EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1521-1535
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Dokdo BE-subalgebras and BE-filters of BE-algebras

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Abstract. With the aim of applying the Dokdo structure to BE-algebra, the notions of (weak) Dokdo BE-subalgebra and Dokdo BE-filter are introduced, and their properties are investigated. The relationship between weak Dokdo BE-subalgebra, Dokdo BE-subalgebra and Dokdo BE-filter is established. The conditions under which Dokdo structure can be weak Dokdo BE-subalgebra and Dokdo BE-filter, and the condition under which weak Dokdo BE-subalgebra can be Dokdo BE-subalgebra are explored. Characterizations of Dokdo BE-filter are provided.

2020 Mathematics Subject Classifications: 03G25, 06F35, 08A72

Key Words and Phrases: Weak Dokdo BE-subalgebra, Dokdo BE-subalgebra, Dokdo BE-filter

1. Introduction

Soft sets and fuzzy sets (interval value, bipolar) are useful tools for solving the problem of maintaining uncertainty in everyday life. Fuzzy sets are an extension of an existing set using fuzzy logic, and interval-valued fuzzy sets are also an extension of fuzzy sets whose membership degree range is a subinterval of $[0, 1]$. As an extension of fuzzy sets, bipolar fuzzy sets whose membership degree range is $[-1, 1]$ are a very useful tool for considering positive information and negative information at the same time. Soft set theory is a generalization of fuzzy set theory. (Bipolar, interval-valued) fuzzy set theory and soft set theory are good mathematical tools for dealing with uncertainty in a parametric manner, and have many applications in medical diagnosis and decision making etc. In the information age, there is a growing need to use hybrid structures in various fields. It has become necessary to study hybrid structures based on logical algebra to present the mathematical tools needed to meet these needs. Hybrid structures dealing with two or more different concepts at the same time have the advantage of reducing the loss of

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4510

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information when addressing uncertainty issues. In line with this background and need, Jun [5] introduced a new type of hybrid structure called Dokdo structure, where "Dokdo" is the name of the most beautiful island in Korea, using the concepts of bipolar fuzzy set, soft set and interval-valued fuzzy and first applied it to the algebraic structure BCK/BCI-algebras (see [5, 6]). In 2007, H. S. Kim and Y. H. Kim [7] introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. They defined and studied the concept of a filter in BE-algebras. In [1, 11], S. S. Ahn et al. and A. Rezaei et al. studied fuzzy BE-algebras. G. Dymek and A. Walendziak [2] developed the theory of fuzzy filters in BE-algebras.

For the purpose of applying the Dokdo structure to BE-algebra, we introduce (weak) Dokdo BE-subalgebra and Dokdo BE-filter and study its characteristics. We investigate the relationship between weak Dokdo BE-subalgebra, Dokdo BE-subalgebra and Dokdo BE-filter. We explore the conditions under which Dokdo structure can be weak Dokdo BE-subalgebra and Dokdo BE-filter, and the condition under which weak Dokdo BE-subalgebra can be Dokdo BE-subalgebra. We discuss the characterization of Dokdo BE-filter.

2. Preliminaries

2.1. Basic concepts about BE-algebras

A *BE-algebra* (see [7]) is defined to be a set X together with a binary operation " $*$ " and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,
(BE2) $(\forall a \in X)\ (a * 1 = 1)$,
(BE3) $(\forall a \in X)\ (1 * a = a)$,
(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

The order relation " $\leq$ " in a BE-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow \ a * b = 1). \tag{1}$$

Every BE-algebra X satisfies the following conditions (see [7]):

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,, \tag{2}$$
$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{3}$$

A BE-algebra X is said to be *self-distributive* (see [7]) if it satisfies:

$$(\forall x, b, c \in X)\,(x * (b * c) = (x * b) * (x * c))\,. \tag{4}$$

A subset A of a BE-algebra X is called

- a *BE-subalgebra* of X (see [7]) if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{5}$$

- an *n-fold weak BE-subalgebra* of X (see [4]) if it satisfies:

$$(\forall a, b \in A)(a^n * b \in A), \tag{6}$$

where n is a natural number with $n \geq 2$ and $a^n * b = a * (a * (\cdots (a * b) \cdots))$ in which a appears n times. The n-fold weak BE-subalgebra with $n = 2$ is called a *weak BE-subalgebra*.

- a *BE-filter* of X (see [7]) if it satisfies:

$$1 \in A, \tag{7}$$

$$(\forall a, b \in X)(a * b \in A,\ a \in A \ \Rightarrow\ b \in A). \tag{8}$$

2.2. Basic concepts about Dokdo structures

Let X be a set. A *bipolar fuzzy set* in X (see [8]) is an object having the form

$$\mathring{\varphi} = \{(a, \varphi^-(a), \varphi^+(a)) \mid a \in X\} \tag{9}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. The bipolar fuzzy set which is described in (9) is simply denoted by $\mathring{\varphi} := (X; \varphi^-, \varphi^+)$.

A bipolar fuzzy set can be reinterpreted as a function:

$$\mathring{\varphi} : X \to [-1, 0] \times [0, 1],\ x \mapsto (\varphi^-(x), \varphi^+(x)).$$

Let U be an initial universe set and X be a set of parameters. For any subset A of X, a pair (φ^s, A) is called a *soft set* over U (see [9]), where φ^s is a mapping described as follows:

$$\varphi^s : A \to 2^U$$

where 2^U is the power set of U. If $A = X$, the soft set (φ^s, A) over U is simply denoted by φ^s only.

A mapping $\tilde{\varphi} : X \to [[0, 1]]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X (see [3, 12]) where $[[0, 1]]$ is the set of all closed subintervals of $[0, 1]$, and members of $[[0, 1]]$ are called *interval numbers* and are denoted by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, etc., where $\tilde{a} = [a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$.

For every two interval numbers $\tilde{a}$ and $\tilde{b}$, we define

$$\tilde{a} \preceq \tilde{b} \text{ (or } \tilde{b} \succeq \tilde{a}) \ \Leftrightarrow\ a^- \leq b^-,\ a^+ \leq b^+, \tag{10}$$

$$\tilde{a} = \tilde{b} \ \Leftrightarrow\ \tilde{a} \preceq \tilde{b},\ \tilde{b} \preceq \tilde{a}, \tag{11}$$

$$\text{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]. \tag{12}$$

Let U be an initial universe set and X a set of parameters. A triple $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is called a *Dokdo structure* (see [5]) in (X, U) if $\mathring{\varphi} : X \to [-1, 0] \times [0, 1]$ is a bipolar fuzzy set in X, $\varphi^s : X \to 2^U$ is a soft set over U and $\tilde{\varphi} : X \to [[0,1]]$ is an interval-valued fuzzy set in X.

The Dokdo structure $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ in (X,U) can be represented as follows:

$$\begin{aligned} Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi}) : X \to ([-1,0] \times [0,1]) \times 2^U \times [[0,1]], \\ x \mapsto (\mathring{\varphi}(x), \varphi^s(x), \tilde{\varphi}(x)) \end{aligned} \tag{13}$$

where $\mathring{\varphi}(x) = (\mathring{\varphi}^-(x), \mathring{\varphi}^+(x))$ and $\tilde{\varphi}(x) = [\tilde{\varphi}_L(x), \tilde{\varphi}_R(x)]$.

Given a Dokdo structure $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ in a Dokdo universe (X,U), we consider the following sets:

$$\begin{aligned}
&\mathring{\varphi}(\max,\min) := \left\{ \tfrac{x}{(y,z)} \in \tfrac{X}{X\times X} \,\middle|\, \begin{array}{l} \mathring{\varphi}^-(x) \leq \max\{\mathring{\varphi}^-(y), \mathring{\varphi}^-(z)\} \\ \mathring{\varphi}^+(x) \geq \min\{\mathring{\varphi}^+(y), \mathring{\varphi}^+(z)\} \end{array} \right\}, \\
&\mathring{\varphi}(s,-) := \{x \in X \mid \mathring{\varphi}^-(x) \leq s\},\ \mathring{\varphi}(t,+) := \{x \in X \mid \mathring{\varphi}^+(x) \geq t\}, \\
&\mathring{\varphi}(s,t) := \mathring{\varphi}(s,-) \cap \mathring{\varphi}(t,+), \\
&\varphi^s_\alpha := \{x \in X \mid \varphi^s(x) \supseteq \alpha\},\ \tilde{\varphi}_{\tilde{a}} := \{x \in X \mid \tilde{\varphi}(x) \succeq \tilde{a}\},
\end{aligned}$$

where $(s,t) \in [-1,0] \times [0,1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.

3. Dokdo BE-subalgebras

Let U be an initial universe set and X a set of parameters. We say that the pair (X,U) is called a *Dokdo BE-universe* if X is a BE-algebra. In what follows, let (X,U) denote the Dokdo BE-universe unless otherwise specified.

Definition 1. *A Dokdo structure $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is called a Dokdo BE-subalgebra of (X,U) if it satisfies:*

$$(\forall x,y \in X)\left(\tfrac{x*y}{(x,y)} \in \mathring{\varphi}(\max,\min)\right), \tag{14}$$

$$(\forall x,y \in X)\left(\varphi^s(x*y) \supseteq \varphi^s(x) \cap \varphi^s(y)\right), \tag{15}$$

$$(\forall x,y \in X)\left(\tilde{\varphi}(x*y) \succeq \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}\right). \tag{16}$$

Example 1. *Let (X,U) be a BE-Dokdo universe in which $U = \mathbb{Z}$ and $X = \{1,2,3,4,5,6\}$ is a BE-algebra (see [1]) with a binary operation "$*$" given in the table below.*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	2	4	4	5
3	1	1	1	4	4	4
4	1	2	3	1	2	3
5	1	1	2	1	1	2
6	1	1	1	1	1	1

Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in $(X, U = \mathbb{Z})$ which is defined as follows:

X	$\mathring{\varphi}(x)$	$\varphi^s(x)$	$\tilde{\varphi}(x)$
1	$(-0.7, 0.8)$	$2\mathbb{Z}$	$[0.4, 0.8]$
2	$(-0.6, 0.5)$	$8\mathbb{Z}$	$[0.3, 0.6]$
3	$(-0.3, 0.2)$	$4\mathbb{Z}$	$[0.1, 0.5]$
4	$(-0.5, 0.4)$	$8\mathbb{N}$	$[0.3, 0.7]$
5	$(-0.4, 0.3)$	$16\mathbb{N}$	$[0.2, 0.6]$
6	$(-0.3, 0.2)$	$16\mathbb{N}$	$[0.1, 0.5]$

It is routine to verify that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo BE-subalgebra of $(X, U = \mathbb{Z})$.

Definition 2. *A Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is called a weak Dokdo BE-subalgebra of (X, U) if it satisfies:*

$$(\forall x, y \in X) \left(\tfrac{x*(x*y)}{(x,y)} \in \mathring{\varphi}(\max, \min) \right), \tag{17}$$

$$(\forall x, y \in X) \left(\varphi^s(x * (x * y)) \supseteq \varphi^s(x) \cap \varphi^s(y) \right), \tag{18}$$

$$(\forall x, y \in X) \left(\tilde{\varphi}(x * (x * y)) \succeq \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \right). \tag{19}$$

Example 2. *Let (X, U) be a BE-Dokdo universe in which $U = \mathbb{Z}$ and $X = \{1, 2, 3, 4\}$ is a BE-algebra (see [10]) with a binary operation "$*$" given in the table below.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	2	2
3	1	1	1	2
4	1	1	2	1

Define a Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in (X, U) as follows:

$$Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi}) : X \to ([-1, 0] \times [0, 1]) \times 2^U \times [[0, 1]],$$
$$x \mapsto \begin{cases} ((-0.46, 0.73), \mathbb{Z}, [0.41, 0.73]) & \text{if } x = 1, \\ ((-0.36, 0.63), \mathbb{N}, [0.32, 0.64]) & \text{otherwise.} \end{cases}$$

It is routine to check that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a weak Dokdo BE-subalgebra of (X, U).

Lemma 1. *Every Dokdo BE-subalgebra is a weak Dokdo BE-subalgebra.*

Proof. The proof is straightforward.

The converse of Lemma 1 may not be true as seen in the following example.

Example 3. *Let (X, U) be a BE-Dokdo universe in which $U = \mathbb{Z}$ and $X = \{1, 2, 3, 4\}$ is a BE-algebra (see [10]) with a binary operation "$*$" given in the table below.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	3
3	1	4	1	2
4	1	3	2	1

Define a Dokdo structure $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ *in* (X, U) *as follows:*

$$Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi}) : X \to ([-1, 0] \times [0, 1]) \times 2^U \times [[0, 1]],$$
$$x \mapsto \begin{cases} ((-0.46, 0.73), \mathbb{Z}, [0.41, 0.73]) & \text{if } x = 1, \\ ((-0.36, 0.63), 2\mathbb{Z}, [0.32, 0.64]) & \text{if } x \in \{2, 4\}, \\ ((-0.27, 0.58), 2\mathbb{N}, [0.29, 0.59]) & \text{if } x = 3. \end{cases}$$

It is routine to check that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ *is a weak Dokdo BE-subalgebra of* (X, U). *But it is not a Dokdo BE-subalgebra of* (X, U) *since* $\frac{2*4}{(2,4)} = \frac{3}{(2,4)} \notin \mathring{\varphi}(\max, \min)$, $\varphi^s(2*4) = \varphi^s(3) = 2\mathbb{N} \not\supseteq 2\mathbb{Z} = \varphi^s(2) \cap \varphi^s(4)$, *or* $\tilde{\varphi}(2 * 4) = \tilde{\varphi}(3) = [0.29, 0.59] \not\succeq [0.32, 0.64] = \text{rmin}\{\tilde{\varphi}(2), \tilde{\varphi}(4)\}$.

We explore the conditions under which the converse of Lemma 1 becomes true.

Theorem 1. *If a weak Dokdo BE-subalgebra* $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ *of* (X, U) *satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} \frac{x*y}{(x*(x*y)),\, x*(x*y))} \in \mathring{\varphi}(\max, \min), \\ \varphi^s(x * y) \supseteq \varphi^s(x * (x * y)), \\ \tilde{\varphi}(x * y) \succeq \tilde{\varphi}(x * (x * y)) \end{array} \right), \tag{20}$$

then $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ *is a Dokdo BE-subalgebra of* (X, U).

Proof. For every $x, y \in X$, we have

$$\mathring{\varphi}^-(x * y) \leq \mathring{\varphi}^-(x * (x * y)) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(y)\}$$

and $\mathring{\varphi}^+(x * y) \geq \mathring{\varphi}^+(x * (x * y)) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(y)\}$. Hence $\frac{x*y}{(x,\, y)} \in \mathring{\varphi}(\max, \min)$. Also, $\varphi^s(x*y) \supseteq \varphi^s(x*(x*y)) \supseteq \varphi^s(x) \cap \varphi^s(y)$ and $\tilde{\varphi}(x*y) \succeq \tilde{\varphi}(x*(x*y)) \succeq \text{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}$. Therefore $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo BE-subalgebra of (X, U).

Proposition 1. *If* $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ *is a weak Dokdo BE-subalgebra of* (X, U), *then*

(i) $\varphi^-(1)$ *is a lower bound of* $\{\varphi^-(x) \mid x \in X\}$,

(ii) $\varphi^+(1)$ *is an upper bound of* $\{\varphi^-(x) \mid x \in X\}$,

(iii) $(\forall x \in X)$ $(\varphi^s(1) \supseteq \varphi^s(x),\ \tilde{\varphi}(1) \succeq \tilde{\varphi}(x))$.

Proof. Let $Dok_{\varphi} := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ be a weak Dokdo BE-subalgebra of (X, U). For every $x \in X$, if we use (BE1) and (BE2), then $\frac{1}{(x,x)} = \frac{x*(x*x)}{(x,x)} \in \mathring{\varphi}(\max, \min)$ which implies that

$$\varphi^-(1) \leq \max\{\varphi^-(x), \varphi^-(x)\} = \varphi^-(x)$$

and

$$\varphi^+(1) \geq \min\{\varphi^+(x), \varphi^+(x)\} = \varphi^+(x).$$

Hence (i) and (ii) are valid. Also, $\varphi^s(1) = \varphi^s(x * (x * x)) \supseteq \varphi^s(x) \cap \varphi^s(x) = \varphi^s(x)$ and $\tilde{\varphi}(1) = \tilde{\varphi}(x * (x * x)) \succeq \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x)\} = \tilde{\varphi}(x)$.

The combination of Lemma 1 and Proposition 1 leads to the following corollary.

Corollary 1. *If $Dok_{\varphi} := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo BE-subalgebra of (X, U), then the results (i), (ii), and (iii) in Proposition 1 are valid.*

Proposition 2. *Every weak Dokdo BE-subalgebra $Dok_{\varphi} := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ of (X, U) satisfies:*

$$(\forall x, y \in X)\left(\tfrac{y}{(y*(y*x),y*(y*x))} \in \mathring{\varphi}(\max, \min) \ \Rightarrow \left\{ \begin{array}{l} \varphi^-(y) = \varphi^-(1) \\ \varphi^+(y) = \varphi^+(1) \end{array} \right. \right), \tag{21}$$

$$(\forall x, y \in X)\left(\varphi^s(y) \supseteq \varphi^s(y * (y * x)) \ \Rightarrow \ \varphi^s(y) = \varphi^s(1)\right), \tag{22}$$

$$(\forall x, y \in X)\left(\tilde{\varphi}(y) \succeq \tilde{\varphi}(y * (y * x)) \ \Rightarrow \ \tilde{\varphi}(y) = \tilde{\varphi}(1)\right). \tag{23}$$

Proof. Assume that $\frac{y}{(y*(y*x),y*(y*x))} \in \mathring{\varphi}(\max, \min)$, $\varphi^s(y) \supseteq \varphi^s(y * (y * x))$ and $\tilde{\varphi}(y) \succeq \tilde{\varphi}(y*(y*x))$ for all $x, y \in X$. If we take $x = 1$ and use (BE2), then $\frac{y}{(1,1)} = \frac{y}{(y*(y*1),y*(y*1))} \in \mathring{\varphi}(\max, \min)$, $\varphi^s(y) \supseteq \varphi^s(y * (y * 1)) = \varphi^s(1)$ and $\tilde{\varphi}(y) \succeq \tilde{\varphi}(y * (y * 1)) = \tilde{\varphi}(1)$. The combination of these and Proposition 1 leads to $\varphi^-(y) = \varphi^-(1)$, $\varphi^+(y) = \varphi^+(1)$, $\varphi^s(y) = \varphi^s(1)$ and $\tilde{\varphi}(y) = \tilde{\varphi}(1)$.

Corollary 2. *Every Dokdo BE-subalgebra $Dok_{\varphi} := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ of (X, U) satisfies* (21), (22) *and* (23).

Theorem 2. *If $Dok_{\varphi} := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a weak Dokdo BE-subalgebra of (X, U), then the nonempty sets $\mathring{\varphi}(s, t)$, φ^s_{α} and $\tilde{\varphi}_{\tilde{a}}$ are weak BE-subalgebras of X for all $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a^-, a^+]$.*

Proof. Let $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a^-, a^+]$ be such that $\mathring{\varphi}(s, t)$, φ^s_{α} and $\tilde{\varphi}_{\tilde{a}}$ are nonempty. Let $x, y \in \mathring{\varphi}(s, t) \cap \varphi^s_{\alpha} \cap \tilde{\varphi}_{\tilde{a}}$. Then $\varphi^-(x) \leq s$, $\varphi^-(y) \leq s$, $\varphi^+(x) \geq t$, $\varphi^+(y) \geq t$, $\varphi^s(x) \supseteq \alpha$, $\varphi^s(y) \supseteq \alpha$, $\tilde{\varphi}(x) \succeq \tilde{a}$ and $\tilde{\varphi}(y) \succeq \tilde{a}$. Hence

$$\varphi^-(x * (x * y)) \leq \max\{\varphi^-(x), \varphi^-(y)\} \leq s,$$
$$\varphi^+(x * (x * y)) \geq \min\{\varphi^+(x), \varphi^+(y)\} \geq t,$$

and so $x * (x * y) \in \mathring{\varphi}(s, t)$. Also we have $\varphi^s(x * (x * y)) \supseteq \varphi^s(x) \cap \varphi^s(y) \supseteq \alpha$ and $\tilde{\varphi}(x * (x * y)) \succeq \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \succeq \tilde{a}$, that is, $x * (x * y) \in \varphi^s_{\alpha}$ and $x * (x * y) \in \tilde{\varphi}_{\tilde{a}}$. Therefore $\mathring{\varphi}(s, t)$, φ^s_{α} and $\tilde{\varphi}_{\tilde{a}}$ are weak BE-subalgebras of X.

Corollary 3. *If* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *is a Dokdo BE-subalgebra of* (X,U)*, then the nonempty sets* $\mathring{\varphi}(s,t)$*,* φ^s_α *and* $\tilde{\varphi}_{\tilde{a}}$ *are weak BE-subalgebras of* X *for all* $(s,t) \in [-1,0] \times [0,1]$*,* $\alpha \in 2^U$ *and* $\tilde{a} = [a^-, a^+]$*.*

The following example shows that the converse of Theorem 2 may not be true.

Example 4. *Let* (X,U) *be the BE-Dokdo universe in Example 1. Define a Dokdo structure* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *in* (X,U) *as follows:*

$$Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi}) : X \to ([-1,0] \times [0,1]) \times 2^U \times [[0,1]],$$
$$x \mapsto \begin{cases} ((-0.85, 0.71),\ \mathbb{Z},\ [0.42, 0.76]) & \text{if } x = 1, \\ ((-0.66, 0.53),\ 4\mathbb{Z},\ [0.29, 0.58]) & \text{if } x = 2, \\ ((-0.44, 0.57),\ 4\mathbb{Z},\ [0.29, 0.58]) & \text{if } x = 3, \\ ((-0.44, 0.53),\ 4\mathbb{Z},\ [0.29, 0.58]) & \text{if } x = 4, \\ ((-0.44, 0.53),\ 4\mathbb{Z},\ [0.29, 0.58]) & \text{if } x = 5, \\ ((-0.72, 0.68),\ 2\mathbb{Z},\ [0.33, 0.72]) & \text{if } x = 6. \end{cases}$$

It is routine to verify that the nonempty sets $\mathring{\varphi}(s,t)$*,* φ^s_α *and* $\tilde{\varphi}_{\tilde{a}}$ *are weak BE-subalgebras of* X *for all* $(s,t) \in [-1,0] \times [0,1]$*,* $\alpha \in 2^U$ *and* $\tilde{a} = [a^-, a^+]$*. But* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *is not a weak Dokdo BE-subalgebra of* (X,U) *because of* $\frac{2*(2*6)}{(2,6)} = \frac{4}{(2,6)} \notin \mathring{\varphi}(\max, \min)$*.*

We provide conditions for a Dokdo structure to be a weak Dokdo BE-subalgebra.

Theorem 3. *Given a Dokdo structure* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *in* (X,U)*, if the nonempty sets* $\mathring{\varphi}(s,-)$*,* $\mathring{\varphi}(t,+)$*,* φ^s_α *and* $\tilde{\varphi}_{\tilde{a}}$ *are weak BE-subalgebras of* X *for all* $(s,t) \in [-1,0] \times [0,1]$*,* $\alpha \in 2^U$ *and* $\tilde{a} = [a^-, a^+]$*, then* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *is a weak Dokdo BE-subalgebra of* (X,U)*.*

Proof. Assume that $\mathring{\varphi}(s,-)$, $\mathring{\varphi}(t,+)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are nonempty weak BE-subalgebras of X for all $(s,t) \in [-1,0] \times [0,1]$, $\alpha \in 2^U$ and $\tilde{a} = [a^-, a^+]$. If there exist $x, y \in X$ such that $\frac{x*(x*y)}{(x,y)} \notin \mathring{\varphi}(\max, \min)$, then

$$\varphi^-(x*(x*y)) > \max\{\varphi^-(x), \varphi^-(y)\} \text{ or } \varphi^+(x*(x*y)) < \min\{\varphi^+(x), \varphi^+(y)\}.$$

It follows that $x, y \in \mathring{\varphi}(s,-) \cap \mathring{\varphi}(t,+)$, $x*(x*y) \notin \mathring{\varphi}(s,-)$ and $x*(x*y) \notin \mathring{\varphi}(t,+)$ for $s := \max\{\varphi^-(x), \varphi^-(y)\}$ and $t := \min\{\varphi^+(x), \varphi^+(y)\}$. This is a contradiction, and thus $\frac{x*(x*y)}{(x,y)} \in \mathring{\varphi}(\max, \min)$ for all $x, y \in X$. For every $x, y \in X$, let $\varphi^s(x) = \alpha_x$, $\varphi^s(y) = \alpha_y$, $\tilde{\varphi}(x) = \tilde{a}$ and $\tilde{\varphi}(y) = \tilde{b}$. If we take $\alpha := \alpha_x \cap \alpha_y$ and $\tilde{c} := \mathrm{rmin}\{\tilde{a}, \tilde{b}\}$, then $x, y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$ and so $x*(x*y) \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$. Hence

$$\varphi^s(x*(x*y)) \supseteq \alpha = \alpha_x \cap \alpha_y = \varphi^s(x) \cap \varphi^s(y)$$

and

$$\tilde{\varphi}(x*(x*y)) \succeq \tilde{c} = \mathrm{rmin}\{\tilde{a}, \tilde{b}\} = \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}.$$

Therefore $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a weak Dokdo BE-subalgebra of (X,U).

4. Dokdo BE-filters

Definition 3. *A Dokdo structure* $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ *in* (X,U) *is called a Dokdo BE-filter of* (X,U) *if it satisfies:*

$$(\forall x \in X)\begin{pmatrix} \frac{1}{(x,x)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(1) \supseteq \varphi^s(x),\ \tilde{\varphi}(1) \succeq \tilde{\varphi}(x) \end{pmatrix}, \tag{24}$$

$$(\forall x, y \in X)\begin{pmatrix} \frac{y}{(x,x*y)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(y) \supseteq \varphi^s(x) \cap \varphi^s(x*y), \\ \tilde{\varphi}(y) \succeq \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x*y)\} \end{pmatrix}. \tag{25}$$

Example 5. *Let* (X,U) *be a BE-Dokdo universe in which* $U = \mathbb{N}$ *and* $X = \{1,2,3,4,5\}$ *is a BE-algebra (see [7]) with a binary operation "$*$" given in the table below.*

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

Let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ *be a Dokdo structure in* $(X, U = \mathbb{N})$ *given in the table below.*

X	$\mathring{\varphi}(x)$	$\varphi^s(x)$	$\tilde{\varphi}(x)$
1	$(-0.6, 0.8)$	$2\mathbb{N}$	$[0.4, 0.9]$
2	$(-0.6, 0.8)$	$2\mathbb{N}$	$[0.4, 0.9]$
3	$(-0.3, 0.6)$	$4\mathbb{N}$	$[0.2, 0.5]$
4	$(-0.5, 0.4)$	$8\mathbb{N}$	$[0.3, 0.7]$
5	$(-0.3, 0.4)$	$8\mathbb{N}$	$[0.2, 0.5]$

Through routine calculations, we can confirm that $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ *in* (X,U) *is a Dokdo BE-filter of* $(X, U = \mathbb{N})$.

Proposition 3. *Every Dokdo BE-filter* $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ *of* (X,U) *satisfies:*

$$(\forall x, y \in X)\left(x \le y \;\Rightarrow\; \begin{cases} \frac{y}{(x,x)} \in \mathring{\varphi}(\max,\min) \\ \varphi^s(y) \supseteq \varphi^s(x),\ \tilde{\varphi}(y) \succeq \tilde{\varphi}(x) \end{cases}\right), \tag{26}$$

$$(\forall x, y, z \in X)\left(x \le y * z \;\Rightarrow\; \begin{cases} \frac{z}{(x,y)} \in \mathring{\varphi}(\max,\min) \\ \varphi^s(z) \supseteq \varphi^s(x) \cap \varphi^s(y) \\ \tilde{\varphi}(z) \succeq \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \end{cases}\right), \tag{27}$$

$$(\forall x, y \in X)\left(\begin{pmatrix} \frac{y*x}{(1,1)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(y*x) = \varphi^s(1), \\ \tilde{\varphi}(y*x) = \tilde{\varphi}(1) \end{pmatrix} \;\Rightarrow\; \begin{cases} \frac{x}{(y,y)} \in \mathring{\varphi}(\max,\min) \\ \varphi^s(x) \supseteq \varphi^s(y), \\ \tilde{\varphi}(x) \succeq \tilde{\varphi}(y) \end{cases}\right). \tag{28}$$

Proof. If $x \leq y$, then $x * y = 1$ and so (26) is derived from the definition of Dokdo BE-filter. Let $x, y, z \in X$ be such that $x \leq y * z$. Then $x * (y * z) = 1$, and so

$$\begin{aligned}\varphi^-(z) &\leq \max\{\varphi^-(y), \varphi^-(y * z)\}\\ &\leq \max\{\varphi^-(y), \max\{\varphi^-(x), \varphi^-(x * (y * z))\}\}\\ &= \max\{\varphi^-(y), \max\{\varphi^-(x), \varphi^-(1)\}\}\\ &= \max\{\varphi^-(x), \varphi^-(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+(z) &\geq \min\{\varphi^+(y), \varphi^+(y * z)\}\\ &\geq \min\{\varphi^+(y), \min\{\varphi^+(x), \varphi^+(x * (y * z))\}\}\\ &= \min\{\varphi^+(y), \min\{\varphi^+(x), \varphi^+(1)\}\}\\ &= \min\{\varphi^+(x), \varphi^+(y)\},\end{aligned}$$

that is, $\frac{z}{(x,y)} \in \mathring{\varphi}(\max, \min)$. Also, we have

$$\begin{aligned}\varphi^s(z) &\supseteq \varphi^s(y) \cap \varphi^s(y * z) \supseteq \varphi^s(y) \cap (\varphi^s(x) \cap \varphi^s(x * (y * z)))\\ &= \varphi^s(y) \cap (\varphi^s(x) \cap \varphi^s(1)) = \varphi^s(y) \cap \varphi^s(x)\end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}(z) &\succeq \mathrm{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y * z)\} \succeq \mathrm{rmin}\{\tilde{\varphi}(y), \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x * (y * z))\}\}\\ &= \mathrm{rmin}\{\tilde{\varphi}(y), \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(1)\}\} = \mathrm{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(x)\}.\end{aligned}$$

Let $x, y \in X$ be such that $\frac{y*x}{(1,1)} \in \mathring{\varphi}(\max, \min)$, $\varphi^s(y * x) = \varphi^s(1)$ and $\tilde{\varphi}(y * x) = \tilde{\varphi}(1)$. It follows that

$$\varphi^-(x) \leq \max\{\varphi^-(y), \varphi^-(y * x)\} \leq \max\{\varphi^-(y), \varphi^-(1)\} = \varphi^-(y)$$

and

$$\varphi^+(x) \geq \min\{\varphi^+(y), \varphi^+(y * x)\} \geq \min\{\varphi^+(y), \varphi^+(1)\} = \varphi^+(y),$$

that is, $\frac{x}{(y,y)} \in \mathring{\varphi}(\max, \min)$. Also, we get

$$\varphi^s(x) \supseteq \varphi^s(y) \cap \varphi^s(y * x) = \varphi^s(y) \cap \varphi^s(1) = \varphi^s(y)$$

and

$$\tilde{\varphi}(x) \succeq \mathrm{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y * x)\} = \mathrm{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(1)\} = \tilde{\varphi}(y).$$

This completes the proof.

Theorem 4. *Every Dokdo BE-filter is a (weak) Dokdo BE-subalgebra.*

Proof. Let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ be a Dokdo BE-filter of (X, U). Since $x \leq y * x$ for all $x, y \in X$, we have $\frac{y*x}{(x,x)} \in \mathring{\varphi}(\max, \min)$, $\varphi^s(y * x) \supseteq \varphi^s(x)$, and $\tilde{\varphi}(y * x) \succeq \tilde{\varphi}(x)$ by (26). It follows from (25) that

$$\begin{aligned}
&\varphi^-(y * x) \leq \varphi^-(x) \leq \max\{\varphi^-(y), \varphi^-(y * x)\} \leq \max\{\varphi^-(x), \varphi^-(y)\},\\
&\varphi^+(y * x) \geq \varphi^+(x) \geq \min\{\varphi^+(y), \varphi^+(y * x)\} \geq \min\{\varphi^+(x), \varphi^+(y)\},\\
&\varphi^s(y * x) \supseteq \varphi^s(x) \supseteq \varphi^s(y) \cap \varphi^s(y * x) \supseteq \varphi^s(x) \cap \varphi^s(y),\\
&\tilde{\varphi}(y * x) \succeq \tilde{\varphi}(x) \succeq \mathrm{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y * x)\} \succeq \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}.
\end{aligned}$$

Therefore $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo BE-subalgebra, and hence a weak Dokdo BE-subalgebra of (X, U).

The converse of Theorem 4 may not be true as seen in the following example.

Example 6. *(i) Let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ be the Dokdo BE-subalgebra of (X, U) which is described in Example 1. It is not a Dokdo BE-filter of (X, U) since $\varphi^s(5) = 16\mathbb{N} \not\supseteq 8\mathbb{N} = \varphi^s(2) \cap \varphi^s(2 * 5)$ or $\frac{3}{(5,5*3)} = \frac{3}{(5,2)} \notin \mathring{\varphi}(\max, \min)$.*

(ii) Let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ be the weak Dokdo BE-subalgebra of (X, U) which is described in Example 3. It is not a Dokdo BE-filter of (X, U) since $\frac{3}{(2,2*3)} = \frac{3}{(2,4)} \notin \mathring{\varphi}(\max, \min)$ or $\tilde{\varphi}(3) = [0.29, 0.59] \not\succeq [0.32, 0.64] = \mathrm{rmin}\{\tilde{\varphi}(2), \tilde{\varphi}(2 * 3)\}$.

Proposition 4. *Let (X, U) be a Dokdo BE-universe in which X is a self-distributive BE-algebra. If $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo BE-filter of (X, U), then the next assertions are equivalent.*

$$(\forall x, y \in X)\left(\begin{array}{l} \frac{y*x}{(y*(y*x),\, y*(y*x))} \in \mathring{\varphi}(\max, \min),\\ \varphi^s(y * x) \supseteq \varphi^s(y * (y * x)),\\ \tilde{\varphi}(y * x) \succeq \tilde{\varphi}(y * (y * x)) \end{array}\right). \tag{29}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} \frac{(z*y)*(z*x)}{(z*(y*x),\, z*(y*x))} \in \mathring{\varphi}(\max, \min),\\ \varphi^s((z * y) * (z * x)) \supseteq \varphi^s(z * (y * x)),\\ \tilde{\varphi}((z * y) * (z * x)) \succeq \tilde{\varphi}(z * (y * x)) \end{array}\right). \tag{30}$$

Proof. Let $x, y, z \in X$. Since X is self-distributive, we have

$$z * (y * x) \leq z * ((z * y) * (z * x)) = z * (z * ((z * y) * x)).$$

Assume that (29) is valid. Using (BE4), (26) and (29), we have

$$\begin{aligned}
&\varphi^-((z * y) * (z * x)) = \varphi^-(z * ((z * y) * x))\\
&\leq \varphi^-(z * (z * ((z * y) * x))) \leq \varphi^-(z * (y * x))
\end{aligned}$$

and

$$\varphi^+((z * y) * (z * x)) = \varphi^+(z * ((z * y) * x))$$

$$\geq \varphi^{+}(z*(z*((z*y)*x))) \geq \varphi^{+}(z*(y*x)),$$

that is, $\frac{(z*y)*(z*x)}{(z*(y*x),\, z*(y*x))} \in \mathring{\varphi}(\max,\min)$. Also, we have

$$\begin{aligned}\varphi^{s}((z*y)*(z*x)) &= \varphi^{s}(z*((z*y)*x)) \\ &\supseteq \varphi^{s}(z*(z*((z*y)*x))) \supseteq \varphi^{s}(z*(y*x)),\end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}((z*y)*(z*x)) &= \tilde{\varphi}(z*((z*y)*x)) \\ &\succeq \tilde{\varphi}(z*(z*((z*y)*x))) \succeq \tilde{\varphi}(z*(y*x)).\end{aligned}$$

Conversely, suppose that (30) is valid. If we put $y := z$ in (30) and use $(BE1)$ and (BE3), then

$$\frac{z*x}{(z*(z*x),\, z*(z*x))} = \frac{1*(z*x)}{(z*(z*x),\, z*(z*x))} = \frac{(z*z)*(z*x)}{(z*(z*x),\, z*(z*x))} \in \mathring{\varphi}(\max,\min),$$

$$\varphi^{s}(z*x) = \varphi^{s}(1*(z*x)) = \varphi^{s}((z*z)*(z*x)) \supseteq \varphi^{s}(z*(z*x))$$

and $\tilde{\varphi}(z*x) = \tilde{\varphi}(1*(z*x)) = \tilde{\varphi}((z*z)*(z*x)) \succeq \tilde{\varphi}(z*(z*x))$. This proves (29).

Proposition 5. *Let (X,U) be a Dokdo BE-universe in which X is a self-distributive BE-algebra. Then every Dokdo BE-filter $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^{s},\, \tilde{\varphi})$ of (X,U) satisfies:*

$$(\forall x,y,z \in X)\left(\begin{array}{l}\frac{y*x}{(y*z,\, z*x)} \in \mathring{\varphi}(\max,\min), \\ \varphi^{s}(y*x) \supseteq \varphi^{s}(y*z) \cap \varphi^{s}(z*x), \\ \tilde{\varphi}(y*x) \succeq \operatorname{rmin}\{\tilde{\varphi}(y*z), \tilde{\varphi}(z*x)\}\end{array}\right). \tag{31}$$

Proof. Using (BE1), (BE2), (BE4) and (4), we have $y*z \leq (z*x)*(y*x)$ for all $x,y,z \in X$. Hence (31) is derived from (27).

Theorem 5. *If a Dokdo structure $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^{s},\, \tilde{\varphi})$ in (X,U) satisfies* (27)*, then it is a Dokdo BE-filter of (X,U).*

Proof. Since $x \leq x*1$ for all $x \in X$, we have $\frac{1}{(x,x)} \in \mathring{\varphi}(\max,\min)$, $\varphi^{s}(1) \supseteq \varphi^{s}(x)$, and $\tilde{\varphi}(1) \succeq \tilde{\varphi}(x)$ by (27). Since $x*y \leq x*y$ for all $x,y \in X$, it follows from (27) that $\frac{y}{(x,x*y)} \in \mathring{\varphi}(\max,\min)$, $\varphi^{s}(y) \supseteq \varphi^{s}(x) \cap \varphi^{s}(x*y)$, and $\tilde{\varphi}(y) \succeq \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x*y)\}$. So, $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^{s},\, \tilde{\varphi})$ is a Dokdo BE-filter of (X,U).

Corollary 4. *If a Dokdo structure $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^{s},\, \tilde{\varphi})$ in (X,U) satisfies* (27)*, then it is a (weak) Dokdo BE-subalgebra of (X,U).*

Theorem 6. *A Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in (X,U) is a Dokdo BE-filter of (X,U) if and only if it satisfies* (24) *and*

$$(\forall x,y,z \in X)\left(\begin{array}{l} \frac{x*z}{(x*(y*z),\,y)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(x*z) \supseteq \varphi^s(x*(y*z)) \cap \varphi^s(y), \\ \tilde{\varphi}(x*z) \succeq \operatorname{rmin}\{\tilde{\varphi}(x*(y*z)), \tilde{\varphi}(y)\} \end{array}\right). \tag{32}$$

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo BE-filter of (X,U) and let $x,y,z \in X$. Then $\varphi^-(x*z) \leq \max\{\varphi^-(y), \varphi^-(y*(x*z))\} = \max\{\varphi^-(y), \varphi^-(x*(y*z))\}$ and $\varphi^+(x*z) \geq \min\{\varphi^+(y), \varphi^+(y*(x*z))\} = \min\{\varphi^+(y), \varphi^+(x*(y*z))\}$, that is, $\frac{x*z}{(x*(y*z),\,y)} \in \mathring{\varphi}(\max,\min)$. Also, we have

$$\varphi^s(x*z) \supseteq \varphi^s(y) \cap \varphi^s(y*(x*z)) = \varphi^s(y) \cap \varphi^s(x*(y*z))$$

and

$$\tilde{\varphi}(x*z) \succeq \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y*(x*z))\} = \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(x*(y*z))\}.$$

Conversely, suppose that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ satisfies (24) and (32). If we put $x = 1$ in (32) and use (BE3), then we get $\frac{z}{(y*z,\,y)} = \frac{1*z}{(1*(y*z),\,y)} \in \mathring{\varphi}(\max,\min)$, $\varphi^s(z) = \varphi^s(1*z) \supseteq \varphi^s(1*(y*z)) \cap \varphi^s(y) = \varphi^s(y*z) \cap \varphi^s(y)$ and

$$\tilde{\varphi}(z) = \tilde{\varphi}(1*z) \succeq \operatorname{rmin}\{\tilde{\varphi}(1*(y*z)), \tilde{\varphi}(y)\} = \operatorname{rmin}\{\tilde{\varphi}(y*z), \tilde{\varphi}(y)\}$$

for all $y,z \in X$. Therefore $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo BE-filter of (X,U).

Theorem 7. *A Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in (X,U) is a Dokdo BE-filter of (X,U) if and only if it satisfies:*

$$(\forall x,y \in X)\left(\begin{array}{l} \frac{y*x}{(x,\,x)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(y*x) \supseteq \varphi^s(x), \\ \tilde{\varphi}(y*x) \succeq \tilde{\varphi}(x) \end{array}\right), \tag{33}$$

$$(\forall x,y,a,b \in X)\left(\begin{array}{l} \frac{(a*(b*x))*x}{(a,\,b)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s((a*(b*x))*x) \supseteq \varphi^s(a) \cap \varphi^s(b), \\ \tilde{\varphi}((a*(b*x))*x) \succeq \operatorname{rmin}\{\tilde{\varphi}(a), \tilde{\varphi}(b)\} \end{array}\right). \tag{34}$$

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo BE-filter of (X,U) and let $x,y,a,b \in X$. Then

$$\varphi^-(y*x) \leq \max\{\varphi^-(x), \varphi^-(x*(y*x))\} = \max\{\varphi^-(x), \varphi^-(1)\} = \varphi^-(x)$$

and

$$\varphi^+(y*x) \geq \min\{\varphi^+(x), \varphi^+(x*(y*x))\} = \min\{\varphi^+(x), \varphi^+(1)\} = \varphi^+(x),$$

that is, $\frac{y*x}{(x,\,x)} \in \mathring{\varphi}(\max,\min)$. Also, we obtain

$$\varphi^s(y*x) \supseteq \varphi^s(x) \cap \varphi^s(x*(y*x)) = \varphi^s(x) \cap \varphi^s(1) = \varphi^s(x)$$

and

$$\tilde{\varphi}(y * x) \succeq \text{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x * (y * x))\} = \text{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(1)\} = \tilde{\varphi}(x).$$

Hence (33) is valid. The following facts can be obtained by using (3), (26), and Theorem 6.

$$\begin{aligned}
&\varphi^-((a * (b * x)) * x) \leq \max\{\varphi^-((a * (b * x)) * (b * x)), \varphi^-(b)\} \leq \max\{\varphi^-(a), \varphi^-(b)\},\\
&\varphi^+((a * (b * x)) * x) \geq \min\{\varphi^+((a * (b * x)) * (b * x)), \varphi^+(b)\} \geq \min\{\varphi^+(a), \varphi^+(b)\},\\
&\varphi^s((a * (b * x)) * x) \supseteq \varphi^s((a * (b * x)) * (b * x)) \cap \varphi^s(b) \supseteq \varphi^s(a) \cap \varphi^s(b),\\
&\tilde{\varphi}((a * (b * x)) * x) \succeq \text{rmin}\{\tilde{\varphi}((a * (b * x)) * (b * x)), \tilde{\varphi}(b)\} \succeq \text{rmin}\{\tilde{\varphi}(a), \tilde{\varphi}(b)\}.
\end{aligned}$$

Thus (34) is valid.

Conversely, suppose that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ satisfies (33) and (34). If we take $y = x$ in (33) and use (BE1), then $\frac{1}{(x,x)} = \frac{x*x}{(x,x)} \in \mathring{\varphi}(\max, \min)$, $\varphi^s(1) = \varphi^s(x * x) \supseteq \varphi^s(x)$, and $\tilde{\varphi}(1) = \tilde{\varphi}(x * x) \succeq \tilde{\varphi}(x)$ for all $x \in X$. Using (BE1), (BE3) and (34), we have

$$\begin{aligned}
&\varphi^-(y) = \varphi^-(1 * y) = \varphi^-(((x * y) * (x * y)) * y) \leq \max\{\varphi^-(x * y), \varphi^-(x)\},\\
&\varphi^+(y) = \varphi^+(1 * y) = \varphi^+(((x * y) * (x * y)) * y) \geq \min\{\varphi^+(x * y), \varphi^+(x)\},\\
&\varphi^s(y) = \varphi^s(1 * y) = \varphi^s(((x * y) * (x * y)) * y) \supseteq \varphi^s(x * y) \cap \varphi^s(x),\\
&\tilde{\varphi}(y) = \tilde{\varphi}(1 * y) = \tilde{\varphi}(((x * y) * (x * y)) * y) \succeq \text{rmin}\{\tilde{\varphi}(x * y), \tilde{\varphi}(x)\}.
\end{aligned}$$

Consequently, $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo BE-filter of (X, U).

5. Conclusion

To apply the Dokdo structure to BE-algebra, we introduced (weak) Dokdo BE-subalgebra and Dokdo BE-filter and study its characteristics. We investigated the relationship between weak Dokdo BE-subalgebra, Dokdo BE-subalgebra and Dokdo BE-filter. We explored the conditions under which Dokdo structure can be weak Dokdo BE-subalgebra and Dokdo BE-filter, and the condition under which weak Dokdo BE-subalgebra can be Dokdo BE-subalgebra. We discussed the characterization of Dokdo BE-filter.

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

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