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Forcing Subsets of Connected Co-Independent Hop Domination in the Edge Corona and Lexicographic Product of Graphs

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Abstract. This study deals with the forcing subsets of a minimum connected co-independent hop dominating sets in graphs. Bounds or exact values of the forcing connected co-independent hop domination numbers of graphs resulting from some binary operations such as edge corona and lexicographic product of graphs are determined.
Some main results generated in this study include: (a) characterization of the minimum connected co-independent hop dominating sets; and (b) characterization of the forcing subsets for these types of sets.

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1. Introduction

Beginning with C. Berge [4] in 1958, the study on domination in graphs was developed. There are now a lot of studies involving domination and its variations. One of its variations is the connected co-independent domination number of graphs that was studied in [7].

Years later, a new domination parameter called hop domination in graph was introduced in [12] by Natarajan and Ayyaswamy and was also studied in [3, 13–15]. A study in 2021 by S. Nanding and H. Rara [11] introduced a new concept of hop domination called the connected co-independent hop domination and generated some characterizations of connected co-independent hop domination in graphs.

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On the other hand, the concept of forcing numbers started from the study of molecular resonance structure which was introduced by Klein and Randic [10] in 1987. Harary et. al [16] first used the name "forcing number" and introduced the concept of the forcing of a perfect match in 1991. Chartrand et. al [5] initiated the investigation on the relation between forcing and domination concepts in 1997 and defined the term "forcing domination number". In 2017, John et. al [9] investigated the forcing connected domination number of a graph, and C. Armada and S. Canoy Jr. [1] investigated the forcing independent domination number of a graph in 2019. Furthermore, in 2018, Canoy et. al [2] investigated the forcing domination number of graphs under some binary operations.

In this study, the forcing subsets of minimum connected co-independent hop dominating sets in graphs are defined and established and some characterizations of forcing subsets of minimum connected co-independent hop dominating sets of graphs resulting from the edge corona and lexicographic product of two graphs are generated. Also, the values or bounds of their corresponding forcing connected co-independent hop domination numbers are determined.

Connected co-independent hop domination in graphs can have real world applications. For an application, in [6], Desormeaux, Haynes, and Henning inspired their research on these concepts through social networking applications. They considered a factory with a large number of employees and needed to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this. In other words, the manager will select some workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible to minimize costs (extra costs for inspectors) and protect privacy (keep the inspectors' identity confidential). To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed in which each worker is represented by a vertex and an edge between two workers represents possible bias, that is, whether the two workers are close friends or enemies. Ideally, an inspector should not be adjacent to any worker who is being inspected.

In connected co-independent hop domination [11], every worker will be inspected by the nearest non-biased inspector. That is, an inspector who is a close friend (or an enemy) of a close friend (or enemy) of a worker. This is to save time and effort of locating a particular worker. Also, the inspectors should be acquainted with each other and all non-inspector workers are neither friends nor enemies, that is, they are not adjacent or there is no edge between them. The connected co-independent hop domination number will give the minimum number of inspectors needed.

In forcing subsets of connected co-independent hop domination, in each respective group of minimum number of inspectors that will inspect the workers in the designated areas of the factory, the members of that particular group of minimum number of inspectors will be assigned only to that distinct group of minimum number of inspectors, that is, it will strengthen the bond of the respective group of minimum number of non-biased inspectors with each other, since they are uniquely assigned to particular groups, and they will trust each other more doing their duties and will have a much easier time doing their job regarding with the respective workers that they are assigned to inspect. The forcing connected co-independent hop domination number will determine the minimum number of members from the respective group of minimum number of inspectors that will be assigned only to that particular group of respective minimum number of inspectors.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [8] for elementary Graph Theoretic concepts.

An *independent set* S in a graph G is a subset of the vertex-set of G such that no two vertices in S are adjacent in G. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a *β-set* of G.

A set $S \subseteq V(G)$ is a *co-independent* set of G if $V(G) \setminus S$ is independent. The minimum cardinality of a co-independent set in G, denoted by $coi(G)$ is called the co-independent number of G. A co-independent set S with $|S| = coi(G)$ is called a *coi-set* of G.

A dominating set $D \subseteq V(G)$ is called a *connected co-independent dominating set* of G if the subgraph $\langle D \rangle$ is connected and $V(G) \setminus D$ is an independent set. The cardinality of such a minimum set D is called *connected co-independent domination number* of G denoted by $\gamma_{c,coi}(G)$. A connected co-independent dominating set D with $|D| = \gamma_{c,coi}(G)$ is called a *$\gamma_{c,coi}$-set* of G.

Let G be a connected graph. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a *γ_h-set*.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

Let G be a connected graph. A hop dominating set $S \subseteq V(G)$ is a *connected co-independent hop dominating set* of G if $\langle S \rangle$ is connected and $V(G) \backslash S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of G, denoted by $\gamma_{ch,coi}(G)$, is called the *connected co-independent hop domination number* of G. A connected co-independent hop dominating set S with $|S| = \gamma_{ch,coi}(G)$ is called a *$\gamma_{ch,coi}$-set* of G.

Let W be a $\gamma_{ch,coi}$-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique $\gamma_{ch,coi}$-set containing S. The *forcing connected co-independent hop domination number* of W is given by $f\gamma_{ch,coi}(W) = \min\{|S| : S \text{ is a forcing subset for } W\}$. The *forcing connected co-independent hop domination number* of G is given by

$$f\gamma_{ch,coi}(G) = \min\{f\gamma_{ch,coi}(W) : W \text{ is a } \gamma_{ch,coi}\text{-set of } G\}.$$

Let W be a *coi*-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique *coi*-set containing S. The *forcing co-independent number* of W is given by $fcoi(W) = \min\{|S| : S \text{ is a forcing subset for } W\}$. The *forcing co-independent number* of G is given by

$$fcoi(G) = \min\{fcoi(W) : W \text{ is a } coi\text{-set of } G\}.$$

Let G be a connected graph and let M be a β-set (maximum independent set) of G. A set $D \subseteq M^c$ is a forcing subset for the complement M^c of M if M is the unique β-set such that M^c contains D. The number $f\beta^c(M) = \min\{|D| : D$ is a forcing subset for $M^c\}$ is defined as the forcing complement of the independence number M. The forcing complement of the independence number of G is given by

$$f\beta^c(G) = \min\{f\beta^c(M) : M \text{ is a } \beta\text{-set of } G\}.$$

The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H.

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

2. Known Results

The following known results are taken from [11].

Remark 1. *Let G be a connected graph of order n. Then $1 \leq \gamma_{ch,coi}(G) \leq n$. Moreover, $\gamma_{ch,coi}(G) = 1$ if and only if $G = K_1$.*

Example 1. *The equations below give the connected co-independent hop domination number of the path P_n and cycle C_n.*

$$\gamma_{ch,coi}(P_n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, 3, \\ n-2 & \text{if } n \geq 4. \end{cases}$$

$$\gamma_{ch,coi}(C_n) = \begin{cases} 3 & \text{if } n = 3, \\ n-1 & \text{if } n \geq 4. \end{cases}$$

Remark 2. *If G is a complete graph, then $\gamma_{ch,coi}(G) = n$.*

Theorem 1. *Let G and H be nontrivial connected graphs with $|V(G)| = n$. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ of $V(G[H])$ is a connected co-independent hop dominating set if and only if*
(i) $S = V(G)$.
(ii) For every $x \in V(G)$ such that $T_x \neq V(H)$, $V(H) \backslash T_x$ is an independent set and $T_y = V(H)$ for every $y \in N_G(x)$ where T_x is a hop dominating set of H if $deg_G(x) = n-1$.

Corollary 1. *Let G be any connected noncomplete graph of order m and H be any nontrivial connected graph of order n. Then*

$$\gamma_{ch,coi}(G[H]) = m(n - \beta(H)) + r(G)\beta(H),$$

where $r(G) = min\{|D| : V(G)\backslash D$ is an independent set$\}$ and $\beta(H)$ is the independence number of H.

3. Forcing Connected Co-Independent Hop Domination Number of Some Special Graphs

Remark 3. Let G be a connected graph. Then

(i) $f\gamma_{ch,coi}(G) = 0$ if and only if G has a unique $\gamma_{ch,coi}$-set, and

(ii) $f\gamma_{ch,coi}(G) = 1$ if and only if G has at least two $\gamma_{ch,coi}$-sets, one of which, say B, contains an element which is not found in any $\gamma_{ch,coi}$-set of G.

Theorem 2. Let G be a connected graph. Then $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$ if and only if for all $\gamma_{ch,coi}$-set B of G and for each $v \in B$, there exists $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is a $\gamma_{ch,coi}$-set of G.

Proof: Suppose that $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$. Let B be a $\gamma_{ch,coi}$-set of G such that $f\gamma_{ch,coi}(G) = |B| = \gamma_{ch,coi}(G)$, that is, B is the only forcing subset for itself. Let $v \in B$. Since $B \setminus \{v\}$ is not a forcing subset for B, there exists a $u_v \in V(G) \setminus B$ such that $\big[B \setminus \{v\}\big] \cup \{u_v\}$ is a $\gamma_{ch,coi}$-set of G.

Conversely, suppose that every $\gamma_{ch,coi}$-set B' of G satisfies the given condition. Let B be a $\gamma_{ch,coi}$-set of G such that $f\gamma_{ch,coi}(G) = f\gamma_{ch,coi}(B)$. Suppose further that B has a forcing subset Q with $|Q| < |B|$, that is, $B = Q \cup P$ where $P = \{z \in B : z \notin Q\}$. Pick $z \in P$. By assumption, there exists $u_z \in V(G) \setminus B$ such that $\big[B \setminus \{z\}\big] \cup \{u_z\} = T$ is a $\gamma_{ch,coi}$-set of G. Thus, $T = Q \cup R$, where $R = \big[P \setminus \{z\}\big] \cup \{u_z\}$, is a $\gamma_{ch,coi}$-set containing Q, a contradiction. Hence, B is the only forcing subset for B. Therefore, $f\gamma_{ch,coi}(G) = \gamma_{ch,coi}(G)$. □

Proposition 1. For any complete graph K_n with $n \geq 1$ vertices, $f\gamma_{ch,coi}(K_n) = 0$.

Proof: By definition of K_n, $V(K_n)$ is the only $\gamma_{ch,coi}$-set of K_n. By Remark 3(i), $f\gamma_{ch,coi}(K_n) = 0$. □

Proposition 2. For any path P_n with $n \geq 1$ vertices,

$$f\gamma_{ch,coi}(P_n) = \begin{cases} 0, & \text{if } n \neq 3, \\ 1, & \text{if } n = 3. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. It can be seen that $f\gamma_{ch,coi}(P_1) = f\gamma_{ch,coi}(P_2) = 0$. Moreover, if $n = 4$, then P_4 has $\gamma_{ch,coi}$-set $B_1 = \{v_2, v_3\}$ which is the only $\gamma_{ch,coi}$-set of P_4. By Remark 3(i), $f\gamma_{ch,coi}(P_4) = 0$. Suppose that $n > 4$, then clearly $B_2 = \{v_2, v_3, v_4, \ldots, v_{n-1}\}$ is the only $\gamma_{ch,coi}$-set of P_n. Thus, by Remark 3(i),

$f\gamma_{ch,coi}(B_2) = 0 = f\gamma_{ch,coi}(P_n)$.

Suppose that $n = 3$. Then P_3 has $\gamma_{ch,coi}$-sets $B_3 = \{v_1, v_2\}$ and $B_4 = \{v_2, v_3\}$ which are the only $\gamma_{ch,coi}$-sets of P_3 with $v_1 \in B_3$ and $v_1 \notin B_4$. Hence, by Remark 3(ii), $f\gamma_{ch,coi}(B_3) = 1 = f\gamma_{ch,coi}(P_3)$. □

Proposition 3. For any cycle C_n with $n \geq 3$ vertices,

$$f\gamma_{ch,coi}(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ n-1, & \text{if } n \geq 4. \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Since $C_3 = K_3$, by Proposition 1, $f\gamma_{ch,coi}(C_3) = 0$. Suppose that $n \geq 4$. Then the $\gamma_{ch,coi}$-sets of C_n are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, $\ldots$, $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(C_n) \setminus B_j$ such that $\big[B_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a $\gamma_{ch,coi}$-set of G. Hence, by Theorem 2, $f\gamma_{ch,coi}(C_n) = n - 1$. □

4. Forcing Connected Co-Independent Hop Domination in the Edge Corona of Graphs

The following two results are taken from the Masteral's Thesis of Sandra A. Nanding. To verify the result, the proof is provided.

Theorem 3. Let G be a connected graph of order $n \geq 3$ and H be any graph. Then $C \subseteq V(G \diamond H)$ is a connected co-independent hop dominating set of $G \diamond H$ if and only if $C = A \cup (\bigcup_{uv \in E(G)} S_{uv})$ where

(i) $A \subseteq V(G)$ is a connected co-independent set of G containing all vertices incident to all the edges of G.

(ii) $S_{uv} = V(H^{uv})$ if $uv \in E(G)$ such that $u \in V(G)\backslash A$ or $v \in V(G)\backslash A$.

(iii) For every $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H^{ab})$, $V(H^{ab})\backslash S_{ab}$ is an independent set in H^{ab}.

Proof: Suppose that C is a connected co-independent hop dominating set of $G \diamond H$. Let $A = C \cap V(G)$ and let $S_{uv} = C \cap V(H^{uv})$ for each $uv \in E(G)$. Then $C = A \cup (\bigcup_{uv \in V(G)} S_{uv})$ where $A \subseteq V(G)$. First, we show that $\langle A \rangle$ is connected. Let $x, y \in A$ with $x \neq y$. If $xy \in E(G)$, then we are done. Suppose that $xy \notin E(G)$. Since $\langle C \rangle$ is connected and $x, y \in C$, there exists an x-y path $[x_1, x_2, ..., x_n]$ in $\langle C \rangle$ where $x = x_1$, $y = x_n$ and $n > 2$. If $x_i \in A$ for all $i \in \{1, 2, ..., n\}$, then the path $[x_1, x_2, ..., x_n]$ is in A. Suppose there exists $x_i \notin A$. Then $x_i \in S_{uv}$ for some edge $uv \in E(G)$. By definition of $G \diamond H$, $u, v \in A$. Hence, $[x_1, ..., u, v, ..., x_n]$ is a path in A, showing that $\langle A \rangle$ is connected. Next, let $u, v \in V(G)\backslash A$ with $u \neq v$. Then $u, v \in V(G \diamond H)\backslash C$. Since $V(G \diamond H)\backslash C$ is independent, $uv \notin E(G \diamond H)$. Since $u, v \in V(G)$, $uv \notin E(G)$ implying that $V(G)\backslash A$ is independent. Now, suppose v is a vertex incident to all the edges of G and $v \notin A$. Then $v \in N_G(w) \cap N_{G \diamond H}(p)$ for all $w \in V(G)$ and for all $p \in V(H^{vw})$. Thus, $N_{G \diamond H}(v, 2) \cap C = \varnothing$, a contradiction since C

is a hop dominating set. Hence, A is a connected co-independent set of G containing all vertices incident to all edges of G, showing that (i) holds.

Let $uv \in E(G)$ with $u \notin A$. Suppose $S_{uv} \neq V(H^{uv})$. Then there exists $x \in V(H^{uv})\backslash S_{uv}$. Hence, $x, u \in V(G \diamond H)\backslash C$ and $xu \in E(G \diamond H)$, a contradiction to the independence of $V(G \diamond H)\backslash C$. Thus, $S_{uv} = V(H^{uv})$ and (ii) holds.

Lastly, let $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H^{ab})$. Since $V(G \diamond H)\backslash C$ is independent and $(V(H^{ab})\backslash S_{ab}) \subseteq V(G \diamond H)\backslash C$, $V(H^{ab})\backslash S_{ab}$ is an independent set in H^{ab}. Hence, (iii) holds.

For the converse, suppose $C = A \cup (\bigcup_{uv \in E(G)} S_{uv})$ where $(i), (ii)$ and (iii) hold. First, we show that C is connected. Let $u, v \in C$ with $u \neq v$. If $uv \in E(G \diamond H)$, then we are done. So, suppose that $uv \notin E(G \diamond H)$. Consider the following cases.

Case 1. $u, v \in A$

By (i), $\langle A \rangle$ is connected. Hence, there exists a u-v path $P[u, v]$ in A. Since $A \subseteq C$, the path $P[u, v]$ is in C.

Case 2. $u \in A$ and $v \in S_{xy}$ for some $xy \in E(G)$

Since $uv \notin E(G \diamond H)$, $u \neq x$ and $u \neq y$. Since $V(G)\backslash A$ is independent by (i), $x \in A$ or $y \in A$, say $x \in A$. If $ux \in E(G)$, then the path $[u, x, v]$ is a u-v path in C. Suppose $ux \notin E(G)$. Since $\langle A \rangle$ is connected by (i) and $u, x \in A$, there exists u-x path $[y_1, y_2, ..., y_k]$ in A where $u = y_1$, $x = y_k$ and $k > 2$. Hence, the path $[y_1, y_2, ..., y_k, v]$ is a u-v path in C.

Case 3. $u, v \in S_{pq}$ for some edge $pq \in E(G)$.

Since $V(G)\backslash A$ is independent by (i), $p \in A$ or $q \in A$. Hence, the path $[u, p, v]$ or $[u, q, v]$ is in C.

In any case, $\langle C \rangle$ is connected.

Next, we show that $V(G \diamond H)\backslash C$ is independent. Let $p, q \in V(G \diamond H)\backslash C$ with $p \neq q$. Consider the following cases.

Case 1. $p \in V(G)\backslash A$ and $q \in V(G)\backslash A$

Since $V(G)\backslash A$ is independent by (i), $pq \notin E(G)$. Thus, $pq \notin E(G \diamond H)$.

Case 2. $p \in V(G)\backslash A$, $q \in V(H^{xy})\backslash S_{xy}$ for some $xy \in E(G)$

Since $S_{xy} \neq V(H^{xy})$, $x, y \in A$ by (ii). Hence, $p \neq x$ and $p \neq y$. By definition of $G \diamond H$, $pq \notin E(G \diamond H)$.

Case 3. $p \in V(H^{xy})\backslash S_{xy}$ and $q \in V(H^{rs})\backslash S_{rs}$ for some distinct edges $xy, rs \in E(G)$

Then, by definition of $G \diamond H$, $pq \notin E(G \diamond H)$.

Case 4. $p, q \in V(H^{zt})\backslash S_{zt}$ for some edge $zt \in E(G)$

Since $V(H^{zt})\backslash S_{zt}$ is independent by (iii), $pq \notin E(G \diamond H)$.

Therefore, in any case, $V(G \diamond H)\backslash C$ is an independent set in $G \diamond H$.

Lastly, we show that C is a hop dominating set of $G \diamond H$. Let $u \in V(G \diamond H)\backslash C$. Consider the following cases.

Case 1. $u \in V(G)\backslash A$

Let $deg_G(u) = 1$. Since $|V(G)| \geq 3$, there exists $vw \in E(G)$ with $u \in N_G(v)\backslash N_G(w)$ or $u \in N_G(w)\backslash N_G(v)$. If $w \in A$, then $w \in N_G(u, 2) \cap A$. If $w \notin A$, then $S_{vw} = V(H^{vw})$ by (ii). Thus, a vertex $p \in N_{G \diamond H}(u, 2) \cap S_{vw}$ exists. Hence, $p \in N_{G \diamond H}(u, 2) \cap C$.

Case 2. $u \in V(H^{xy})\backslash S_{xy}$ for some $xy \in E(G)$
By (ii), $x, y \in A$. Since $|V(G)| \geq 3$, there exist $z \in V(G) \cap N_G(x)$ or $z \in V(G) \cap N_G(y)$. If $z \in A$, then $z \in N_{G\diamond H}(u,2) \cap C$. If $z \notin A$, then $S_{yz} = V(H^{yz})$. Hence, a vertex $w \in N_{G\diamond H}(u,2) \cap S_{yz}$ or $w \in N_{G\diamond H}(u,2) \cap S_{xz}$.

Therefore, in any case, C is a hop dominating set of $G \diamond H$.

Accordingly, C is a connected co-independent hop dominating set of $G \diamond H$. □

Corollary 2. Let G be a connected graph of order $n \geq 3$ of size p and H be any graph of order m. Then $\gamma_{ch,coi}(G \diamond H) = n + p(m - \beta(H))$.

Proof: Let $C_o = A \cup (\bigcup_{uv \in V(G)} S_{uv})$ be a $\gamma_{ch,coi}$-set of $G \diamond H$. Then conditions $(i), (ii)$ and (iii) of Theorem 3 hold where $A = V(G)$ and $S_{uv} = V(H^{uv})\backslash S^*$ where S^* is any independent set of H^{uv}. Thus,

$$\begin{aligned}\gamma_{ch,coi}(G \diamond H) &= |A| + p|S_{uv}| \\ &= n + p(|V(H^{uv})| - |S^*|) \\ &\geq n + p(m - \beta(H)).\end{aligned}$$

Let T be a β-set of H and $S_{uv} = V(H^{uv})\backslash T$ for each $uv \in E(G)$. Then $C = V(G) \cup (\bigcup_{uv \in E(G)} S_{uv})$ is a connected co-independent hop dominating set of $G \diamond H$ by Theorem 3. Hence,

$$\begin{aligned}\gamma_{ch,coi}(G \diamond H) &\leq |C| \\ &= |V(G)| + p|S_{uv}| \\ &= n + p|V(H^{uv})\backslash T| \\ &= n + p(m - \beta(H)).\end{aligned}$$

Therefore, $\gamma_{ch,coi}(G \diamond H) = n + p(m - \beta(H))$. □

Remark 3 and Theorem 2 hold if the $\gamma_{ch,coi}$-set is replace by *coi*-set. Thus, we have the following Remark and Theorem.

Remark 4. Let G be a connected graph. Then

(i) $fcoi(G) = 0$ if and only if G has a unique *coi*-set, and

(ii) $fcoi(G) = 1$ if and only if G has at least two *coi*-sets, one of which, say D, contains an element which is not found in any *coi*-set of G.

Theorem 4. Let G be a connected graph. Then $fcoi(G) = coi(G)$ if and only if for all *coi*-set D of G and for each $z \in D$, there exists $u_z \in V(G) \setminus D$ such that $\left[D \setminus \{z\}\right] \cup \{u_z\}$ is a *coi*-set of G.

Proof: Suppose that $fcoi(G) = coi(G)$. Let D be a *coi*-set of G such that $fcoi(G) = |D| = coi(G)$, that is, D is the only forcing subset for itself. Let $z \in D$.

Since $D \setminus \{z\}$ is not a forcing subset for D, there exists a $u_z \in V(G) \setminus D$ such that $\big[D \setminus \{z\}\big] \cup \{u_z\}$ is a *coi*-set of G.

Conversely, suppose that every *coi*-set $D^{'}$ of G satisfies the given condition. Let D be a *coi*-set of G such that $fcoi(G) = fcoi(D)$. Suppose further that D has a forcing subset A with $|A| < |D|$, that is, $D = A \cup C$ where $C = \{v \in D : v \notin A\}$. Pick $v \in C$. By assumption, there exists $u_v \in V(G) \setminus D$ such that $\big[D \setminus \{v\}\big] \cup \{u_v\} = R$ is a *coi*-set of G. Thus, $R = A \cup T$, where $T = \big[C \setminus \{v\}\big] \cup \{u_v\}$, is a *coi*-set containing A, a contradiction. Hence, D is the only forcing subset for D. Therefore, $fcoi(G) = coi(G)$. □

Example 2. The formulas below give the co-independent number of the complete graph K_n, path P_n and cycle C_n.

$$coi(K_n) = \begin{cases} 1, & \text{if } n = 1, \\ n-1, & \text{if } n \geq 2. \end{cases}$$

$$coi(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ \dfrac{n}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

$$coi(C_n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \geq 4 \text{ and } n \text{ is even}, \\ \dfrac{n+1}{2}, & \text{if } n \geq 3 \text{ and } n \text{ is odd}. \end{cases}$$

Proposition 4. For any complete graph K_n with $n \geq 1$ vertices,

$$fcoi(K_n) = \begin{cases} 0, & \text{if } n = 1, \\ n-1, & \text{if } n \geq 2. \end{cases}$$

Proof: Suppose that $V(K_n) = \big\{v_1, v_2, \ldots, v_n\big\}$. It can be seen that $fcoi(K_1) = 0$. If $n = 2$, then K_2 has only two *coi*-sets $R_1 = \{v_1\}$ and $R_2 = \{v_2\}$ with $v_1 \in R_1$ and $v_1 \notin R_2$. By Remark 4(ii), $fcoi(K_2) = n - 1 = 1$.

Suppose that $n > 2$. Then the *coi*-sets of K_n are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, $\ldots$, $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(K_n) \setminus B_j$ such that $\big[B_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a *coi*-set of G. Hence, by Theorem 4, $fcoi(K_n) = n - 1$. □

Proposition 5. For any path P_n with $n \geq 1$ vertices,

$$fcoi(P_n) = \begin{cases} 0, & \text{if } n = 1, 3 \text{ and } n \geq 5 \text{ is odd}, \\ 1, & \text{if } n = 2, 4 \text{ and } n \geq 6 \text{ is even}. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. It can be verified that $fcoi(P_1) = fcoi(P_3) = 0$ and $fcoi(P_2) = 1$. If $n = 4$, then P_4 has *coi*-sets $B_1 = \{v_1, v_3\}$, $B_2 = \{v_2, v_4\}$ and $B_3 = \{v_2, v_3\}$ which are the only *coi*-sets of P_4 with $v_4 \in B_2$ and $v_4 \notin B_1, B_3$. Thus, by Remark 4(ii), $fcoi(P_4) = 1$.

Now, suppose that $n \geq 5$ and n is odd, then clearly $B = \{v_2, v_4, v_6, \ldots, v_{n-3}, v_{n-1}\}$ is the only *coi*-set of P_n. Thus, by Remark 4(i), $fcoi(B) = 0 = fcoi(P_n)$. Next, suppose that $n \geq 6$ and n is even. Then P_n has *coi*-sets $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $S_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ which are the only *coi*-set of P_n with $v_3 \in S_1$ and $v_3 \notin S_2$. Hence, by Remark 4(ii), $fcoi(P_n) = 1$. $\square$

Proposition 6. For any cycle C_n with $n \geq 3$ vertices,

$$fcoi(C_n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. If $n = 3$, then the *coi*-sets of C_3 are $Q_1 = \{v_1, v_2\}$, $Q_2 = \{v_2, v_3\}$ and $Q_3 = \{v_1, v_3\}$. Clearly, for each $v_i \in Q_j$ where $i, j \in \{1, 2, 3\}$, there exists $v_k \in V(C_3) \setminus Q_j$, where $k \in \{1, 2, 3\}$ and $i \neq k$ such that $\big[Q_j \setminus \{v_i\}\big] \cup \{v_k\}$ is a *coi*-set of G. Thus, by Theorem 4, $fcoi(C_3) = 2$. Now, suppose that n is even. Then $B_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $B_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ are the only *coi*-sets of C_n with $v_3 \in B_1$ and $v_3 \notin B_2$. Thus, by Remark 4(ii), $fcoi(B_1) = 1 = fcoi(C_n)$. Next, suppose that $n > 3$ and n is odd. Then

$$\begin{aligned} R_1 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}, \\ R_2 &= \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\}, \\ R_3 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_n\} \text{ and,} \\ R_4 &= \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\} \end{aligned}$$

are *coi*-sets of C_n. Hence, no vertex of C_n is contained in a unique *coi*-set. Thus, $fcoi(C_n) \geq 2$. Clearly, $\{v_1, v_n\}$ is uniquely contained in R_1. Therefore, $fcoi(R_1) = 2 = fcoi(C_n)$. $\square$

The next result is a restatement of Corollary 2.

Corollary 3. Let G be a connected graph of order $n \geq 3$ of size p and H be any graph. Then $C \subseteq V(G \diamond H)$ is a $\gamma_{ch,coi}$-set of $G \diamond H$ if and only if $C = V(G) \cup \Big(\bigcup_{uv \in E(G)} S_{uv}\Big)$ where S_{uv} is a co-independent set of H^{uv} for each $u, v \in V(G)$ such that $uv \in E(G)$, and $S_{uv} \neq V(H^{uv})$. In particular, $\gamma_{ch,coi}(G \diamond H) = |V(G)| + p\big(coi(H)\big)$.

Theorem 5. Let G be a nontrivial connected graph of order $n \geq 3$ of size p and H be any graph. Then

$$f\gamma_{ch,coi}(G \diamond H) = \begin{cases} 0, & \text{if } H \text{ has a unique } coi\text{-set,} \\ p\big[fcoi(H)\big], & \text{if } H \text{ has no unique } coi\text{-set.} \end{cases}$$

Proof: Suppose H has a unique *coi*-set. For each $uv \in E(G)$, where $u, v \in V(G)$, let $R_{uv} \subseteq V(H^{uv})$ be the unique *coi*-set of H^{uv}. By Corollary 3, $C = V(G) \cup \Big(\bigcup_{uv \in E(G)} R_{uv} \Big)$ is the unique $\gamma_{ch,coi}$-set of $G \diamond H$. Thus, by Remark 3(i), $f\gamma_{ch,coi}(G \diamond H) = 0$. On the other hand, suppose that H does not have a unique *coi*-set. For every $uv \in E(G)$, where $u, v \in V(G)$, let $W_{uv} \subseteq V(H^{uv})$ be a *coi*-set of H^{uv} with $fcoi(H^{uv}) = fcoi(W_{uv})$, and let $R_{W_{uv}} \subseteq W_{uv}$ be a forcing subset for W_{uv} with $fcoi(W_{uv}) = |R_{W_{uv}}|$. Then by Corollary 3, $S_W = V(G) \cup \Big(\bigcup_{uv \in E(G)} W_{uv} \Big)$ is a $\gamma_{ch,coi}$-set of $G \diamond H$. Let $D = \bigcup_{uv \in E(G)} R_{W_{uv}}$. Then D is a forcing subset for S_W. Thus,

$$f\gamma_{ch,coi}(G \diamond H) \leq f\gamma_{ch,coi}(S_W) \leq |D| = p\big[fcoi(H)\big].$$

Next, let C' be a $\gamma_{ch,coi}$-set of $G \diamond H$ such that $f\gamma_{ch,coi}(G \diamond H) = f\gamma_{ch,coi}(C')$. Then by Corollary 3, let $C' = V(G) \cup \Big(\bigcup_{uv \in E(G)} Q_{uv} \Big)$ where Q_{uv} is a *coi*-set of H^{uv} for each $uv \in E(G)$ for which $u, v \in V(G)$. Now, we let D' be a forcing subset for C' such that $f\gamma_{ch,coi}(C') = |D'|$. Suppose that there exists $ab \in E(G)$, for each $a, b \in V(G)$ such that $D' \cap Q_{ab} = D_{ab}$ is not a forcing subset for Q_{ab}. Let $Q^{'}_{ab}$ be a *coi*-set of H^{ab} with $Q^{'}_{ab} \neq Q_{ab}$. Then

$$C'' = V(G) \cup \Big(\bigcup_{uv \in E(G) \setminus \{ab\}} Q_{uv} \Big) \cup Q^{'}_{ab}$$

is a $\gamma_{ch,coi}$-set of $G \diamond H$ with $C' \neq C''$ and $D' \subseteq C''$, a contradiction. Thus, $C_{uv} = D' \cap Q_{uv}$ is a forcing subset for Q_{uv} for each $uv \in E(G)$ where $u, v \in V(G)$. Let $C_0 = \bigcup_{uv \in E(G)} C_{uv}$. Then

$$f\gamma_{ch,coi}(G \diamond H) = |D'| \geq |C_0| = \sum_{uv \in E(G)} |C_{uv}| \geq \sum_{uv \in E(G)} fcoi(H^{uv}) = |E(G)| fcoi(H).$$

Therefore, $f\gamma_{ch,coi}(G \diamond H) = p\big[fcoi(H)\big]$. □

Example 3. For complete graph K_3 and path P_5, $f\gamma_{ch,coi}(K_3 \diamond P_5) = 0$ since P_5 has a unique *coi*-set.

Example 4. For cycle C_3 and path P_2, $f\gamma_{ch,coi}(C_3 \diamond P_2) = 3\big[fcoi(P_2)\big] = 3 \cdot 1 = 3$ since P_2 has no unique *coi*-sets.

5. Forcing Connected Co-Independent Hop Domination in the Lexicographic Product of Graphs

The following is a restatement of Corollary 1.

Theorem 6. Let G and H be any nontrivial connected graphs of orders $m > 2$ and $n > 2$, respectively with $\gamma(H) \neq 1$. Then $\gamma_{ch,coi}(G[H]) = mn - \beta(G)\beta(H)$.

Proof: Suppose that M is a β-set of G and N is a β-set of H. Set $T_x = V(H) \setminus N$ if $x \in M$ and else set $T_x = V(H)$. Then in view of Theorem 1,

$$S = \bigcup_{x \in V(G)} \Big(\{x\} \times T_x \Big)$$

is a connected co-independent hop dominating set of $G[H]$. Hence,

$$\begin{aligned}
\gamma_{ch,coi}(G[H]) &\leq |S| \\
&= \left| \bigcup_{x \in M} \Big(\{x\} \times (V(H) \setminus N) \Big) \right| + \left| \bigcup_{x \in V(G) \setminus M} \Big(\{x\} \times V(H) \Big) \right| \\
&= |M|\big(|V(H)| - |N|\big) + \big(|V(G)| - |M|\big)|V(H)| \\
&= \beta(G)\big(n - \beta(H)\big) + \big(m - \beta(G)\big)n \\
&= \big(\beta(G)\big)n - \beta(G)\beta(H) + mn - \big(\beta(G)\big)n \\
&= mn - \beta(G)\beta(H).
\end{aligned}$$

Conversely, suppose that $S_0 = \bigcup_{x \in V(G)} \big(\{x\} \times R_x\big)$ is a $\gamma_{ch,coi}$-set of $G[H]$. Let $D = \big\{x \in V(G) : R_x = V(H)\big\}$. We claim that $V(G) \setminus D$ is an independent set of G. Suppose on the contrary. Then there exists vertices $x, y \in V(G) \setminus D$ such that $xy \in E(G)$ and pick $a, b \in V(H) \setminus R_x$ for which $ab \in E(H)$. This implies that $(x, a)(x, b), (y, a)(y, b) \in E(G[H])$, a contradiction to the fact that S_0 is a $\gamma_{ch,coi}$-set of $G[H]$. Thus, $V(G) \setminus D$ is an independent set of G. Hence, $|V(G) \setminus D| \leq \beta(G)$ and so

$$(1) \qquad |D| \geq m - \beta(G).$$

On the other hand, by condition (ii) in Theorem 1, we have $V(H) \setminus R_x$ is an independent set of H. Thus, $|V(H) \setminus R_x| \leq \beta(H)$ and so

$$(2) \qquad |R_x| \geq n - \beta(H).$$

Therefore, inequalities (1) and (2) imply

$$\begin{aligned}
\gamma_{ch,coi}(G[H]) &= |S_0| \\
&= \left| \bigcup_{x \in D} \Big(\{x\} \times V(H) \Big) \right| + \left| \bigcup_{x \in V(G) \setminus D} \Big(\{x\} \times R_x \Big) \right| \\
&= |D||V(H)| + \big(|V(G)| - |D|\big)|R_x| \\
&\geq |D|n + \big(m - |D|\big)\big(n - \beta(H)\big) \\
&= |D|n + mn - m\beta(H) - |D|n + |D|\beta(H) \\
&\geq mn - m\beta(H) + \big(m - \beta(G)\big)\beta(H) \\
&= mn - m\beta(H) + m\beta(H) - \beta(G)\beta(H)
\end{aligned}$$

$$= mn - \beta(G)\beta(H).$$

Therefore, $\gamma_{ch,coi}(G[H]) = mn - \beta(G)\beta(H)$. □

As a consequence of Theorem 6, the next result follows.

Corollary 4. Let G and H be nontrivial connected graphs with orders $m > 2$ and $n > 2$, respectively, and $\gamma(H) \neq 1$. Then $S \subseteq V(G[H])$ is a $\gamma_{ch,coi}$-set of $G[H]$ if and only if

$$S = \bigcup_{x \in A} \big(\{x\} \times T_x\big) \cup \big[\big(V(G) \setminus A\big) \times V(H)\big]$$

for some β-set A of G such that $V(H) \setminus T_x$ is a β-set of H for all $x \in A$.

Theorem 7. Let G and H be connected graphs of orders $m > 2$ and $n > 2$, respectively, where H has a unique β-set and $\gamma(H) \neq 1$. Then

$$f\gamma_{ch,coi}(G[H]) = \begin{cases} 0, & \text{if } G \text{ has a unique } \beta\text{-set}, \\ f\beta^c(G), & \text{if } G \text{ has no unique } \beta\text{-set}. \end{cases}$$

Proof: By Corollary 4, $S = \big[A \times B\big] \cup \big[A^c \times V(H)\big]$ is a $\gamma_{ch,coi}$-set of $G[H]$ for some β-set A of G such that $V(H) \setminus B$ is a β-set of H. Suppose that H has unique β-set, say R. If G has a unique β-set M, then $G[H]$ has a unique $\gamma_{ch,coi}$-set

$$\big[M \times R\big] \cup \big[M^c \times V(H)\big].$$

By Remark 3(i), $f\gamma_{ch,coi}(G[H]) = 0$. Now, suppose that G has no unique β-set. Let A be a β-set of G and let D_{A^c} be a forcing subset for the complement A^c of A such that $f\beta^c(G) = f\beta^c(A) = |D_{A^c}|$. By Corollary 4,

$$S = \big[A \times R\big] \cup \big[A^c \times V(H)\big]$$

is a $\gamma_{ch,coi}$-set of $G[H]$. We claim that $D_{A^c} \times \{a\}$ is a forcing subset for S for every $a \in V(H) \setminus R$. Suppose on the contrary. Then there exists a $\gamma_{ch,coi}$-set S' of $G[H]$ with $S' \neq S$ such that $D_{A^c} \times \{a\} \subseteq S'$. Then $S' = \big[A' \times R\big] \cup \big[(A')^c \times V(H)\big]$ for some β-set A' of G. Since $S' \neq S$, $A' \neq A$ and $D_{A^c} \times \{a\} \subseteq (A')^c \times V(H)$. Thus, $D_{A^c} \subseteq (A')^c$ a contradiction since D_{A^c} is a forcing subset for A^c. Hence, $D_{A^c} \times \{a\}$ is a forcing subset for S. Thus,

$$f\gamma_{ch,coi}(G[H]) \leq f\gamma_{ch,coi}(S) \leq |D_{A^c} \times \{a\}| = |D_{A^c}| = f\beta^c(G).$$

Let $S_0 = \big[A_0 \times R\big] \cup \big[A_0^c \times V(H)\big]$ be a $\gamma_{ch,coi}$-set of $G[H]$ such that $f\gamma_{ch,coi}(G[H]) = f\gamma_{ch,coi}(S_0)$. By Corollary 4, A_0 is β-set of G. Let F_0 be a forcing subset for S_0 with $f\gamma_{ch,coi}(S_0) = |F_0|$. Let $F_0 = \bigcup_{x \in K} \big(\{x\} \times T_x\big)$. We claim that $K \subseteq A_0^c$

and $T_x \subseteq V(H)$. Let $K = K_1 \cup K_2$ where $K_1 \cap K_2 = \varnothing$. Suppose that $K \subseteq A_0$ or $K_1 \subseteq A_0$ and $K_2 \subseteq A_0^c$. Let B_0 be a β-set of G with $B_0 \neq A_0$. Consider the $\gamma_{ch,coi}$-set

$$S_0' = \big[B_0 \times R\big] \cup \big[B_0^c \times V(H)\big].$$

Then $S_0' \neq S_0$. If $A_0 \cap B_0 = \varnothing$, then $K \subseteq B_0^c$ and

$$F_0 \subseteq K \times R \subseteq B_0^c \times V(H) \subseteq S_0'$$

or

$$\begin{aligned} F_0 &\subseteq \big[K_1 \times R\big] \cup \big[K_2 \times V(H)\big] \\ &\subseteq (K_1 \cup K_2) \times V(H) \\ &\subseteq B_0^c \times V(H) \\ &\subseteq S_0'. \end{aligned}$$

On the other hand, if $A_0 \cap B_0 \neq \varnothing$ and $K_1 \subseteq B_0$ and $K_2 \subseteq B_0^c$, then

$$\begin{aligned} F_0 &\subseteq \big[K \times R\big] \\ &\subseteq \big[K_1 \times R\big] \cup \big[K_2 \times V(H)\big] \\ &\subseteq \big[B_0 \times R\big] \cup \big[B_0^c \times V(H)\big] \\ &= S_0'. \end{aligned}$$

In either case, we have a contradiction since F_0 is a forcing subset for S_0. Hence, $K \subseteq A_0^c$ and $T_x \subseteq V(H)$. This implies that K is a forcing subset for A_0^c. Choose any $x \in K$ and $a \in T_x$. Then $F_a = K \times \{a\} \subseteq F_0$. Thus,

$$f\gamma_{ch,coi}(G[H]) = f\gamma_{ch,coi}(S_0) = |F_0| \geq |F_a| = |K| \geq f\beta^c(A_0) = f\beta^c(G).$$

Therefore, $f\gamma_{ch,coi}(G[H]) = f\beta^c(G)$. □

Example 5. For paths P_3 and P_5, $f\gamma_{ch,coi}(P_3[P_5]) = 0$ since P_5 has a unique β-set.

Example 6. For paths P_4 and P_3, $f\gamma_{ch,coi}(P_4[P_3]) = f\beta^c(P_4) = 2$ since P_4 has no unique β-sets.

Theorem 8. Let G and H be connected graphs of orders $m > 2$ and $n > 2$, respectively, where G and H have no unique β-sets and $\gamma(H) \neq 1$. Then

$$f\gamma_{ch,coi}(G[H]) \leq f\beta^c(H) + f\beta^c(G)\big[f\beta^c(H) + 1\big].$$

Proof: Suppose that G and H do not have unique β-sets. Let A be a β-set of G and F_{A^c} be a forcing subset for the complement A^c of A such that $f\beta^c(G) = f\beta^c(A) = |F_{A^c}|$. Then for all $x \in A$, let $B = V(H) \setminus T_x$ be a β-set of H and F_{B^c} be a forcing subset for the complement B^c of B such that $f\beta^c(H) = f\beta^c(B) = |F_{B^c}|$. By Corollary 4,

$$S = \bigcup_{x \in A} \big(\{x\} \times T_x\big) \cup \big[A^c \times V(H)\big]$$

is a $\gamma_{ch,coi}$-set of $G[H]$. We claim that $S_0 = \big(\{x\} \times F_{B^c}\big) \cup \big(F_{A^c} \times [F_{B^c} \cup \{y\}]\big)$ for each $x \in A$ and $y \in B$ is a forcing subset for S. Suppose on the contrary. Then there exists a $\gamma_{ch.coi}$-set $S^{'} \neq S$ such that $S_0 \subseteq S^{'}$. By Corollary 4,

$$S^{'} = \bigcup_{x \in A^{'}} \big(\{x\} \times T_x^{'}\big) \cup \big[(A^{'})^c \times V(H)\big]$$

for some β-set $A^{'}$ of G such that $B^{'} = V(H) \setminus T_x^{'}$ is a β-set of H for all $x \in A^{'}$. Since $S^{'} \neq S$, either one of the following holds:

(i) $A^{'} = A$ and $B^{'} \neq B$
(ii) $A^{'} \neq A$ and $B^{'} = B$
(iii) $A^{'} \neq A$ and $B^{'} \neq B$.

Suppose (i) holds. Then $S^{'} = \bigcup_{x \in A} \big(\{x\} \times T_x^{'}\big) \cup \big[A^c \times V(H)\big]$. Since $S_0 \subseteq S^{'}$, $F_{B^c} \subseteq T_x^{'} = V(H) \setminus B^{'} = (B^{'})^c$, a contradiction since F_{B^c} is a forcing subset for B^c. If (ii) holds, then $F_{A^c} \subseteq (A^{'})^c$. This is again a contradiction since F_{A^c} is a forcing subset for A^c. If (iii) holds, then $F_{B^c} \subseteq (B^{'})^c$ and $F_{A^c} \subseteq (A^{'})^c$, a contradiction. Therefore, S_0 is a forcing subset for S. Thus,

$$f\gamma_{ch,coi}(G[H]) \leq f\gamma_{ch,coi}(S) = |S_0| = f\beta^c(H) + f\beta^c(G)\big[f\beta^c(H) + 1\big]. \quad \square$$

Example 7. For cycle C_4 and path P_4,

$$f\gamma_{ch,coi}(C_4[P_4]) \leq f\beta^c(P_4) + f\beta^c(C_4)\big[f\beta^c(P_4) + 1\big] = 2 + 2[2+1] = 8$$

since both C_4 and P_4 have no unique β-sets.

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