{n+\alpha-\nu}},\end{aligned}$$

which is the desired Fourier series of $E_n^{(\alpha)}(x;a,b,c)$.

Taking $\alpha = 1$, the Fourier series in Theorem 3.6 reduces to that in Theorem 2.4.
For $\alpha = 2$, the Fourier series is given by

$$\frac{E_n^{(2)}(x;a,b,c)}{2^2(n!)} = -\sum_{k\in\mathbb{Z}}(-n-1)B_0^{(2)}(x;a,b,c)\frac{e^{t_k(x\ln c-2\ln b)}}{t_k^{n+2}} + B_1^{(2)}(x;a,b,c)\frac{e^{t_k(x\ln c-2\ln b)}}{t_k^{n+1}},$$

where

$$B_0^{(2)}(x;a,b,c) = \frac{1}{B^2}, \tag{15}$$

$$B_1^{(2)}(x;a,b,c) = \frac{x\ln c}{B^2} + \frac{\ln ab-(\ln b)^2-\ln b\ln a-(\ln a)^2}{B^2}. \tag{16}$$

Lemma 3.7. *Let a, b, c be positive real numbers with $b > a$. Let $N, n, \alpha \in \mathbb{Z}^+$, $N > 1$ and C_N be the circle about zero with raidus $R = \frac{(2N+1)\pi-\varepsilon}{B}$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$. For*

$$0 < x < \left(\alpha\ln a - \frac{B}{\pi-\varepsilon}\right)\Big/\ln c, \quad \ln c > 0$$

we have

$$\lim_{N\to+\infty}\int_{C_N}\frac{c^{xt}}{(b^t+a^t)^\alpha}\frac{dt}{t^{n-\alpha+1}} = 0.$$

Proof. This follows from Lemma 3.4.

Lemma 3.8. *For $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}, \nu, \alpha \in \mathbb{Z}^+$ with fixed $\nu \geq \alpha$,*

$$G_\nu^{(\alpha)}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} G_l^{(\alpha)}(0; a, b, c)(x \ln c)^{\nu - l}.$$

Proof. The proof is done similarly as that of Lemma 3.2.

Theorem 3.9. *Let a, b, c be positive real numbers with $b > a$. Let $N, n, \alpha \in \mathbb{Z}^+$ with $n \geq \alpha \geq 2$, $N > 1$ and C_N be the circle about zero of radius $R = ((2N+1)\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$. The Fourier series of the Genocchi-type polynomials $G_n^{(\alpha)}(x; a, b, c)$ of order α is given by*

$$\frac{G_n^{(\alpha)}(x; a, b, c)}{n!} = -\frac{2^\alpha}{(\alpha-1)!} \sum_{k \in \mathbb{Z}} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} \, B_\nu^{(\alpha)}(x; a, b, c) \frac{e^{t_k (x \ln c - \alpha \ln b)}}{t_k^{n-\nu}}.$$

Proof. Applying the Cauchy Integral Formula to (3),

$$\frac{G_n^{(\alpha)}(x; a, b, c)}{n!} = \frac{2^\alpha}{2\pi i} \int_C \frac{c^{xt}}{(b^t + a^t)^\alpha} \frac{dt}{t^{n-\alpha+1}},$$

where C is a circle about zero of radius $< \pi/B$. Let

$$h_{\alpha(t)} = \frac{c^{xt}}{(b^t + a^t)^\alpha t^{n-\alpha+1}}.$$

This function has a pole of order $n - \alpha + 1$ at $t = 0$ and a pole of order α at the zeros of $b^t + a^t$. These poles are given by $t_k = (2k+1)\pi i/B$, $k \in \mathbb{Z}$. Applying the Residue Theorem and taking the limit as $N \to +\infty$,

$$\lim_{N \to +\infty} \frac{1}{2\pi i} \int_C h_\alpha(t) dt = Res(h_\alpha(t), t = 0) + \sum_{k \in \mathbb{Z}} Res(h_\alpha(t), t = t_k).$$

It follows from Lemma 3.7 that

$$\frac{G_n^{(\alpha)}(x; a, b, c)}{n! \, 2^\alpha} = -\sum_{k \in \mathbb{Z}} Res(h_\alpha(t), t = t_k), \tag{17}$$

where

$$Res(h_\alpha(t), t = t_k) = \frac{1}{(\alpha-1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left((t - t_k)^\alpha \frac{c^{xt}}{(b^t + a^t)^\alpha} \cdot \frac{1}{t^{n+1-\alpha}} \right).$$

From (14),

$$Res(h_\alpha(t), t = t_k) = \frac{e^{-\alpha t_k \ln b}}{(\alpha-1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left(c^{xt} t^{-n+\alpha-1} \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^\alpha}{n!} \right).$$

Following the computation in the Euler-type polynomials,

$$Res(h_\alpha(t), t = t_k) = \frac{e^{t_k(x\ln c - \alpha \ln b)}}{(\alpha-1)!} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu}\, t_k^{-n+\nu} B_\nu^{(\alpha)}(x; a, b, c). \tag{18}$$

Substituting (18) to (17) gives the desired Fourier series.

Taking $\alpha = 1$, the Fourier series in Theorem 3.9 reduces to that in Theorem 2.5. Taking $\alpha = 2$ and $n = 4$, the series gives

$$\frac{G_4^{(2)}(x; a, b, c)}{2^2(4!)} = -\sum_{k\in\mathbb{Z}} \left\{ -3B_0^{(2)}(x; a, b, c) \frac{e^{(2k+1)\pi i(x\ln c - 2\ln b)}}{((2k+1)\pi i)^4} + B_1^{(2)}(x; a, b, c) \frac{e^{(2k+1)\pi i(x\ln c - 2\ln b)}}{((2k+1)\pi i)^3} \right\}$$

where $B_0^{(2)}(x; a, b, c)$ and $B_1^{(2)}(x; a, b, c)$ are given in (15) and (16), respectively.

4. Some Remarks

The Fourier series expansions obtained in this paper for $B_n^{(\alpha)}(x; a, b, c)$, $E_n^{(\alpha)}(x; a, b, c)$ and $G_n^{(\alpha)}(x; a, b, c)$ are useful in establishing the asymptotic formulas of these polynomials. It would then be interesting to investigate the asymptotic behavior of these polynomials.

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