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Positive implicative makgeolli ideals of BCK-algebras

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Abstract. The concept of a positive implicative makgeolli ideal in BCK-algebras is introduced, and its properties are investigated. The relationship between a makgeolli ideal and a positive implicative makgeolli ideal is established. The conditions under which a makgeolli ideal can be a positive implicative makgeolli ideal are explored. Characterizations of a positive implicative makgeolli ideal are discussed, and the extension property for a positive implicative makgeolli ideal is established.

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Key Words and Phrases: BCK-soft universe, makgeolli structure, makgeolli ideal, positive implicative makgeolli ideal.

1. Introduction

Many problems that need to be solved in the real world often involve inherently uncertain, inaccurate, and ambiguous factors. Zadeh [24] pointed out Various problems in system identification involve characteristics which are essentially non-probabilistic in nature, and he introduced fuzzy set theory as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [25]. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of interval mathematics, theory of vague sets, and theory of rough sets. But, Molodtsov [21] pointed out all of these theories have their own difficulties. Maji et al. [18] and Molodtsov [21] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [21] introduced the concept

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of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches, and he pointed out several directions for the applications of soft sets. Globally, interest in soft set theory and its application has been growing rapidly in recent years. Soft set theory has been applied to algebraic structures, for example, groups, rings, fields and modules (see [1, 3–5, 14]), and BCK/BCI-algebras etc. (see [9–13, 15–17, 22, 23]). In 2019, Ahn et al. [2] introduced the notion of makgeolli structures as a hybrid structure based on fuzzy set and soft set theory, and applied it to BCK/BCI-algebras.

In this paper, we introduce the notion of a positive implicative makgeolli ideal in BCK-algebras, and investigate its properties. We establish the relationship between a makgeolli ideal and a positive implicative makgeolli ideal. We explore the conditions under which a makgeolli ideal can be a positive implicative makgeolli ideal. We discusse the characterization of positive implicative makgeolli ideal, and construct the extension property for a positive implicative makgeolli ideal.

2. Preliminaries

2.1. Preliminaries on BCK-algebras

BCI/BCK-algebra is an important type of logical algebra introduced by K. Iséki (see [7] and [8]), and it has been extensively investigated by several researchers. See the books [6, 20] for further information regarding BCI-algebras and BCK-algebras.

In this section, we recall the definitions and basic results required in this paper.

Let X be a set with a special element "0" and a binary operation " $*$ ". If it satisfies the following conditions:

(I1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,

(I2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,

(I3) $(\forall a \in X)\ (a * a = 0)$,

(I4) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

(K) $(\forall a \in X)\ (0 * a = 0)$,

then it is called a *BCK-algebra*, and it is denoted by $(X, *)_0$.

The order relation " $\leq$ " in a BCK-algebra $(X, *)_0$ is defined as follows:

$$(\forall a, b \in X)(a \leq b\ \Leftrightarrow\ a * b = 0). \tag{1}$$

Every BCK-algebra $(X, *)_0$ satisfies the following conditions (see [19, 20]):

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2}$$

$$(\forall a, b, c \in X)\,(a \leq b \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a)\,, \tag{3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{4}$$

A BCK-algebra $(X, *)_0$ is said to be *positive implicative* (see [20]) if $(a * c) * (b * c) = (a * b) * c$ for all $a, b, c \in X$.

A subset C of a BCK-algebra $(X, *)_0$ is called an *ideal* of $(X, *)_0$ (see [6, 20]) if it satisfies:

$$0 \in C, \tag{5}$$

$$(\forall a, b \in X)(a * b \in C,\, b \in C \;\Rightarrow\; a \in C). \tag{6}$$

A subset C of a BCK-algebra $(X, *)_0$ is called a *positive implicative ideal* of $(X, *)_0$ (see [20]) if it satisfies (5) and

$$(\forall a, b, c \in X)((a * b) * c \in C,\, b * c \in C \;\Rightarrow\; a * c \in C). \tag{7}$$

2.2. Preliminaries on makgeolli structures

Let X be a universal set and E a set of parameters. We say that the pair (X, E) is a *soft universe.*

Definition 1 ([2])**.** *Let (X, E) be a soft universe and let C and D be subsets of E. A makgeolli structure on X (related to C and D) is a structure of the form:*

$$\mathcal{M}_{(C,D,X)} := \{\langle (a, b, x); M_C(a), G_D(b), \xi(x)\rangle \mid (a, b, x) \in C \times D \times X\} \tag{8}$$

where $M_C := (M, C)$ and $G_D := (G, D)$ are soft sets over X and ξ is a fuzzy set in X.

For the sake of simplicity, the makgeolli structure in (8) will be denoted by $\mathcal{M}_{(C,D,X)} = (M_C, G_D, \xi)$. The makgeolli structure $\mathcal{M}_{(C,C,X)} = (M_C, G_C, \xi)$ on X related to a subset C of E is simply denoted by $\mathcal{M}_{(C,X)} = (M_C, G_C, \xi)$. If $C = D = E$, we use the notation $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ as the makgeolli structure of (X, E).

By a *BCK/BCI-soft universe*, we mean a soft universe (X, E) in which X and E are BCK/BCI-algebras with binary operations "$*$" and "$\looparrowright$", respectively.

Definition 2 ([2])**.** *Let (X, E) be a BCK/BCI-soft universe. A makgeolli structure $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is called a makgeolli ideal of (X, E) if it satisfies:*

$$\begin{cases} (\forall a \in E)\,(M_E(0) \supseteq M_E(a),\;\; G_E(0) \subseteq G_E(a))\,. \\ (\forall x \in X)\,(0/\xi(x) \in \xi)\,. \end{cases} \tag{9}$$

$$\begin{cases} (\forall a, b \in E)\begin{pmatrix} M_E(a) \supseteq M_E(a \looparrowright b) \cap M_E(b) \\ G_E(a) \subseteq G_E(a \looparrowright b) \cup G_E(b) \end{pmatrix}. \\ (\forall x, y, z \in X)(\forall t, r \in (0, 1])\begin{pmatrix} \{x * y\}/t \in \xi,\;\; y/r \in \xi \\ \Rightarrow\;\; x/\min\{t, r\} \in \xi \end{pmatrix}. \end{cases} \tag{10}$$

Lemma 1 ([2])**.** *Let (X, E) be a BCK/BCI-soft universe. Every makgeolli ideal $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ of (X, E) satisfies the following assertions.*

(i) $\left\{\begin{array}{l}(\forall a,b\in E)\left(a\leq b\ \Rightarrow\left\{\begin{array}{l}M_E(a)\supseteq M_E(b)\\ G_E(a)\subseteq G_E(b)\end{array}\right.\right).\\ (\forall x,y\in X)\,(x\leq y\ \Rightarrow\ \xi(x)\geq\xi(y*z)).\end{array}\right.$

(ii) $\left\{\begin{array}{l}(\forall a,b,c\in E)\left(a\looparrowright b\leq c\ \Rightarrow\left\{\begin{array}{l}M_E(a)\supseteq M_E(b)\cap M_E(c)\\ G_E(a)\subseteq G_E(b)\cup G_E(c)\end{array}\right.\right).\\ (\forall x,y,z\in X)\,(x*y\leq z\ \Rightarrow\ \xi(x)\geq\min\{\xi(y*z),\xi(z)\}).\end{array}\right.$

Let (X,E) be a BCK/BCI-soft universe. Given a makgeolli structure $\mathcal{M}_{(X,E)}:=(M_E, G_E, \xi)$ on (X,E), consider the following sets:

$$\begin{aligned}&\mathcal{E}(M_E;\alpha):=\{a\in E\mid M_E(a)\supseteq\alpha\},\\ &\mathcal{E}(G_E;\beta):=\{b\in E\mid G_E(b)\subseteq\beta\},\\ &\mathcal{X}(\xi;t):=\{x\in X\mid \xi(x)\geq t\}\end{aligned}$$

where α and β are subsets of X and $t\in[0,1]$.

3. Positive implicative makgeolli ideals

In what follows, let (X,E) be a BCK-soft universe unless otherwise specified.

Definition 3. *A makgeolli structure $\mathcal{M}_{(X,E)}:=(M_E, G_E, \xi)$ is called a positive implicative makgeolli ideal of (X,E) if it satisfies (9) and*

$$(\forall a,b,c\in E)\left(\begin{array}{l}M_E(a\looparrowright c)\supseteq M_E((a\looparrowright b)\looparrowright c)\cap M_E(b\looparrowright c)\\ G_E(a\looparrowright c)\subseteq G_E((a\looparrowright b)\looparrowright c)\cup G_E(b\looparrowright c)\end{array}\right). \tag{11}$$

$$(\forall x,y,z\in X)(\forall t,r\in(0,1])\left(\begin{array}{c}\{(x*y)*z\}/t\in\xi,\ \{y*z\}/r\in\xi\\ \Rightarrow\ \{x*z\}/\min\{t,r\}\in\xi\end{array}\right). \tag{12}$$

Note that the condition (12) is equivalent to the following assertion.

$$(\forall x,y,z\in X)\,(\xi(x*z)\geq\min\{\xi((x*y)*z),\xi(y*z)\}). \tag{13}$$

In fact, suppose that the condition (12) is valid. Since $\{(x*y)*z\}/\xi((x*y)*z)\in\xi$ and $\{y*z\}/\xi(y*z)\in\xi$, it follows from (12) that $\{x*z\}/\min\{\xi((x*y)*z),\xi(y*z)\}\in\xi$. Hence $\xi(x*z)\geq\min\{\xi((x*y)*z),\xi(y*z)\}$. Conversely, assume that the condition (13) is valid. Let $x,y,z\in X$ and $t,r\in(0,1]$ be such that $\{(x*y)*z\}/t\in\xi$ and $\{y*z\}/r\in\xi$. Then $\xi((x*y)*z)\geq t$ and $\xi(y*z)\geq r$. It follows from (13) that $\xi(x*z)\geq\min\{\xi((x*y)*z),\xi(y*z)\}\geq\min\{t,r\}$, that is, $\{x*z\}/\min\{t,r\}\in\xi$.

Example 1. *Consider a BCK-soft universe (X,E) in which $X:=\{0,1,2,3,4\}$ and $E:=\{0,1,2,3\}$ with binary operations "$*$" and "$\looparrowright$", respectively, given by Table 1.*
Let $\mathcal{M}_{(X,E)}:=(M_E, G_E, \xi)$ be a makgeolli structure on (X,E) defined as follows:

$$(M_E,G_E):E\to\mathcal{P}(X)\times\mathcal{P}(X),\ x\mapsto\left\{\begin{array}{ll}(X,\{2\}) & \text{if } x=0,\\ (\{0,1,3,4\},\{0,2\}) & \text{if } x=1,\\ (\{1,3,4\},X) & \text{if } x=2,\\ (\{1,4\},\{0,2,3\}) & \text{if } x=3,\end{array}\right.$$

Table 1: Cayley table for the binary operations "$\looparrowright$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$\looparrowright$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	3	3	0

$$\xi : X \to [0,1],\ y \mapsto \begin{cases} 0.78 & \text{if } y = 0, \\ 0.63 & \text{if } y = 1, \\ 0.51 & \text{if } y = 2, \\ 0.44 & \text{if } y = 3, \\ 0.59 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E).

Proposition 1. *Every positive implicative makgeolli ideal $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ of (X, E) satisfies:*

$$\begin{cases} (\forall a, b \in E) \begin{pmatrix} M_E(a \looparrowright b) \supseteq M_E((a \looparrowright b) \looparrowright b) \\ G_E(a \looparrowright b) \subseteq G_E((a \looparrowright b) \looparrowright b) \end{pmatrix}, \\ (\forall x, y \in X)\, (\xi((x * y) \geq \xi((x * y) * y))\,. \end{cases} \tag{14}$$

Proof. If we replace c and z with b and y in (11) and (12), respectively, and use (I3) and (9), then we have (14).

We discuss the relationship between positive implicative makgeolli ideal and makgeolli ideal.

Theorem 1. *Every positive implicative makgeolli ideal is a makgeolli ideal.*

Proof. Let $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ be a positive implicative makgeolli ideal of (X, E). If we replace c and z with 0 in (11) and (12), then we can get (10). Hence $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a makgeolli ideal of (X, E).

The following example shows that the converse of Theorem 1 may not be true.

Example 2. *Consider a BCK-soft universe (X, E) in which $X := \{0, 1, 2, 3, 4\}$ and $E := \{0, 1, 2, 3\}$ with binary operations "$*$" and "$\looparrowright$", respectively, given by Table 2.*
Let $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ be a makgeolli structure on (X, E) defined as follows:

$$(M_E, G_E) : E \to \mathcal{P}(X) \times \mathcal{P}(X),\ x \mapsto \begin{cases} (X, \{2, 4\}) & \text{if } x = 0, \\ (\{1, 3\}, \{0, 1, 2, 4\}) & \text{if } x = 1, \\ (\{1, 3\}, \{0, 1, 2, 4\}) & \text{if } x = 2, \\ (\{0, 1, 3, 4\}, \{0, 2, 4\}) & \text{if } x = 3, \end{cases}$$

Table 2: Cayley table for the binary operations "$\looparrowright$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

$\looparrowright$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

$$\xi : X \to [0,1],\ y \mapsto \begin{cases} 0.83 & \text{if } y = 0, \\ 0.52 & \text{if } y = 1, \\ 0.77 & \text{if } y = 2, \\ 0.52 & \text{if } y = 3, \\ 0.69 & \text{if } y = 4. \end{cases}$$

It is routine to check that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ *is a makgeolli ideal of* (X,E). *But it is not a positive implicative makgeolli ideal of* (X,E) *since* $G_E(2 \looparrowright 1) = G_E(1) = \{0,1,2,4\} \nsubseteq \{2,4\} = G_E((2 \looparrowright 1) \looparrowright 1) \cup G_E(1 \looparrowright 1)$ *or* $\{(3*1)*1\}/0.76 = 0/0.76 \in \xi$ *and* $\{1*1\}/0.79 = 0/0.79 \in \xi$, *but* $\{3*1\}/\min\{0.76, 0.79\} = 1/0.76 \notin \xi$.

We explore the conditions under which the converse of Theorem 1 can be established.

Theorem 2. *In a BCK-soft universe* (X,E) *in which* X *and* E *are positive implicative BCK-algebras, every makgeolli ideal is a positive implicative makgeolli ideal.*

Proof. Straightforward.

Theorem 3. *If a makgeolli ideal* $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ *of* (X,E) *satisfies the condition* (14), *then it is a positive implicative makgeolli ideal of* (X,E).

Proof. Let $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ be a makgeolli ideal of (X,E) that satisfies the condition (14). The combination of (I1) and (4) derive

$$((a \looparrowright c) \looparrowright c) \looparrowright (b \looparrowright c) \leq (a \looparrowright c) \looparrowright b = (a \looparrowright b) \looparrowright c$$

and $((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$ for all $a,b,c \in E$ and $x,y,z \in X$, and so

$$\begin{aligned} &M_E(((a \looparrowright c) \looparrowright c) \looparrowright (b \looparrowright c)) \supseteq M_E((a \looparrowright b) \looparrowright c), \\ &G_E(((a \looparrowright c) \looparrowright c) \looparrowright (b \looparrowright c)) \subseteq G_E((a \looparrowright b) \looparrowright c), \\ &\xi(((x*z)*z)*(y*z)) \geq \xi((x*y)*z) \end{aligned}$$

by Lemma 1(i). It follows from (10) and (14) that

$$\begin{aligned} M_E(a \looparrowright c) &\supseteq M_E((a \looparrowright c) \looparrowright c) \\ &\supseteq M_E(((a \looparrowright c) \looparrowright c) \looparrowright (b \looparrowright c)) \cap M_E(b \looparrowright c) \end{aligned}$$

$$\supseteq M_E((a \looparrowright b) \looparrowright c) \cap M_E(b \looparrowright c),$$

$$\begin{aligned} G_E(a \looparrowright c) &\subseteq G_E((a \looparrowright c) \looparrowright c) \\ &\subseteq G_E(((a \looparrowright c) \looparrowright c) \looparrowright (b \looparrowright c)) \cup G_E(b \looparrowright c) \\ &\subseteq G_E((a \looparrowright b) \looparrowright c) \cup G_E(b \looparrowright c), \end{aligned}$$

and

$$\begin{aligned} \xi(x * z) &\geq \xi((x * z) * z) \\ &\geq \min\{\xi(((x * z) * z) * (y * z)), \xi(y * z)\} \\ &\geq \min\{\xi((x * y) * z), \xi(y * z)\}. \end{aligned}$$

Hence $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E).

Theorem 4. *A makgeolli structure $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ on (X, E) is a positive implicative makgeolli ideal of (X, E) if and only if it is a makgeolli ideal of (X, E) that satisfies the following condition.*

$$\begin{cases} (\forall a, b, c \in E) \left(\begin{array}{l} M_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \supseteq M_E((a \looparrowright b) \looparrowright c) \\ G_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \subseteq G_E((a \looparrowright b) \looparrowright c) \end{array} \right), \\ (\forall x, y, z \in X)\, (\xi((x * z) * (y * z)) \geq \xi(((x * y) * z))\,. \end{cases} \tag{15}$$

Proof. Let $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ be a makgeolli ideal of (X, E) that satisfies (15). If we put $b := c$ and $y := z$ in (15), then

$$\begin{aligned} &M_E(a \looparrowright c) = M_E((a \looparrowright c) \looparrowright 0) = M_E((a \looparrowright c) \looparrowright (c \looparrowright c)) \supseteq M_E((a \looparrowright c) \looparrowright c), \\ &G_E(a \looparrowright c) = G_E((a \looparrowright c) \looparrowright 0) = G_E((a \looparrowright c) \looparrowright (c \looparrowright c)) \subseteq G_E((a \looparrowright c) \looparrowright c), \\ &\xi(x * z) = \xi((x * z) * 0) = \xi((x * z) * (z * z)) \geq \xi((x * z) * z) \end{aligned}$$

for all $a, c \in E$ and $x, z \in X$ by (I3) and (2). It follows from Theorem 3 that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E).

Conversely, assume that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E). Then it is a makgeolli ideal of (X, E) by Theorem 1. Since

$$((a \looparrowright (b \looparrowright c)) \looparrowright c) \looparrowright c = ((a \looparrowright c) \looparrowright (b \looparrowright c)) \looparrowright c \leq (a \looparrowright b) \looparrowright c$$

and $((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z$ for all $a, b, c \in E$ and $x, y, z \in X$, we have

$$\begin{aligned} &M_E((a \looparrowright c) \looparrowright (b \looparrowright c)) = M_E((a \looparrowright (b \looparrowright c)) \looparrowright c) \\ &\supseteq M_E(((a \looparrowright (b \looparrowright c)) \looparrowright c) \looparrowright c) \supseteq M_E((a \looparrowright b) \looparrowright c), \end{aligned}$$

$$G_E((a \looparrowright c) \looparrowright (b \looparrowright c)) = G_E((a \looparrowright (b \looparrowright c)) \looparrowright c)$$

$$\subseteq G_E(((a \looparrowright (b \looparrowright c)) \looparrowright c) \looparrowright c) \subseteq G_E((a \looparrowright b) \looparrowright c),$$

and

$$\begin{aligned} \xi((x * z) * (y * z)) &= \xi((x * (y * z)) * z) \\ &\geq \xi(((x * (y * z)) * z) * z) \geq \xi((x * y) * z) \end{aligned}$$

by (4), Lemma 1(i) and Proposition 1. Hence (15) is valid.

Theorem 5. *A makgeolli structure* $\mathcal{M}_{(X,E)} := (M_E,\ G_E,\ \xi)$ *on* (X, E) *is a positive implicative makgeolli ideal of* (X, E) *if and only if it satisfies* (9) *and*

$$\begin{cases} (\forall a, b, c \in E) \left(\begin{array}{l} M_E(a \looparrowright b) \supseteq M_E(((a \looparrowright b) \looparrowright b) \looparrowright c) \cap M_E(c) \\ G_E(a \looparrowright b) \subseteq G_E(((a \looparrowright b) \looparrowright b) \looparrowright c) \cup G_E(c) \end{array} \right), \\ (\forall x, y, z \in X)\, (\xi(x * y) \geq \min\{\xi(((x * y) * y) * z), \xi(z)\}) \,. \end{cases} \tag{16}$$

Proof. Let $\mathcal{M}_{(X,E)} := (M_E,\ G_E,\ \xi)$ be a positive implicative makgeolli ideal of (X, E). Then it is a makgeolli ideal of (X, E) by Theorem 1, and so the condition (9) is valid. Using (I3), (2), (4), (10) and (15), we get

$$\begin{aligned} M_E(a \looparrowright b) &\supseteq M_E((a \looparrowright b) \looparrowright c) \cap M_E(c) = M_E(((a \looparrowright c) \looparrowright b) \looparrowright (b \looparrowright b)) \cap M_E(c) \\ &\supseteq M_E(((a \looparrowright c) \looparrowright b) \looparrowright b) \cap M_E(c) = M_E(((a \looparrowright b) \looparrowright b) \looparrowright c) \cap M_E(c), \end{aligned}$$

$$\begin{aligned} G_E(a \looparrowright b) &\subseteq G_E((a \looparrowright b) \looparrowright c) \cup M_E(c) = G_E(((a \looparrowright c) \looparrowright b) \looparrowright (b \looparrowright b)) \cup G_E(c) \\ &\subseteq G_E(((a \looparrowright c) \looparrowright b) \looparrowright b) \cup G_E(c) = G_E(((a \looparrowright b) \looparrowright b) \looparrowright c) \cup G_E(c) \end{aligned}$$

and

$$\begin{aligned} \xi(x * y) &\geq \min\{\xi((x * y) * z), \xi(z)\} = \min\{\xi(((x * z) * y) * (y * y)), \xi(z)\} \\ &\geq \min\{\xi(((x * z) * y) * y), \xi(z)\} = \min\{\xi(((x * y) * y) * z), \xi(z)\} \end{aligned}$$

for all $a, b, c \in E$ and $x, y, z \in X$. Therefore (16) is valid.

Conversely, assume that $\mathcal{M}_{(X,E)} := (M_E,\ G_E,\ \xi)$ satisfies (9) and (16). Then

$$M_E(a) = M_E(a \looparrowright 0) \supseteq M_E(((a \looparrowright 0) \looparrowright 0) \looparrowright c) \cap M_E(c) = M_E(a \looparrowright c) \cap M_E(c),$$

$$G_E(a) = G_E(a \looparrowright 0) \subseteq G_E(((a \looparrowright 0) \looparrowright 0) \looparrowright c) \cup G_E(c) = G_E(a \looparrowright c) \cup G_E(c),$$

and

$$\xi(x) = \xi(x * 0) \geq \min\{\xi(((x * 0) * 0) * z), \xi(z)\} = \min\{\xi(x * z), \xi(z)\}$$

for all $a, c \in E$ and $x, z \in X$. Hence $\mathcal{M}_{(X,E)} := (M_E,\ G_E,\ \xi)$ is a makgeolli ideal of (X, E). If we put $c = 0 = z$ in (16) and use (2), then

$$M_E(a \looparrowright b) \supseteq M_E(((a \looparrowright b) \looparrowright b) \looparrowright 0) \cap M_E(0)$$

$$= M_E((a \looparrowright b) \looparrowright b) \cap M_E(0) = M_E((a \looparrowright b) \looparrowright b),$$

$$\begin{aligned} G_E(a \looparrowright b) &\subseteq G_E(((a \looparrowright b) \looparrowright b) \looparrowright 0) \cup G_E(0) \\ &= G_E((a \looparrowright b) \looparrowright b) \cup G_E(0) = G_E((a \looparrowright b) \looparrowright b) \end{aligned}$$

and $\xi(x * y) \geq \min\{\xi(((x * y) * y) * 0), \xi(0)\} = \min\{\xi((x * y) * y), \xi(0)\} = \xi((x * y) * y)$ for all $a, b \in E$ and $x, y \in X$. Therefore $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ is a positive implicative makgeolli ideal of (X, E) by Theorem 3.

Lemma 2. *If a makgeolli structure $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ on (X, E) satisfies the assertion (ii) in Lemma 1, then it is a makgeolli ideal of (X, E).*

Proof. Since $0 \looparrowright a \leq a$ and $0 * x \leq x$ for all $a \in E$ and $x \in X$, we have $M_E(0) \supseteq M_E(a) \cap M_E(a) = M_E(a)$, $G_E(0) \subseteq G_E(a) \cup G_E(a) = G_E(a)$, and $\xi(0) \geq \min\{\xi(x), \xi(x)\} = \xi(x)$, i.e., $0/\xi(x) \in \xi$ by the condition (ii) in Lemma 1. Since $a \looparrowright (a \looparrowright b) \leq b$ and $x*(x*y) \leq y$ for all $a, b \in E$ and $x, y \in X$, it follows from the condition (ii) in Lemma 1 that $M_E(a) \supseteq M_E(a \looparrowright b) \cap M_E(b)$, $G_E(a) \subseteq G_E(a \looparrowright b) \cup G_E(b)$, and $\xi(x) \geq \min\{\xi(x*y), \xi(y)\}$ for all $a, b \in E$ and $x, y \in X$. Consequently, $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ is a makgeolli ideal of (X, E).

The next corollary is derived by the combination of Theorem 4 and Lemma 2.

Corollary 1. *If a makgeolli structure $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ on (X, E) satisfies* (15) *and the assertion (ii) in Lemma 1, then $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ is a positive implicative makgeolli ideal of (X, E).*

Theorem 6. *A makgeolli structure $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ on (X, E) is a positive implicative makgeolli ideal of (X, E) if and only if it satisfies:*

$$\left\{ \begin{array}{l} (\forall a, b, x, y \in E) \left(\begin{array}{l} ((a \looparrowright b) \looparrowright b) \looparrowright x \leq y \\ \quad \Rightarrow \left\{ \begin{array}{l} M_E(a \looparrowright b) \supseteq M_E(x) \cap M_E(y) \\ G_E(a \looparrowright b) \subseteq G_E(x) \cup G_E(y) \end{array} \right. \end{array} \right). \\ (\forall x, y, a, b \in X)\, (((x * y) * y) * a \leq b \;\Rightarrow\; \xi(x * y) \geq \min\{\xi(a), \xi(b)\}) . \end{array} \right. \tag{17}$$

Proof. Assume that $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ is a positive implicative makgeolli ideal of (X, E). Then it is a makgeolli ideal of (X, E). Let $a, b, x, y \in E$ and $x, y, a, b \in X$ be such that $((a \looparrowright b) \looparrowright b) \looparrowright x \leq y$ and $((x * y) * y) * a \leq b$. Then

$$\begin{aligned} &M_E(a \looparrowright b) \supseteq M_E((a \looparrowright b) \looparrowright b) \supseteq M_E(x) \cap M_E(y), \\ &G_E(a \looparrowright b) \subseteq G_E((a \looparrowright b) \looparrowright b) \subseteq G_E(x) \cup G_E(y), \\ &\xi(x * y) \geq \xi((x * y) * y) \geq \min\{\xi(a), \xi(b)\} \end{aligned}$$

by Proposition 1 and the assertion (ii) in Lemma 1.

Conversely, let $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ be a makgeolli structure on (X, E) that satisfies (17). Let $a, b, c \in E$ and $x, y, z \in X$ be such that $a \looparrowright b \leq c$ and $x * y \leq z$. Then $((a \looparrowright$

$0) \looparrowright 0) \looparrowright b \leq c$ and $((x * 0) * 0) * y \leq z$, and so $M_E(a) = M_E(a \looparrowright 0) \supseteq M_E(b) \cap M_E(c)$, $G_E(a) = G_E(a \looparrowright 0) \subseteq G_E(b) \cup G_E(c)$ and $\xi(x) = \xi(x * 0) \geq \min\{\xi(y), \xi(z)\}$ by (2) and (17). Hence $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a makgeolli ideal of (X, E) by Lemma 2. Since $(((a \looparrowright b) \looparrowright b) \looparrowright ((a \looparrowright b) \looparrowright b)) \looparrowright 0 = 0$ and $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ for all $a, b \in E$ and $x, y \in X$, It follows from (9) and (17) that

$$\begin{aligned}
&M_E(a \looparrowright b) \supseteq M_E((a \looparrowright b) \looparrowright b) \cap M_E(0) = M_E((a \looparrowright b) \looparrowright b),\\
&G_E(a \looparrowright b) \subseteq G_E((a \looparrowright b) \looparrowright b) \cup G_E(0) = G_E((a \looparrowright b) \looparrowright b),\\
&\xi(x * y) \geq \min\{\xi((x * y) * y), \xi(0)\} = \xi((x * y) * y).
\end{aligned}$$

Consequently, $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E) by Theorem 3.

Theorem 7. *A makgeolli structure $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ on (X, E) is a positive implicative makgeolli ideal of (X, E) if and only if it satisfies:*

$$\begin{cases}
(\forall a, b, c, x, y \in E) \left(\begin{array}{l} ((a \looparrowright b) \looparrowright c) \looparrowright x \leq y \\ \quad \Rightarrow \begin{cases} M_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \supseteq M_E(x) \cap M_E(y) \\ G_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \subseteq G_E(x) \cup G_E(y) \end{cases} \end{array} \right). \\
(\forall x, y, z, a, b \in X) \left(\begin{array}{l} ((x * y) * z) * a \leq b \\ \quad \Rightarrow \ \xi((x * z) * (y * z)) \geq \min\{\xi(a), \xi(b)\} \end{array} \right).
\end{cases} \tag{18}$$

Proof. Assume that $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ is a positive implicative makgeolli ideal of (X, E). Then it is a makgeolli ideal of (X, E) (see Theorem 1). Let $((a \looparrowright b) \looparrowright c) \looparrowright x \leq y$ for all $a, b, c, x, y \in E$, and let $((x * y) * z) * a \leq b$ for all $x, y, z, a, b \in X$. The combination of the assertion (ii) in Lemma 1 and Theorem 4 leads to

$$\begin{aligned}
&M_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \supseteq M_E((a \looparrowright b) \looparrowright c) \supseteq M_E(x) \cap M_E(y),\\
&G_E((a \looparrowright c) \looparrowright (b \looparrowright c)) \subseteq G_E((a \looparrowright b) \looparrowright c) \subseteq G_E(x) \cup G_E(y),\\
&\xi((x * z) * (y * z)) \geq \xi((x * y) * z) \geq \min\{\xi(a), \xi(b)\}.
\end{aligned}$$

Conversely, let $\mathcal{M}_{(X,E)} := (M_E, G_E, \xi)$ be a makgeolli structure on (X, E) that satisfies (18). Let $a, b, x, y \in E$ be such that $((a \looparrowright b) \looparrowright b) \looparrowright x \leq y$, and let $x, y, a, b \in X$ be such that $((x * y) * y) * a \leq b$. Using (I3), (2) and (18), we have

$$\begin{aligned}
&M_E(a \looparrowright b) = M_E((a \looparrowright b) \looparrowright (b \looparrowright b)) \supseteq M_E(x) \cap M_E(y),\\
&G_E(a \looparrowright b) = G_E((a \looparrowright b) \looparrowright (b \looparrowright b)) \subseteq G_E(x) \cup G_E(y),\\
&\xi(x * y) = \xi((x * y) * (y * y)) \geq \min\{\xi(a), \xi(b)\}.
\end{aligned}$$

It follows from Theorem 6 that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E).

Theorem 8. *A makgeolli structure $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ on (X, E) is a positive implicative makgeolli ideal of (X, E) if and only if the sets $\mathcal{E}(M_E; \alpha)$ and $\mathcal{E}(G_E; \beta)$ are positive implicative ideals of E, and the set $\mathcal{X}(\xi; t)$ is a positive implicative ideal of X for all $\alpha, \beta \in \mathcal{P}(X)$ and $t \in [0, 1]$.*

Proof. Assume that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E). It is clear that $0 \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\beta) \cap \mathcal{X}(\xi;t)$ for all $\alpha, \beta \in \mathcal{P}(X)$ and $t \in [0,1]$. Let $a, b, c \in E$ be such that $(a \looparrowright b) \looparrowright c \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\beta)$ and $b \looparrowright c \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\beta)$. Then

$$M_E(a \looparrowright c) \supseteq M_E((a \looparrowright b) \looparrowright c) \cap M_E(b \looparrowright c) \supseteq \alpha,$$
$$G_E(a \looparrowright c) \subseteq G_E((a \looparrowright b) \looparrowright c) \cup G_E(b \looparrowright c) \subseteq \beta,$$

and so $a \looparrowright c \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\beta)$. Let $x, y, z \in X$ be such that $(x * y) * z \in \mathcal{X}(\xi;t)$ and $y * z \in \mathcal{X}(\xi;t)$. Then $\xi(x * z) \geq \min\{\xi((x * y) * z), \xi(y * z)\} \geq t$, and thus $x * z \in \mathcal{X}(\xi;t)$. Therefore $\mathcal{E}(M_E;\alpha)$ and $\mathcal{E}(G_E;\beta)$ are positive implicative ideals of E, and $\mathcal{X}(\xi;t)$ is a positive implicative ideal of X for all $\alpha, \beta \in \mathcal{P}(X)$ and $t \in [0,1]$.

Conversely, suppose that $\mathcal{E}(M_E;\alpha)$ and $\mathcal{E}(G_E;\beta)$ are positive implicative ideals of E, and $\mathcal{X}(\xi;t)$ is a positive implicative ideal of X for all $\alpha, \beta \in \mathcal{P}(X)$ and $t \in [0,1]$. Then $\mathcal{E}(M_E;\alpha)$ and $\mathcal{E}(G_E;\beta)$ are subalgebras of E, and $\mathcal{X}(\xi;t)$ is a subalgebra of X. Let $(a_1, b_1, x_1), (a_2, b_2, x_2) \in E \times E \times X$ be such that $\mathcal{M}_{(X,E)}(a_1, b_1, x_1) := (M_E(a_1),\, G_E(b_1),\, \xi(x_1)) = (\alpha_1, \beta_1, t_1)$ and $\mathcal{M}_{(X,E)}(a_2, b_2, x_2) := (M_E(a_2),\, G_E(b_2),\, \xi(x_2)) = (\alpha_2, \beta_2, t_2)$. If we take $(\alpha, \beta, t) := (\alpha_1 \cap \alpha_2, \beta_1 \cup \beta_2, \min\{t_1, t_2\})$, then $a_1, a_2 \in \mathcal{E}(M_E;\alpha)$, $b_1, b_2 \in \mathcal{E}(G_E;\beta)$ and $x_1, x_2 \in \mathcal{X}(\xi;t)$. Hence $a_1 \looparrowright a_2 \in \mathcal{E}(M_E;\alpha)$, $b_1 \looparrowright b_2 \in \mathcal{E}(G_E;\beta)$ and $x_1 * x_2 \in \mathcal{X}(\xi;t)$. If we put $a_1 = a_2$, $b_1 = b_2$, and $x_1 = x_2$, then $0 \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\beta) \cap \mathcal{X}(\xi;t)$, and so $M_E(0) \supseteq \alpha = M_E(a)$, $G_E(0) \subseteq G_E(b)$ and $\xi(0) \geq \xi(x)$ for all $(a, b, x) \in E \times E \times X$. Let $a, b, c \in E$ and $x, y, z \in X$ be such that $M_E((a \looparrowright b) \looparrowright c) = \alpha_1$, $M_E(b \looparrowright c) = \alpha_2$ $G_E((a \looparrowright b) \looparrowright c) = \beta_1$, $G_E(b \looparrowright c) = \beta_2$, $\xi((x * y) * z) = t_1$, and $\xi(y * z) = t_2$. If we take $\alpha = \alpha_1 \cap \alpha_2$, $\beta = \beta_1 \cup \beta_2$ and $t = \min\{t_1, t_2\}$, then $(a \looparrowright b) \looparrowright c \in \mathcal{E}(M_E;\alpha)$, $b \looparrowright c \in \mathcal{E}(M_E;\alpha)$, $(a \looparrowright b) \looparrowright c \in \mathcal{E}(G_E;\alpha)$, $b \looparrowright c \in \mathcal{E}(G_E;\alpha)$, $(x * y) * z \in \mathcal{X}(\xi;t)$, and $y * z \in \mathcal{X}(\xi;t)$. It follows that $a \looparrowright c \in \mathcal{E}(M_E;\alpha) \cap \mathcal{E}(G_E;\alpha)$ and $x * z \in \mathcal{X}(\xi;t)$. Hence

$$M_E(a \looparrowright c) \supseteq \alpha = \alpha_1 \cap \alpha_2 = M_E((a \looparrowright b) \looparrowright c) \cap M_E(b \looparrowright c),$$
$$G_E(a \looparrowright c) \subseteq \beta = \beta_1 \cup \beta_2 = G_E((a \looparrowright b) \looparrowright c) \cup G_E(b \looparrowright c),$$
$$\xi(x * z) \geq t = \min\{t_1, t_2\} = \min\{\xi((x * y) * z), \xi(y * z)\}.$$

Therefore $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E).

Note that a makgeolli ideal might not be a positive implicative makgeolli ideal (see Example 2). But we have the following extension property for a positive implicative makgeolli ideal.

Theorem 9. *Let $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ and $\mathcal{N}_{(X,E)} := (N_E, H_E, \eta)$ be makgeolli ideals of (X, E) such that $M_E(0) = N_E(0)$, $G_E(0) = H_E(0)$, $\xi(0) = \eta(0)$, $M_E(a) \subseteq N_E(a)$, $G_E(b) \supseteq H_E(b)$ and $\xi(x) \leq \eta(x)$ for all $(a, b, x) \in E \times E \times X$. If $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E), then so is $\mathcal{N}_{(X,E)} := (N_E, H_E, \eta)$.*

Proof. Assume that $\mathcal{M}_{(X,E)} := (M_E,\, G_E,\, \xi)$ is a positive implicative makgeolli ideal of (X, E). Using (I3), (4), Theorem 4 and the given assumption, we have

$$N_E(0) = M_E(0) = M_E(((a \looparrowright b) \looparrowright c) \looparrowright ((a \looparrowright b) \looparrowright c))$$

$$
\begin{aligned}
&= M_E(((a \looparrowright b) \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c)\\
&= M_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright b) \looparrowright c)\\
&\subseteq M_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c) \looparrowright (b \looparrowright c))\\
&\subseteq N_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c) \looparrowright (b \looparrowright c))\\
&= N_E(((a \looparrowright c) \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright (b \looparrowright c))\\
&= N_E(((a \looparrowright c) \looparrowright (b \looparrowright c)) \looparrowright ((a \looparrowright b) \looparrowright c)),
\end{aligned}
$$

$$
\begin{aligned}
H_E(0) = G_E(0) &= G_E(((a \looparrowright b) \looparrowright c) \looparrowright ((a \looparrowright b) \looparrowright c))\\
&= G_E(((a \looparrowright b) \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c)\\
&= G_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright b) \looparrowright c)\\
&\supseteq G_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c) \looparrowright (b \looparrowright c))\\
&\supseteq H_E(((a \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright c) \looparrowright (b \looparrowright c))\\
&= H_E(((a \looparrowright c) \looparrowright ((a \looparrowright b) \looparrowright c)) \looparrowright (b \looparrowright c))\\
&= H_E(((a \looparrowright c) \looparrowright (b \looparrowright c)) \looparrowright ((a \looparrowright b) \looparrowright c))
\end{aligned}
$$

and

$$
\begin{aligned}
\eta(0) = \xi(0) &= \xi(((x * y) * z) * ((x * y) * z))\\
&= \xi(((x * y) * ((x * y) * z)) * z)\\
&= \xi(((x * ((x * y) * z)) * y) * z)\\
&\leq \xi(((x * ((x * y) * z)) * z) * (y * z))\\
&\leq \eta(((x * ((x * y) * z)) * z) * (y * z))\\
&= \eta(((x * z) * ((x * y) * z)) * (y * z))\\
&= \eta(((x * z) * (y * z)) * ((x * y) * z)).
\end{aligned}
$$

It follows from (9) and (10) that

$$
\begin{aligned}
&N_E((a \looparrowright c) \looparrowright (b \looparrowright c))\\
&\supseteq N_E(((a \looparrowright c) \looparrowright (b \looparrowright c)) \looparrowright ((a \looparrowright b) \looparrowright c)) \cap N_E((a \looparrowright b) \looparrowright c)\\
&\supseteq N_E(0) \cap N_E((a \looparrowright b) \looparrowright c) = N_E((a \looparrowright b) \looparrowright c),
\end{aligned}
$$

$$
\begin{aligned}
&H_E((a \looparrowright c) \looparrowright (b \looparrowright c))\\
&\subseteq H_E(((a \looparrowright c) \looparrowright (b \looparrowright c)) \looparrowright ((a \looparrowright b) \looparrowright c)) \cup H_E((a \looparrowright b) \looparrowright c)\\
&\subseteq H_E(0) \cup H_E((a \looparrowright b) \looparrowright c) = H_E((a \looparrowright b) \looparrowright c),
\end{aligned}
$$

and

$$
\begin{aligned}
&\eta((x * z) * (y * z))\\
&\geq \min\{\eta(((x * z) * (y * z)) * ((x * y) * z)), \eta((x * y) * z)\}
\end{aligned}
$$

$$\geq \min\{\eta(0), \eta((x * y) * z)\} = \eta((x * y) * z)$$

for all $a, b, c \in E$ and $x, y, z \in X$. Therefore $\mathcal{N}_{(X,E)} := (N_E, H_E, \eta)$ is a positive implicative makgeolli ideal of (X, E) by Theorem 4.

4. Conclusions

A fuzzy set is an extension of an existing set using fuzzy logic. Soft set theory is a generalization of fuzzy set theory. Fuzzy and soft set theory are good mathematical tools for dealing with uncertainty in a parametric manner. Ahn et al. [2] introduced the concept of makgeolli structures as a hybrid structure using fuzzy and soft set theory, and applied it to BCK/BCI-algebras. In this article, we introduced the notion of a positive implicative makgeolli ideal in BCK-algebras, and investigated its properties. We established the relationship between makgeolli ideal and positive implicative makgeolli ideal, and explored the conditions under which makgeolli ideal can be positive implicative makgeolli ideal. We discussed the characterization of positive implicative makgeolli ideal, and constructed the extension property for a positive implicative makgeolli ideal.

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