EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1593-1596
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Diophantine Equation $(p + 4n)^x + p^y = z^2$

Wachirarak Orosram[1,*], Kitsanuphong Makonwattana[1], Saichon Khongsawat[1]

[1] *Department of Mathematics, Faculty of Science, Buriram Rajabhat University, Buriram 31000, Thailand*

Abstract. In this paper, we study the Diophantine equation $(p + 4n)^x + p^y = z^2$, where n is a non-negative integer and $p, p+4n$ are prime numbers such that $p \equiv 7 \pmod{12}$. We show that the non-negative integer solutions of such equation are $(x, y, z) \in \{(0, 1, \sqrt{p+1})\} \cup \{(1, 0, 2\sqrt{n + \frac{p+1}{4}})\}$, where $\sqrt{p+1}$ and $\sqrt{n + \frac{p+1}{4}}$ are integers.

2020 Mathematics Subject Classifications: 11D61

Key Words and Phrases: exponential Diophantine equation, Catalan's conjecture

---

1. Introduction

A problem related to the Diophantine equation has been investigated by many researchers. It is considered one of the significant problems in elementary number theory. The proving method mainly uses a property in the integer system and algebraic number theory. Some of which appear in a higher system of the integer called the ring of integers. In 2011, Suvarnamani [10] considered a Diophantine equation $2^x + p^y = z^2$ when $p > 2$ and p is a prime number. The result showed that $(x, y, z) = (3, 0, 3)$ is a solution of the equation for all prime $p > 2$. If $p = 3$, then $(x, y, z) = (4, 2, 5)$ is also a solution of the equation. If $p = 1 + 2^{k+1}$ for some non-negative integer k, then $(x, y, z) = (2k, 1, 1 + 2k)$. In 2012, the Diophantine equation $4^x + p^y = z^2$, where x, y and z are non-negative integers and p is a positive prime number was studied by Chotchaisthit [2]. The study revealed that the equation has no non-negative integer solution. In 2014, Suvarnamani [11] proved that the equation $p^x + (p+1)^y = z^2$ has a unique non-negative integer solution $(p, x, y, z) = (3, 1, 0, 2)$ when p is an odd prime number. In 2016, Hoque [6] proved that there are exactly two solutions to $(M_{pq})^x + (M_{pq} + 1)^y = z^2$, where $p, q \in \mathbb{Z}$ such that $p > 0$, $q > 1$ and $M_{pq} = p^q - 1$. In 2018, Kumar et al. [7] showed that the non-linear diophantine equation $p^x + (p + 6)^y = z^2$ has no solution. Moreover, Fernando [4] showed

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4508

Email addresses: Wachirarak.tc@bru.ac.th (W. Orosram),
620112210001@bru.ac.th (K. Makonwattana)620112210034@bru.ac.th (S. Khongsawat)

that a Diophantine equation $p^x + (p+8)^y = z^2$ has no positive-integer solution, when $p, p+8$ are primes such that $p > 3$. In 2019, Kumar et al. [8] proved that the solution of an exponential Diophantine equation $p^x + (p+12)^y = z^2$ has no non-negative integer solution, when p and $p+12$ are prime numbers such that p is in the form of $6n+1$. In 2020, Burshtein [1] proved that a Diophantine equation $p^x+(p+12)^y = z^2$ has no positive integer solution (x, y, z), when p is a prime number such that $p+5 = 2^{2u}$. In 2021, Dokchan and Pakapongpun [3] studied a Diophantine equation $p^x + (p+20)^y = z^2$, when p and $p+20$ are primes and showed that the equation has no positive integer solution (x, y, z). In the same year, Gayo and Bacani [5] solved the Diophantine equation $M_p^x + (M_q + 1)^y = z^2$ when M_p and M_q are Mersenne primes .

In this work, we give solutions of the Diophantine equations $1+b^y = z^2, 1+(d+4t)^x = z^2$ where b, t, d are positive integers. Then, we extend to the solutions of the Diophantine equation $(p+4n)^x + p^y = z^2$ where $p, p+4n$ are prime numbers such that $p \equiv 7 \pmod{12}$ and n is a positive integer such that $n \equiv 0, 1 \pmod 3$.

2. Main results

Proposition 1. *(Catalan's conjecture)* $(a, b, x, y) = (3, 2, 2, 3)$ *is the unique solution of the Diophantine equation* $a^x - b^y = 1$*, where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

This proposition was proved in 2004 by Mihailescu [9].

Lemma 1. *Let* b *be a positive integer. The non-negative integer solutions to the Diophantine equation* $1 + b^y = z^2$ *is* $(y, z) = (1, \sqrt{b+1})$ *if* $\sqrt{b+1})$ *is a positive integer.*

Proof. Let b be a positive integer. We have $z^2 - b^y = 1$. By proposition 1, it is sufficient to consider the case $b = 1$, $z \le 1$ or $y \le 1$. Hence, it remains to consider the following cases of b, y and z. If $b = 1$, then we have $z^2 = 2$, which is impossible. If $z = 0$ or $z = 1$, then there is no solution. If $y = 0$, then we have $z^2 = 2$ which is impossible. If $y = 1$, then we have $z^2 = b + 1$ or $z = \sqrt{b+1}$. Thus, we have $(y, z) = (1, \sqrt{b+1})$.

Corollary 1. *Let* p *be a prime number such that* $p \equiv 7 \pmod{12}$*. The non-negative integer solutions to the Diophantine equation* $1 + p^y = z^2$ *is* $(y, z) = (1, \sqrt{p+1})$ *if* $\sqrt{p+1})$ *is a positive integer.*

Lemma 2. *Let* t *and* d *be positive integers. The non-negative integer solutions of the Diophantine equation* $1 + (d+4t)^x = z^2$ *is* $(x, z) = \left(1, 2\sqrt{t + \frac{d+1}{4}}\right)$ *if* $\sqrt{t + \frac{d+1}{4}}$ *is a positive integer.*

Proof. Let t, d be positive integers such that $\sqrt{t + \frac{d+1}{4}}$ is a positive integer. We have $z^2 - (d+4t)^x = 1$. By proposition 1, it is sufficient to consider only the case that $z \le 1$ or $x \le 1$. Hence, we consider the following cases of z and x. For $z = 0$ and $z = 1$, there is no solution. If $x = 0$, then we have $z^2 = 2$, which is impossible. If $x = 1$, then we have

$z^2 = 4t + d + 1$. Thus $z = 2\sqrt{t + \frac{d+1}{4}}$ where $\sqrt{t + \frac{d+1}{4}}$ is a positive integer. Therefore, $(x, z) = (1, 2\sqrt{t + \frac{d+1}{4}})$.

Corollary 2. *Let n be a positive integer and $p, (p+4n)$ be prime numbers such that $n \equiv 0, 1 \pmod 3$ and $p \equiv 7 \pmod{12}$. The non-negative integer solutions of the Diophantine equation $1 + (p+4n)^x = z^2$ is $(x, z) = \left(1, 2\sqrt{n + \frac{p+1}{4}}\right)$ if $\sqrt{n + \frac{p+1}{4}}$ is a positive integer.*

Theorem 1. *Let n be a positive integer such that $n \equiv 0, 1 \pmod 3), p \equiv 7 \pmod{12}$. If $\sqrt{p+1}$ and $\sqrt{n + \frac{p+1}{4}}$ are also integers, then all of the non-negative integer solutions to the Diophantine equation $(p + 4n)^x + p^y = z^2$ are given by $(x, y, z) \in \{(0, 1, \sqrt{p+1})\} \cup \{(1, 0, 2\sqrt{n + \frac{p+1}{4}})\}$, where p and $p + 4n$ are prime number.*

Proof. Since p is a prime number such that $p \equiv 7 \pmod{12}$, it is clear that $p \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 3$. Let (x, y, z) be a non-negative integer solution of $(p+4n)^x + p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 1, \sqrt{p+1})$ or $(x, y, z) = \left(1, 0, 2\sqrt{n + \frac{p+1}{4}}\right)$. Suppose $x > 0$ and $y > 0$. We consider the following cases.

Case 1. x and y are even numbers. Since $(p + 4n)^x + p^y = z^2$, it follows that z is even. So $z^2 \equiv 0 \pmod 4$. Note that $(p + 4n)^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$. Thus $(p + 4n)^x + p^y \equiv 2 \pmod 4$ which contradicts with $z^2 \equiv 0 \pmod 4$.

Case 2. x and y are odd numbers. Since $(p + 4n)^x \equiv 3 \pmod 4$ and $p^y \equiv 3 \pmod 4$, it follows that $(p + 4n)^x + p^y \equiv 2 \pmod 4$ which contradicts with $z^2 \equiv 0 \pmod 4$.

Case 3. x is an even number and y is an odd number. Let $x = 2k, k \geq 1$ and $y = 2s+1, s \geq 0$. We have $(p+4n)^{2k}+p^{2s+1} = z^2$, or equivalently $p^{2s+1} = z^2-(p+4n)^{2k} = [z + (p + 4n)^k][z - (p + 4n)^k]$. Thus, there exist non-negative integers α, β such that $p^\alpha = z+(p+4n)^k$ and $p^\beta = z-(p+4n)^k$, where $\alpha > \beta$ and $\alpha+\beta = 2s+1$. Then, we have $2(p + 4n)^k = p^\beta(p^{\alpha-\beta} - 1)$. This implies that $\beta = 0$. We have $2(p + 4n)^k = (p^{2s+1} - 1)$, which is impossible because $2(p + 4n)^k \equiv 1, 2 \pmod 3$ but $(p^{2s+1} - 1) \equiv 0 \pmod 3$.

Case 4. x is an odd number and y is an even number. Let $x = 2k + 1, k \geq 0$ and $y = 2s, s \geq 1$. We have $(p + 4n)^{2k+1} + p^{2s} = z^2$, or equivalently $(p + 4n)^{2k+1} = z^2 - p^{2s} = (z + p^s)(z - p^s)$. Thus, there exist non-negative integer α, β such that $(p + 4n)^\alpha = z + p^s$ and $(p + 4n)^\beta = z - p^s$ where $\alpha > \beta$ and $\alpha + \beta = 2k + 1$. Then, we have $2(p)^s = (p + 4n)^\beta[(p + 4n)^{\alpha-\beta} - 1]$. This implies that $\beta = 0$. We have $2(p)^s = (p + 4n)^{2k+1} - 1$, which is impossible because $2(p)^s \equiv 2 \pmod 3$ but $(p + 4n)^{2k+1} - 1 \equiv 0, 1 \pmod 3$.

Acknowledgements

The authors wish to thank the referees for their kind suggestions and comments to improve the article.

References

[1] N. Burshtein. On the Diophantine equation $p^x + (p+12)^y = z^2$ when $p+5 = 2^{2u}$. *Annals of Pure and Applied Mathematic*, 18(1):41–44, 2020.

[2] S. Chotchaisthit. On The Diophantine Equation $4^x + p^y = z^2$ where p is a prime number. *American Journal Mathematics and Sciences*, 1(1):191–193, 2012.

[3] R. Dokchan and A. Pakapongpun. On the Diophantine $p^x + (p+20)^y = z^2$ when p and $p+20$ are primes. *International Journal of Mathematics and Computer Science*, 16(1):179–183, 2021.

[4] N. Fernando. On the Solvability of the Diophantine equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p+8$ are primes. *Annals of Pure and Applied Mathematics*, 18(1):9–13, 2018.

[5] W. S. Gayo and J. B. Bacani. On the Diophantine equation $M_p^x + (M_q+1)^y = z^2$. *European Journal of Pure and Applied Mathematics*, 14(2):396–403, 2021.

[6] A. Hoquea. On the Diophantine equation $(M_{pq})^x + (M_{pq}+1)^y = z^2$. *European Journal of Pure and Applied Mathematics*, 9(2):240–243, 2016.

[7] S. Kumar, S. Gupta and H. Kishan. On the non-linear Diophantine equation $p^x + (p+6)^y = z^2$. *Annals of Pure and Applied Mathematics*, 18(1):125–128, 2018.

[8] S. Kumar, S. Gupta and H. Kishan. On the solution of exponential Diophantine equation $p^x + (p+8)^y = z^2$. *International Journal of Mathematics and Computer Science*, 11(1):1–19, 2019.

[9] P. Mihailescu. Primary cyclotomic units and a proof of Catalan's conjecture. *Journal für die Reine and Angewandte Mathematik*, 572:167–195, 2004.

[10] A. Suvarnamani. On the Diophantine equation $p^x + (p+1)^y = z^2$. *International Journal of Mathematical Sciences and Applications*, 1(3):1415–1419, 2011.

[11] A. Suvarnamani. On the Diophantine equation $p^x + (p+1)^y = z^2$. *International Journal of Pure and Applied Mathematics*, 94(5):689–692, 2014.