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A family of optimal cubic-order multiple-root solvers and their dynamcis

Hwajoon Kim[1], Arjun Kumar Rathie[2], Young Hee Geum[3,*]

[1] *Department of IT Engineering, Kyungdong University, Yangju 11458, Korea,*
[2] *Department of Mathematics, Vedant College of Engineering and Technology, Rajasthan Technical University, Village: Tulsi, Post: Jakhmund, District: Bundi, Rajasthan State, India*
[3] *Department of Applied Mathematics, Dankook University, Cheonan, Chungnam, 31116, Korea*

Abstract. The complex dynamical analysis of the cubic-order iterative family is proposed to draw the fractal images via Möbius conjugacy map applied to a quadratic polynomial $(z-A)^m(z-B)^m$. The resulting dynamics is clearly visualized through various stability surfaces and parameter spaces using Mathematica.

2020 Mathematics Subject Classifications: 65H05, 65H99

Key Words and Phrases: Multiple-zero solver, cubic-order, conjugacy, parameter space

1. Introduction

Most nonlinear equations are used in computer science, engineering, medicine and biology. In order to find the solutions for these nonlinear equations, numerical iterative schemes are sought. According to the form and property of the problem, the iteration scheme is one of the generally used methods. With the aid of initial values and recurrence equations, initial guesses are modified continuously until desired accuracy is gotten. In the past decade, researchers[1, 3, 8, 15, 18] have studied the development of the higher-order solver to locate the roots of nonlinear equations. Numerous methods were suggested based on various considerations and theories [4, 11–13]. The optimal cubic-order methods are designed[7]

$$\begin{cases} y_n = x_n - m(1-t)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{mf(y_n)}{t^m f'(x_n)}, \end{cases} \tag{1}$$

*Corresponding author.
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Email addresses: math@kduniv.ac.kr (H.J.Kim), arjunkumarrathie@gmail.com (A.K. Rathie), conpana@empal.com (Y.H.Geum)

with the multiplicity index $m \in \mathbb{N}$ of the sought root and the free parameter $t \in \mathbb{C}$.

Let $R : S \to S$ be an operator with S is the Riemann sphere.The orbit of a point $z_1 \in S$ is defined as the set of images of z_1 by $\{z_1, R(z_1), \ldots R^n(z_1), \ldots\}$. If $R(z_a) = z_a$, a point $z_a \in S$ is a fixed point of R. A point z_b is called a critical point if $R'(z_b) = 0$. The following definition and theorem are important to build the conjugacy map[20] and to visualize the relevant dynamics.

Definition 1. *Let $f : X \to X$ and $g : Y \to Y$ be two analytic functions. We define that the functions f and g are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, where $\circ$ denotes function composition. Then the map h is called a conjugacy [19].*

Theorem 1. *Let f and g be defined in Definition 1. Then the following hold[9]:*

(a) $g = h \circ f \circ h^{-1}$ and $g^n = h \circ f^n \circ h^{-1}$.

(b) If f is topologically conjugate to g via h and ν is a fixed point of g, then $h^{-1}(\nu)$ is a fixed point of f. If f and g are invertible, then the topological conjugacy h maps an orbit of f onto an orbit of g and the order of points is preserved.

2. Conjugacy Maps

A nonlinear equation (1) is reconstructed in a generic form[2, 5, 9, 14] as a discrete dynamical system

$$x_{n+1} = R_f(x_n), \tag{2}$$

where R_f is the iteration function.
We have the following result for discrete system as follows:

$$z_{n+1} = R_f(z_n) = z_n - \frac{mf(y_n)}{t^m f'(z_n)}, \tag{3}$$

where $y_n = z_n - m(1-t)\frac{f(z_n)}{f'(z_n)}$.

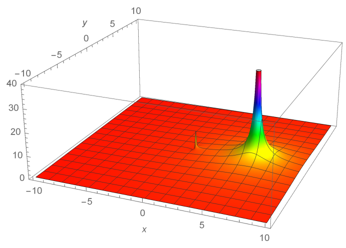
Using Möbius conjugacy map $M(z) = \frac{z-A}{z-B}$ and its inverse $M^{-1}(z) = \frac{Bz-A}{z-1}$ with z, $A \neq B,\ A,\ B \in \mathbb{C} \cup \{\infty\}$ [16, 20], R_f in (3) is conjugated to J satisfying

$$J(z;t) = \frac{z(-r_1 r_2 + r_3\,(1+z)\,t^m)}{-r_1 r_2 z + r_3\,(1+z)\,t^m}, \tag{4}$$

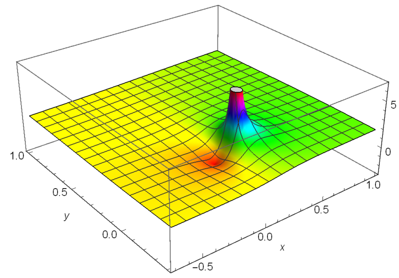
where $r_1 = (t+z)^m,\ r_2 = (1+tz)^m$ and $r_3 = (1+z)^{2m}$.

From (4), two points $z = 0$ and $z = \infty$ are fixed points of the conjugate map $J(z;t)$, regardless of t-values. And $z = 1$ is a strange fixed point of J (that is not a root of $f(z) = [(z-A)(z-B)]^m$) from the fact of $J(1;t) = 1$, regardless of t-values. That is, J is dependent on t however independent of A and B.

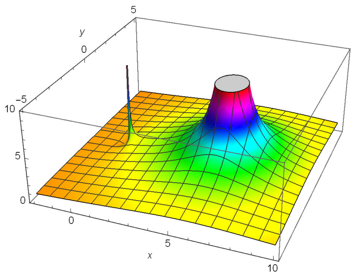
With Mathematica[17], we find $J(z;t)$ as follows:



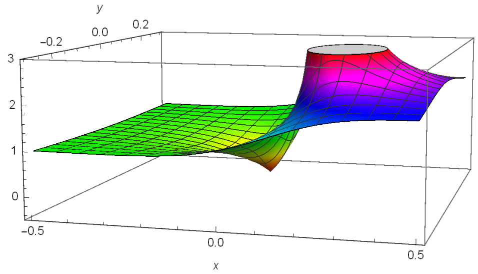
(a) $|\Re(t)| \leq 10, \ |\Im(t)| \leq 10$



(b) $0.7 \leq |\Re(t)| \leq 1, \ -0.4 \leq |\Im(t)| \leq 1$



(c) $-2 \leq |\Re(t)| \leq 10, \ -5 \leq |\Im(t)| \leq 5$



(d) $-2 \leq |\Re(t)| \leq 10, \ -5 \leq |\Im(t)| \leq 5$

Figure 1: Stability surfaces for $m = 1$

$$J(z;t) = \begin{cases} \frac{z(t(1+z)^3-(t+z)(1+tz))}{t(1+z)^3-z(t+z)(1+tz)}, & \text{if} \quad \text{m=1} \\ \frac{z(t^2(1+z)^5-(t+z)^2(1+tz)^2)}{t^2(1+z)^5-z(t+z)^2(1+tz)^2}, & \text{if} \quad \text{m=2} \end{cases} \tag{5}$$

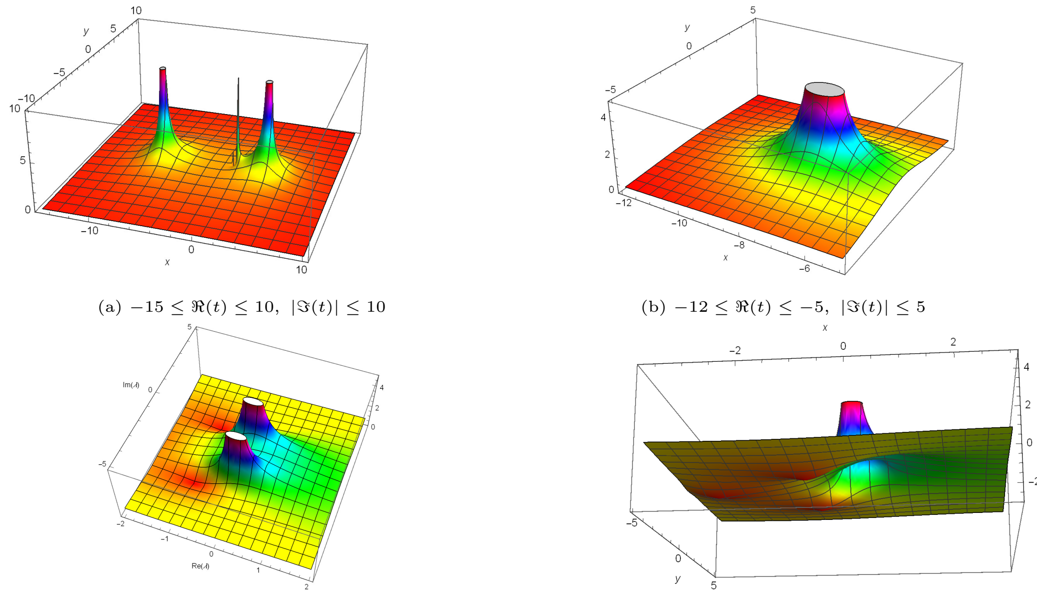
We find the fixed points of the iteration scheme $J(z;\lambda)$. Let $\phi(z;t) = z - J(z;t)$, whose zeros are the sought fixed points of J. We know that $z = 0$ and $z = 1$ are the zeros of ϕ. Hence $\phi(z;t)$ is expressed as the following form:

$$\phi(z;t) = \frac{r_1 r_2 (z-1) z}{r_1 r_2 z - r_3 t^m (z+1)}, \tag{6}$$

To investigate the dynamics behind iterative map (3) applied to a quadratic polynomial raised to the power of m, $f(z) = (z-A)^m (z-B)^m$, we find out the fixed points of J and their stability. From the fact that $M(z)$ is a fixed point of J for a fixed point z of R_p with its inverse $M^{-1}(z) = \frac{zB-A}{z-1}$, we calculate the explicit form of $\phi(z;t) = z - J(z;t)$ for $m \in \{1,2\}$ below:

$$\phi(z;t) = \begin{cases} \frac{z(z-1)(t+z)(1+tz)}{z^2+t^2z^2-t(1+2z+3z^2)}, & \text{if} \quad \text{m=1} \\ \frac{z(z-1)(t+z)^2(1+tz)^2}{-t^2(1+z)^5+z(t+z)^2(1+tz)^2}, & \text{if} \quad \text{m=2} \end{cases} \tag{7}$$

Theorem 2. *Let $m = 1$. Then the following hold:*
(a) If $t = -1$, then $\phi(z;t) = ((-1+z)^3 z)/(1+2z+5z^2)$ and the strange fixed points z are $z = 0$ and $z = 1$.



(a) $-15 \leq \Re(t) \leq 10, \ |\Im(t)| \leq 10$

(b) $-12 \leq \Re(t) \leq -5, \ |\Im(t)| \leq 5$

(c) $|\Re(t)| \leq 2, \ |\Im(t)| \leq 5$

(d) $-3.7 \leq |\Re(t)| \leq 3, \ |\Im(t)| \leq 5$

Figure 2: Stability surfaces for $m = 2$

(b) If $t = 0$, then $\phi(z;t) = 1 - z$ and the strange fixed point is $z = 1$.
(c) If $t = 1$, then $\phi(z;t) = z(1-z)$ and the strange fixed points are $z = 0$ and $z = 1$.
(d) Let $\varphi = (t+z)(1+tz)$ with $t \notin \{-1, 0, 1\}$. Then $\varphi(1/z) = z^{-2}\varphi(z)$ holds for $z \neq 0$. Hence, if $z \neq 0$ is a root of $\varphi(z;t)$, then $1/z$ is also a root of $\varphi(z;t)$.

Proof. After an accurate computation and careful algebraic treatments with the aid of Mathematica, (a), (b) and (c) follow . For the proof of (d) follows from the fact that $\varphi(1/z) = (t + 1/z)(1 + t/z) = ((t+z)(1+tz))/z^2 = z^{-2}\varphi(z)$.

Let $z \notin \{0, 1\}$ be a root of $\phi(z;t)$ for $m = 1, \ 2$. Suppose the numerator and denominator of $\phi(z;t)$ have no common factors for some suitable t-values. Then the roots of $\phi(z;t)$ are explicitly found.
Differentiating J in (4), we require

$$J'(z;t) = \frac{r_3 t^m(-r_1^{\frac{-1+m}{m}} r_2 t + r_3 t^m + k_1 z + k_2 z^2 + r_1 r_2 k_3 z^3)}{(r_1 r_2 z - r_3 t^m(1+z))^2}, \tag{8}$$

where $k_1 = -m r_1 r_2^{\frac{-1+m}{m}} t + 2 r_3 t^m + r_1^{\frac{-1+m}{m}} r_2(-1+m(-1+2t))$, $\ k_2 = r_3 t^m - r_1^{\frac{-1+m}{m}} r_2(2m(-1+t)+t)$ and $\ k_3 = -(1+m) r_1^{-1/m} + m r_2^{-\frac{1}{m}} t$.
Computing $J'(z;t)$ for $m = 1$ and $m = 2$, we have

$$J'(z;t) = \begin{cases} \frac{-2tz((1+z)^2)(1-3t+t^2+(-1-t^2)z+(1-3t+t^2)z^2)}{(t+2tz-(1+(-3+t)t)z^2)^2}, & \text{if} \quad \text{m=1} \\ \frac{-t^2 z(1+z)^4(a+bz+cz^2+bz^3+az^4)}{(-t^2-4t^2z+(2t-10t^2+2t^3)z^2+(1-6t^2+t^4)z^3+(2t-5t^2+2t^3)z^4)^2}, & \text{if} \quad \text{m=2} \end{cases} \tag{9}$$

where $a = 4t - 10t^2 + 4t^3$, $b = 3 - 8t + 2t^2 - 8t^3 + 3t^4$, $b = -4 + 16t - 36t^2 + 16t^3 - 4t^4$ and $c = 3 - 8t + 2t^2 - 8t^3 + 3t^4$, to study the stability of the fixed points which are in Figures 1-2.

The critical points of the iterative scheme are given by the roots of the derivative of J, $J'(z,t) = 0$. The points $z = 0$ and $z = \infty$ are critical points related with the roots a and b of the quadratic polynomial $(z - a)(z - b)$. When $m = 1$, the critical points are $z = 0$, $z = \infty$ and $z = \pm 1$. When $m = 2$, 4 roots ξ can be found numerically for a given t.

3. Dynamical analysis and numerical results

This section describes the complex dynamics involved in the parameter space. The following theorem is used to find useful properties of symmetry in the parameter space.

Theorem 3. *Let $z(t)$ be a free critical point of $J(z;t)$ dependent upon parameter t. Then that parameter space is symmetric about its horizontal axis. [10]*

Theorem 4. *Let z be a critical point. Then the following holds [10]:*
(a) $J'(z;t) = J'(1/z;t)$.
(b) If $z \neq 0$ is a critical point, then so is $1/z$.

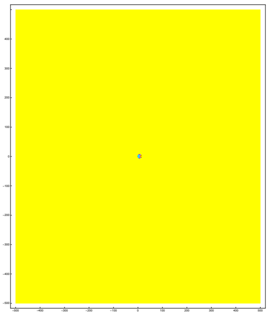
When $m = 2$, the orbit behavior of two branches $cp_1(t) = \xi_1$ and $cp_2(t) = \xi_2$ of the free critical points under the action of $J(z;t)$. The orbit of two branches $cp_3(t) = \frac{1}{cp_1(t)}$ and $cp_4(t) = \frac{1}{cp_2(t)}$ is similarly described.

Let $\mathcal{P} = \{t \in \mathbb{C} :$ a critical orbit of z under $J(z;t)$ converges to a number $\nu_p \in \overline{\mathbb{C}}\}$ be the parameter space. If the number ν_p is a finite constant, there is finite periods in the orbit. Otherwise, the orbit is not periodic however bounded or goes to infinity.

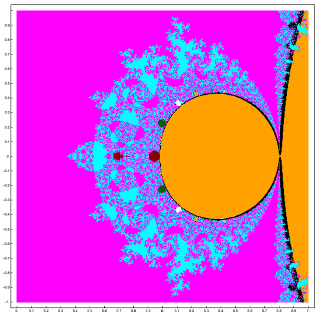
Table 1: Coloring scheme for a q-periodic orbit with $q \in \mathbb{N} \cup \{0\}$

q	C_q
$q = 1$	$C_1 = \begin{cases} \text{magenta, for fixed point } \infty \\ \text{cyan, for fixed point } 0 \\ \text{yellow, for fixed point } 1 \\ \text{red, for other strange fixed point ,} \end{cases}$
$2 \leq q \leq 68$	C_2 = orange, C_3 = light green, C_4 = dark red, C_5 = dark blue, C_6 = dark green, C_7 = dark yellow, C_8 = floral white, C_9 = light pink, C_{10} = khaki, C_{11} = dark orange, C_{12} = turquoise, C_{13} = lavender, C_{14} = thistle, C_{15} = plum, C_{16} = orchid, C_{17} = medium orchid, C_{18} = blue violet, C_{19} = dark orchid, C_{20} = purple, C_{21} = power blue, C_{22} = sky blue, C_{23} = deep sky blue, C_{24} = dodger blue, C_{25} = royal blue, C_{26} = medium spring green, C_{27} = spring green, C_{28} = medium sea green, C_{29} = sea green, C_{30} = forest green, C_{31} = olive drab, C_{32} = bisque, C_{33} = moccasin, C_{34} = light salmon, C_{35} = salmon, C_{36} = light coral, C_{37} = Indian red, C_{38} = brown, C_{39} = fire brick, C_{40} = peach puff, C_{41} = wheat, C_{42} = sandy brown, C_{43} = tomato, C_{44} = orange red, C_{45} = chocolate, C_{46} = pink, C_{47} = pale violet red, C_{48} = deep pink, C_{49} = violet red, C_{50} = gainsboro, C_{51} = light gray, C_{52} = dark gray, C_{53} = gray, C_{54} = charteruse, C_{55} = electric indigo, C_{56} = electric lime, C_{57} = lime, C_{58} = silver, C_{59} = teal, C_{60} = pale turquoise, C_{61} = sandy brown, C_{62} = honeydew, C_{63} = misty rose, C_{64} = lemon chiffon, C_{65} = lavender blush, C_{66} = gold, C_{67} = crimson, C_{68} = tan.
$q = 0^*$ or $q > 69$	C_q = black.

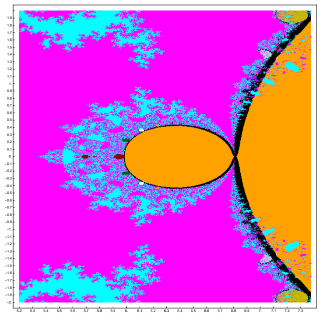
$*$: $q = 0$ implies that the orbit is non-periodic but bounded.



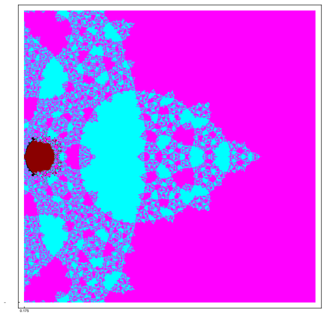
(a) $|\Re(t)| \leq 500, \ |\Im(t)| \leq 500$



(b) $5 \leq \Re(t) \leq 7, \ |\Im(t)| \leq 1$

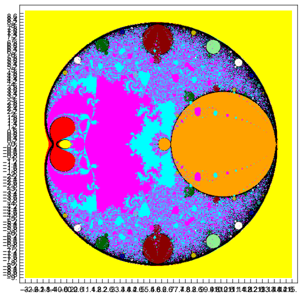


(c) $|\Re(t)| \leq 2, \ |\Im(t)| \leq 5$

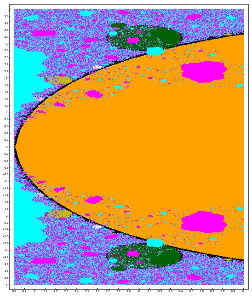


(d) $-3.7 \leq |\Re(t)| \leq 3, \ |\Im(t)| \leq 5$

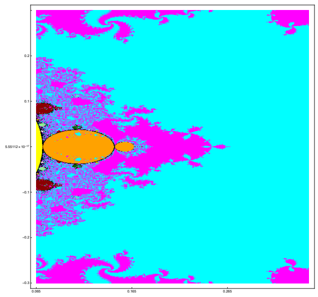
Figure 3: Parameter spaces associated with free critical points cp_1 for $m = 1$.



(a) $-3 \leq \Re(t) \leq 15, \ |\Im(t)| \leq 10$



(b) $6.8 \leq \Re(t) \leq 9, \ |\Im(t)| \leq 4$

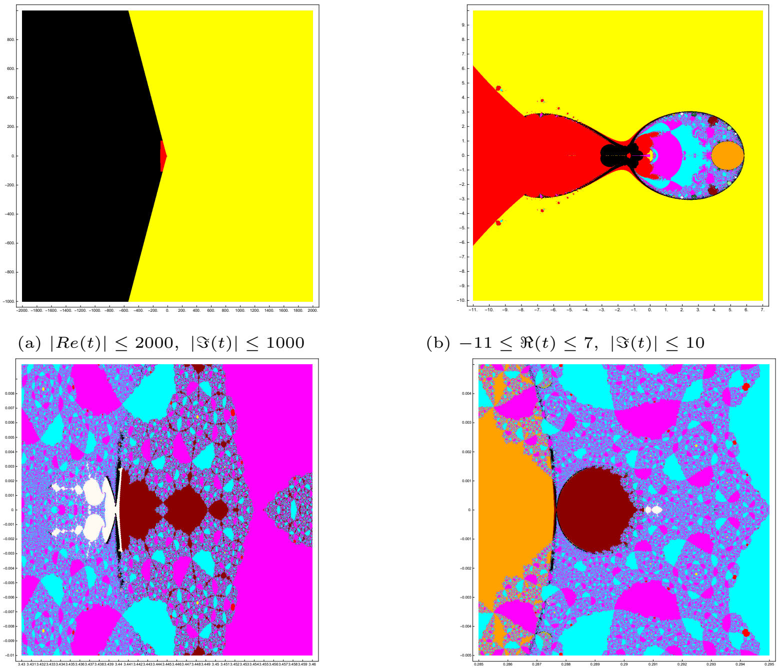


(c) $0.065 \leq \Re(t) \leq 0.35, \ |\Im(t)| \leq 0.3$

Figure 4: Parameter spaces associated with free critical points cp_2 for $m = 1$.

We describe a systematic method coloring a point $t \in \mathcal{P}$ depending on the period of the orbit of z under $J(z;t)$ for $t \in \mathcal{P}$. Then the point t is drawn in corresponding color C_k if t induces a k-periodic orbit with $k \in \mathbb{N} \cup \{0\}$ under $J(z;t)$. We accept the desired k-periodic convergence of an orbit associated with $\mathcal{P}$ after a maximum of 1000-2000 iteration number[17] and with a tolerance of 10^{-6}. We use color C_q according to the color palette shown in Table 1.

In Figures 3–6, we have shown the parameter spaces $\mathcal{P}$ related with $cp_j(t)$, $(1 \leq j \leq 2)$. A point $t \in \mathcal{P}$ is painted according to the coloring scheme shown in Table 1. In terms of numerical phenomena, every point of the parameter space $\mathcal{P}$ whose color is none of cyan(root $z = a$), magenta(root $z = b$), yellow or red is not a better choice of t. Let $\mathcal{P}_i$



(a) $|Re(t)| \leq 2000, \ |\Im(t)| \leq 1000$

(b) $-11 \leq \Re(t) \leq 7, \ |\Im(t)| \leq 10$

(c) $0.1 \leq \Re(t) \leq 0.3, \ |\Im(t)| \leq 0.1$

(d) $0.285 \leq \Re(t) \leq 0.295, \ |\Im(t)| \leq 0.005$

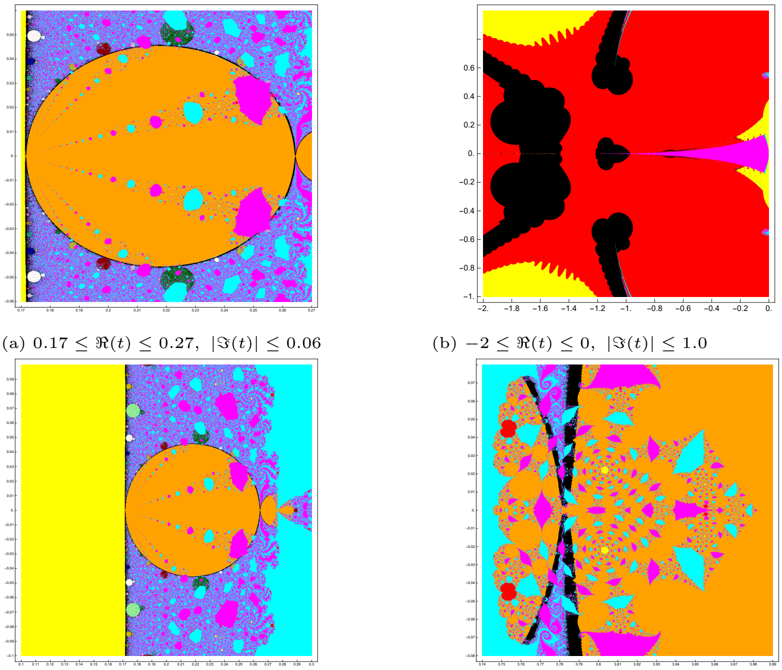
Figure 5: Parameter spaces associated with free critical points cp_1 for $m = 2$.

denote the parameter space related with branch cp_i for $1 \leq i \leq 4$. We find the complicated but beautiful pattern with the behavior that from $n(\neq 1) \in N$-periodic orbit is budding at period-1 component and 6-periodic component is budding at period-3 component.

Based on the theoretical result, we compare the proposed scheme (1) with the Dong's third-order method[6] as follows:

$$x_{n+1} = z_n - \left(1 - \frac{1}{\sqrt{m}}\right)^{1-m} \frac{f(z_n)}{f'(x_n)}, \ z_n = x_n - \sqrt{m}\frac{f(x_n)}{f'(x_n)}.$$

To plot the complex dynamics of the proposed method (g1) and Dong's scheme (d1) with the basins of attraction, we take the test functions having multiple roots with multiplicity $m = 4, 6$. In this statistical data for the basin of attraction, abbreviations cpu, tcon, avg and tdiv denote the value of CPU time for convergence, the value of total convergent points, the value of average iteration number for convergence and the value of divergent points. As the first example, we select the polynomial $p_1(z) = (z^3 - z)^4$ with roots $z = 0, \pm 1$ of multiplicity $m = 4$. The method g1 is better in view of cpu and avg. As the next instance, the polynomial $p_2(z) = (z^2 - 3z + 5)^6$ has the roots $z = 1.5 \pm 1.65831i$. The method g1 is better in view of avg and d1 is better in view of tcon. As can be seen in Figure 7, the picture (c) has shown some black point. The results are listed in Table 2 and Figure 7 .



(a) $0.17 \le \Re(t) \le 0.27,\ |\Im(t)| \le 0.06$

(b) $-2 \le \Re(t) \le 0,\ |\Im(t)| \le 1.0$

(c) $0.1 \le \Re(t) \le 0.3,\ |\Im(t)| \le 0.1$

(d) $3.74 \le \Re(t) \le 3.89,\ |\Im(t)| \le 0.08$

Figure 6: Parameter spaces associated with free critical points cp_2 for $m = 2$.

4. Conclusion

Given the multiplicity m, the complex dynamics were described by means of an Möbius conjugacy map applied to a polynomial of the form $f(z) = (z - A)^m(z - B)^m$ with the stability analysis of strange fixed points.

Futures studies deal with the visualization of different types of numerical methods accurately by improving the current research. In addition, we will investigate the parameter space and the basins of attraction of the developed multiple-root finder in detail. We will observe the beautiful fractal that occurs in numerical methods from a variety of perspectives.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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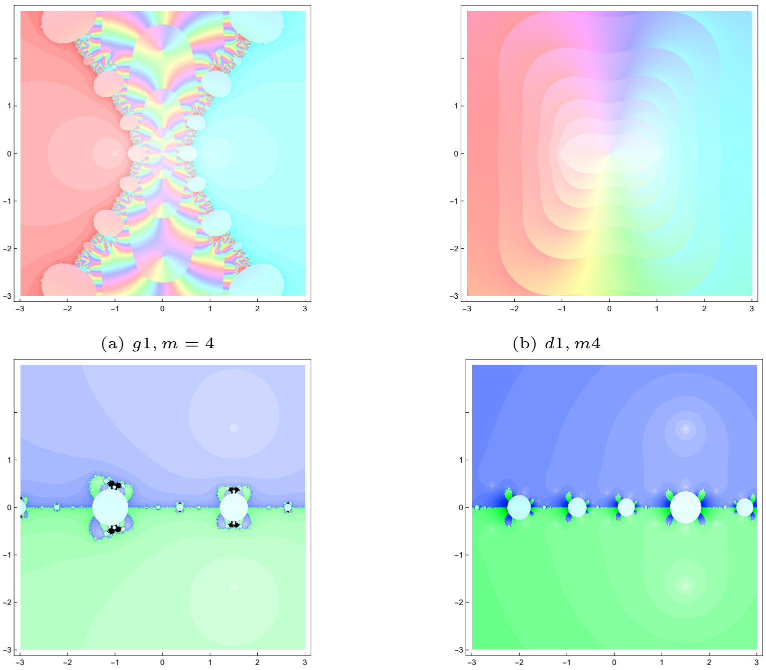
Table 2: Typical Example

p_m	Method	cpu	tcon	avg	tdiv
p_1	g1	54.406	360,000	5.83743	0
	d1	58.407	360,000	6.32902	0
p_2	g1	64.312	360,000	5.63653	1156
	d1	75.516	360,000	9.84352	0



(a) $g1, m = 4$ (b) $d1, m4$

(c) $g1, m = 6$ (d) $d1, m = 6$

Figure 7: Basin of attraction

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