EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1637-1648
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Involution t – Clean Rings with Applications

Shaimaa H. Ahmad[1], Mohammed Al-Neima[2,*], Ahmed Ali[1], Raida D. Mahmood[1]

[1] *Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq*
[2] *Department of Civil Engineering, College of Engineering, University of Mosul, Mosul, Iraq*

Abstract. A new class of rings was introduced, in which every element in the ring is a sum of involution and tripotent elements. This class called involution t-clean rings, which is a generalization of invo-clean rings and subclass of clean rings. Some properties of this class are investigated. For an application in graph theory, a new graph is defined as t-clean graph of involution t-clean ring. The set of vertices is ordered pairs of involution and tripotent element, which is the sum of them is involution t-clean element. The two vertices are adjacent if and only if the sum of involution elements is zero or the product of the tripotent elements is zero. The graphs are connecting, has diameter one and girth three.

2020 Mathematics Subject Classifications: 16E50, 16U99, 05C25

Key Words and Phrases: Clean ring, invo-clean ring, tripotent element, Hosoya polynomial, Wiener index

1. Introduction

Throughout this paper, the ring R is associative with identity. The symbols $U(R)$ is the set of unit elements in R, $Idm(R)$ is the set of idempotent elements in R, $invo(R)$ the subset of $U(R)$ consisting of all involution elements of R, $Tri(R) = \{t \in R : t^3 = t\}$, $N_2(R) = \{x \in R : x^2 = 0\}$. The aim of this paper is to generalize the concept of invo-clean ring,the purpose to obtain more general results by generalizing some results of invo-clean ring. This prompts researchers to study many of the properties of this generalization.

The concept of clean ring, first defined by Nicholson in 1977 [9], a ring R is called clean, if for all $x \in R$ and written as $x = u + e$, where $u \in U(R)$ and $e \in Idm(R)$. If $ue = eu$, get that R is called strongly clean. Recall that a ring R is called invo-clean, if every $x \in R$ can be written as $x = u + e$, where $u \in invo(R)$ and $e \in Idm(R)$. If, $ue = eu$, get that R is strongly invo-clean.Which was first defined Danchev see [6].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4530

Email addresses: shaymaahatim@uomosul.edu.iq (Shaimaa H. Ahmad), mohammedmth@uomosul.edu.iq (Mohammed Al-Neima), ahmedgraph@uomosul.edu.iq (Ahmed Ali), raida.1961@uomosul.edu.iq (Raida D. Mahmood)

Every idempotent element is invo-clean, because $e = (2e-1)+(1-e)$, where $(2e-1)^2 = 1$ and $(1-e)^2 = 1-e$. Moreover, Z_2, Z_3, Z_4, Z_6, Z_8 are invo-clean rings. Every invo-clean ring is clean, but the opposite is untrue. (Examples Z_5, Z_10).

As usual, $M_2(R)$ stand for the 2×2 matrix ring. $T_2(R)$ is 2×2 upper triangular matrix ring. We write Z_n for rings of integers modulo n. In [11], the ring R is regular if every $a \in R, a \in aRa$. The ring R is called r-clean if all elements of R can be written as the summation of a regular and idempotent elements [2].

Zero divisor graph is concept give the relation between the commutative rings and graph theory, which was first defined by Beck in [4]. In the same way, Habibi, Celikel and Abdioglu in [8] start study of clean graph defined in clean ring. For an application to invo-t-clean ring, a new graph is defined; depended on an involution t-clean ring.

In graph theory, the graph is an order pair denoted by $G = (V, F)$ of two sets V (is non-empty) and F (may be empty) such that $F \subseteq V \times V.G' = (V', F')$ is a sub-graph of $G = (V, F)$ written as $G' \subseteq G$ if $V' \subseteq V$ and $F' \subseteq F$. The order of G is the number of vertices, denoted by $|G|$ and the number of edges is denoted by $\|G\|$. The girth is the length of the shortest cycle in the graph G, denoted by $g(G)$. Each edge incident on two vertices, the number of the edges incident on a vertex v is said to be the degree of v and it's denoted by $deg_G(v)$ or $deg(V)$. The numbers $\delta(G)$ and $\Delta(G)$ are minimum and maximum degrees r of a graph G respectively [5]. The average degree of G is defined by: $ad(G) = \frac{1}{|G|}\sum_{\forall v\in V} deg(v)$. From clearly that $\delta(G) \leq ad(G) \leq \Delta(G)$.

The distance $d(v, u)$ in connected graph G is positive number represent. The length shortest $(v-u)$- path in G. The maximum distance between any two distinct vertices in G is the diameter of G and is denoted by $diam(G)$. The sum of the lengths of the shortest $(v-u)$- path in G is called Wiener index [12], that is $W(G) = \frac{1}{2}\sum_{\forall v,u\in V} d(v, u)$. The average distance is define by: $D(G) = \frac{2W(G)}{|G|(|G|-1)}$.

Let $d(G, k)$ be the number of vertex pairs at distance k in a connected graph G, where $k = 0, 1, \cdots, diam(G)$. Then the Hosoya polynomial [7] of G is defined as:

$H(G, t) = \sum_{k=1}^{diam(G)} d(G, k)t^k$. From clearly that:

$W(G) = \frac{\mathrm{d}}{\mathrm{dt}}H(G, t)|_{t=1} = \sum_{k=0}^{diam(G)} kd(G, k)$.

$\sum_{k=0}^{diam(G)} d(G, k) = \frac{|G|(|G|+1)}{2}$

$\sum_{k=1}^{diam(G)} d(G, k) = \frac{|G|(|G|-1)}{2}$

$ad(G) = \frac{2\|G\|}{|G|}, d(G, 0) = |G|\, and d(G, 0) = \|G\|$.

2. Invo-t-clean Rings

The section describes the structure of the invo-t-clean rings.

Definition 2.1. *The ring R is called involution t-clean (for short invo-t-clean), if all $a \in R$ can be expressed as $a = u + t$, where $u \in invo(R)$ and $t \in Tri(R)$. An invo-t-clean ring with $ut = tu$ is strongly invo-t-clean.*

Example 2.2. *1. Let $n \geq 2$ and integer. Then the ring Z_n is invo-t-clean if and only if $n = 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$.*

2. Let $M_2(Z_2), M_2(Z_3)$ and the upper triangular matrices $T_2(Z_2), T_2(Z_3)$ are invo-t-clean rings.

Remark 2.3. *1. Every invo-t-clean ring is also clean, but the reverse is not true, for a commutative example Z_7. For non-commutative example $M_2(Z_4), T_2(Z_4)$. The elements $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ are not invo-t-clean in, $T_2(Z_4)$, where*

$$U_2(T_2(Z_4)) = \left\{ \begin{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}, \\ \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} \end{matrix} \right\}$$

$$Tri(T_2(Z_4)) = \left\{ \begin{matrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, & \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} & \end{matrix} \right\}$$

2. Every invo-clean ring is invo-t-clean, but the reverse is not true. For commutative example Z_5, the element 3 is not invo-clean. For non-commutative example $M_2(Z_3), T_2(Z_3)$. The elements $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not invo-clean in $T_2(Z_3)$, where

$$U_2(T_2(Z_3)) = \left\{ \begin{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, & \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \\ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, & \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \end{matrix} \right\}$$

$$Id(T_2(Z_3)) = \left\{ \begin{matrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \end{matrix} \right\}$$

We begin with two useful technicalities.

Proposition 2.4. *Let* 3 *be an invo-t-clean element in a ring* R. *Then* $15t = 0$ *and* $120 = 0$.

Proof. *Since* $3 = u + t$, *where* $u \in invo(R)$ *and* $t \in Tri(R)$.*Thus*
$ut = (3 - t)t = 3t - t^2 = t(3 - t) = tu$, *so* $ut = tu$. *Now*
$24 = 3^3 - 3 = (u + t)^3 - (u + t) = 8 \cdot 3 = 8(u + t)$,
$u^3 + 3u^2t + 3ut^2 + t^3 - ut = 8(u + t)$
$u + 3t + 3ut^2 + t - u - t = 8(u + t)$.
$-3t - 3ut^2 + 8u + 8t = 0$. *Multiply both sided from the right by* t
$-3t^2 - 3ut^3 + 8ut + 8t^2 = 5t(t + u) = 0$. *Hence* $15t = 0$
In addition, $24 = 3t + 3ut^2 = 3t + 3(3 - t)t^2 = 3t + 9t^2 - 3t^3 = 9t^2$.
So $120 = 5 \cdot 24 = 5 \cdot 9t^2 = 3(15t)t = 0$.■

Proposition 2.5. *The homomorphic image of invo-t-clean rings are invo-t-clean.*

Proof. *Let* $f : R \longrightarrow S$ *be a homomorphic mapping from an invo-t-clean ring* R *into* S. *Then for all* $y \in S$ *there* $x \in R$ *with* $y = f(x)$ *and* $x = u + t$ *where* $u \in invo(R)$ *and* $t \in Tri(R)$. *Now* $y = f(x) = f(u+t) = f(u)+f(t), (f(t))^3 = f(t^3) = f(t) \in Tri(R)$. *Since* $u \in U_2(R)$,*then* $(f(u))^2 = f(u^2) = f(1) = 1$, *so* $f(u) \in U_2(R)$ $y = f(x) = f(u) + f(t)$. *Therefore* S *is invo-t-clean ring.* ■

Corollary 2.6. *Let* R *be an invo-t-clean ring and* I *is an ideal of* R. *Then* R/I *is an invo-t-clean.*

Since the involution element is also a tripotent element. The following proposition explains that the nilpotent element of index 2 will be a sum of two involution elements if it is invo-t-clean.

Proposition 2.7. *For a ring* R. *Let* $a \in N_2(R)$.

1. *If* a *is an invo-t-clean element, then* $t \in invo(R)$.
2. *If* a *is a strongly invo-t-clean element, then* $2a = 0$.

Proof. *(1) Since* $a \in N_2(R)$, *then* $a^2 = 0$ *and* $a = u + t$, *where* $u \in invo(R)$ *and* $t \in Tri(R)$. $0 = a^2 = (u+t)^2 = 1+ut+tu+t^2$. *Multiply both sided from the right and left by* t. $2t^2 + tut^2 + t^2ut = 0$ *Since* $a^2 = 0$, *then* $0 = (ua)^2u + ta^2t = u(u + t)^2u + t(u + t)^2u$ $= 1 + tu + ut + ut^2u + t^2 + tut^2 + t^2ut + t^2 = -t^2 + ut^2u$. *So* $t^2 = ut^2u$ *implies* $ut^2 = t^2u$.
$1 + ut + tu + t^2 = 0 = 2t^2 + tut^2 + t^2ut = 2t^2 + tu + ut$ $t^2 = 1$.
(2): $0 = a^2 = (u + t)^2 = 1 + 2ut + t^2$. *Multiply both sided from the right by* t, $0 = 2ut^2 + 2t = 2t(u + t) = 2ta = 2t^2a = 2a$ *from(1).* ■

Remark 2.8. *The invo-t-clean elements in* $N_2(Z_n)$ *is either* 0 *or* $0, \frac{n}{2}$*.For example*

1. *For an invo-t-clean ring take* Z_{12}*, then every elements in* $N_2(Z_{12}) = 0, 6$ *is invo-t-clean where* $6 = 1 + 5 = 7 + 11$*, where* $invo(Z_{12}) = \{1, 5, 7, 11\}$*,* $Tri(Z_{12}) = \{0, 1, 3, 4, 5, 7, 8, 9, 11\}$.

2. *For a ring which is not invo-t-clean take* Z_{36}*,* $N_2(Z_{36}) = 0, 6, 12, 24, 30$ *the only invo-t-clean elements in* $N_2(Z_{36})$ *is* $0, 18$*,* $18 = 1 + 17 = 19 + 35$*, where* $invo(Z_{36}) = \{1, 17, 19, 35\}$*,* $Tri(Z_{36}) = \{0, 1, 8, 9, 17, 19, 27, 28, 35\}$.

In noncommutative ring R, the invo-t-clean elements in $N_2(R)$ need not strongly invo-t-clean elements, but it is still satisfies the proposition 2.7, which every elements in $N_2(R)$ is sum of two involution elements, for example the ring $T_2(Z_3)$

$N_2(T_2(Z_3)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \right\}$ where $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Proposition 2.9. *Let* a *is a strongly invo-t-clean element in* R*. Then*

1. $-a$ *is strongly invo-t-clean element.*

2. au *is strongly invo-t-clean element.*

3. at *is a sum of idempotent and tripotent elements.*

4. *If* 2 *is invertible. Then* $a^2 - 1$ *is r-clean element.*

5. *If* $char(R) = 4$*. Then* $a = (u - 2t) + 3t, u - 2t \in invo(R)$ *and* $3t \in Tri(R)$.

6. *If* $char(R) = 6$*. Then* $a^2 \in Idm(R)$.

Proof. *(1), (2) and (3) is clearly.*
(4) since a *is strongly invo-t-clean element, then there exists* $u \in invo(R)$ *and* $t \in Tri(R)$ *such that* $a = u + t$ *and* $ut = tu$*,* $a^2 - 1 = (u + t)^2 - 1 = 1 + 2ut + t^2 - 1 = 2ut + t^2$*. For proving* $2ut$ *is regular. Set* $b = ut2^{-1}$*,* $(2ut)b(2ut) = (2ut)(ut2^{-1})(2ut) = 2ututut = 2ut$*. Therefore,* $a^2 - 1$ *is a r-clean element.*
(5) since $a = u + t = (u - 2t) + 3t$*,* $(u - 2t)^2 = u^2 - 4ut + 4t^2 = 1, u - 2t \in invo(R)$*,* $(3t)^3 = 27t^3 = 3t \in Tri(R)$.
(6) since $a = u + t, a - t = u$*,* $(a - t)^2 = u^2 = 1$*,* $a^2 - 2at + t^2 = 1$*, multiply by* t*,* $a^2t - 2at^2 + t^3 = t, a^2t = 2at^2$ *multiply by* t *,* $a^2t^2 = 2at$*,* $a^2 - 2at = 1 - t^2$*,* $(a^2 - 2at)^2 = (1 - t^2)^2$*,* $a^4 - 4a^3t + 4a^2t^2 = 1 - t^2$*,* $a^4 - 4a(t(a)^2t^2) + 4a^2t^2 = 1 - t^2$*,* $a^4 - 4at(2at) + 4(2at) = 1 - t^2$*,* $a^4 - 8a^2t^2 + 8at = 1 - t^2$*,* $a^4 - 16at + 8at = a^2 - 2at$*,* $a^4 - 6at = a^2, so a^4 = a^2$. ■

Proposition 2.10. *If R is a ring with $J(R)$ is strongly invo-t-clean. Then $J(R)$ is nil with index of nilpotent at most 3, and $char(J(R)) = 4$.*

Proof. *Let $a \in J(R)$, since $J(R)$ is strongly invo-t-clean, then there exists $u \in invo(R)$ and $t \in Tri(R)$ such that $a = u + t$, $-u + a = t \in U(R) + J(R) = U(R)$, so $t \in U(R) \cap J(R) = U_2(R)$, $t^2 = 1$. Now $a^2 = (u+t)^2 = 2 + 2ut = 2u(u+t) = 2ua$, multiply $a^2 = 2ua$ both side by a, $a^3 = 2ua^2 = 2u(2ua) = 4a$, replacing a by $2a$, $(8a)^3 = 8a, 8a(1 - a^2) = 0$, since $a \in J(R), 1 - a^2 \in U(R)$, get that $8a = 0$. For $a^2 = 2ua$ replacing a by $2a$, $4a^2 = 4ua$. by multiply $a^2 = 2ua$ by 4 get $(4a)^2 = 8ua$. So $8ua = 4ua, 4ua = 0, 4a = 0$. $a^3 = 4a = 0$.* ■

3. Applications in Graph Theory

Definition 3.1. *The graph of invo-t-clean ring R, which is denoted by $Cl_t(R)$ has vertex set $V(Cl_t(R)) = \{(u,t) : u \in invo(R) \& t \in Tri(R)\}$ and has the edge set*

$F(Cl_t(R)) = \{h_1h_2 : h_1 = (u_1,t_1), h_2 = (u_2,t_2), u_1 + u_2 = 0$ or $t_1 \cdot t_2 = 0$, $u_i \in invo(R)$, $t_i \in Tri(R)$, $u_i + t_i$ is an invo-t-clean element $i = 1,2$ $\}$.

For example, let $V(Cl_t(Z_4)) = \{[1,0],[3,0],[1,1],[1,3],[3,1],[3,3]\}$ Then,

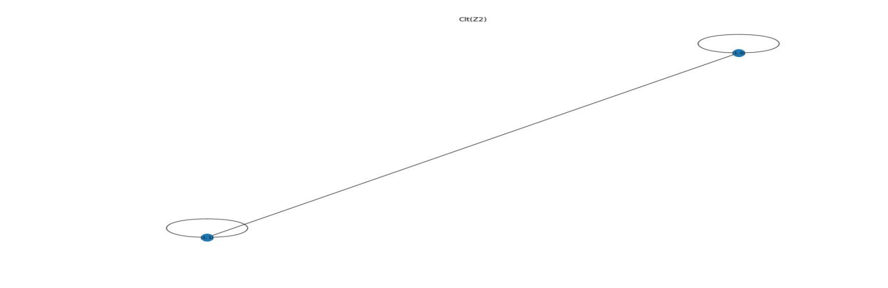
$F(Cl_t(Z_4)) = \{[1,0][1,0],[1,0][3,0],[1,0][1,1],[1,0][3,1],[1,0][1,3],[1,0][3,3],[3,0][3,0], [3,0][1,1],[3,0][3,1],[3,0][1,3],[3,0][3,3],[1,1][3,1],[1,1][3,3],[1,3][3,1],[1,3][3,3]\}$.

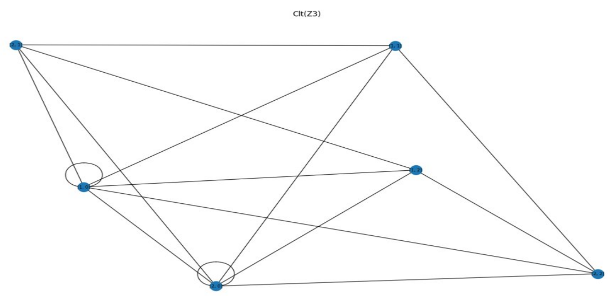
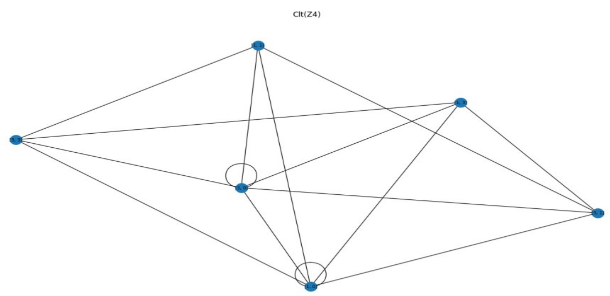
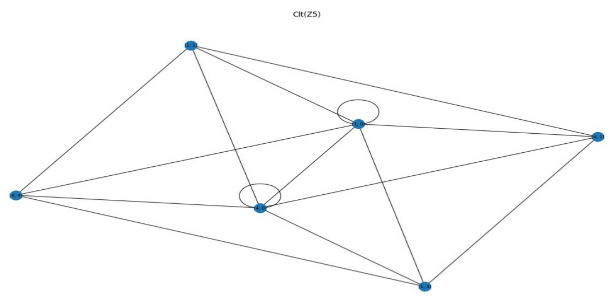
From the above example, we see that there is a loop at vertices $[1,0]$ and $[3,0]$ and because it is not important in graph theory, we will ignore it, and the girth of $Cl_t(Z_4)$ is three. There are many recent studies that connected the ring theory with the graph theory to review the paper, see [3] [10] [1].

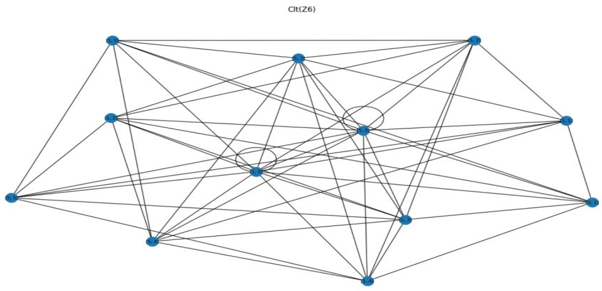
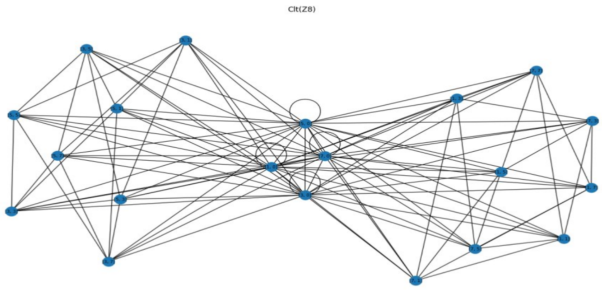
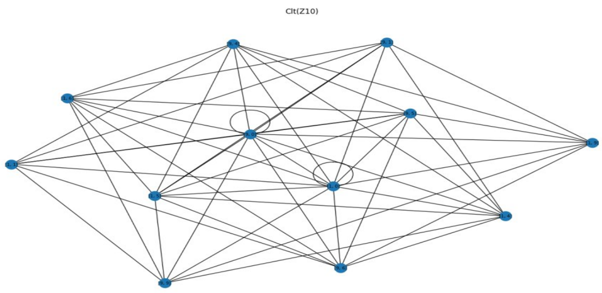
Using the Python Programming Language, we get all connected graphs and the result from definition 3.1, and then we get some invariant properties in graph theory, such as: Hosoya polynomial, Wiener index, average distance and average degree.

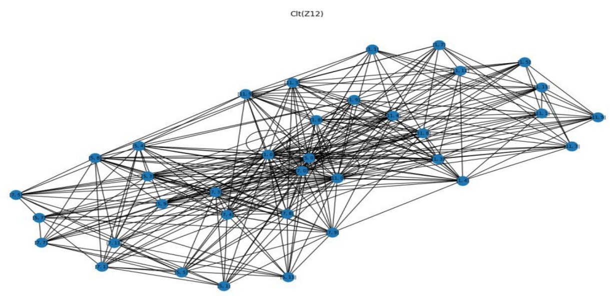
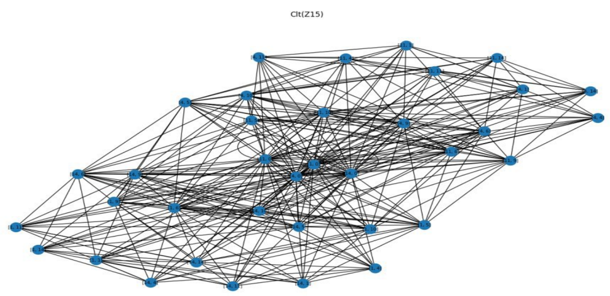
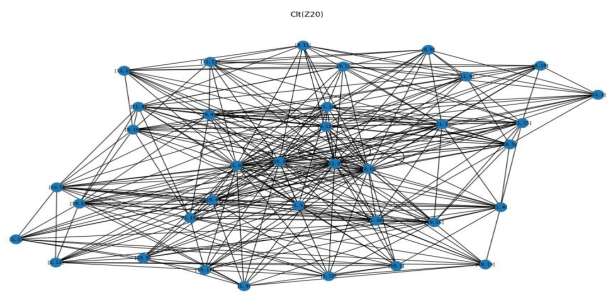
In the next table, the graphs structures corresponding invo-t-clean ring Z_n, $n = 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$.

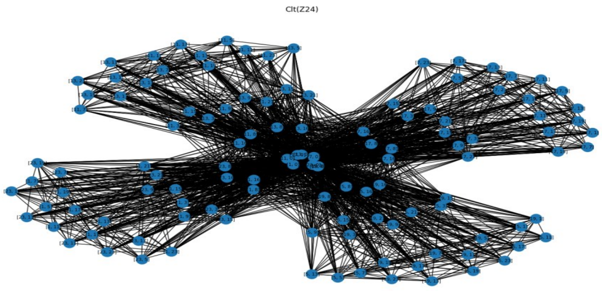
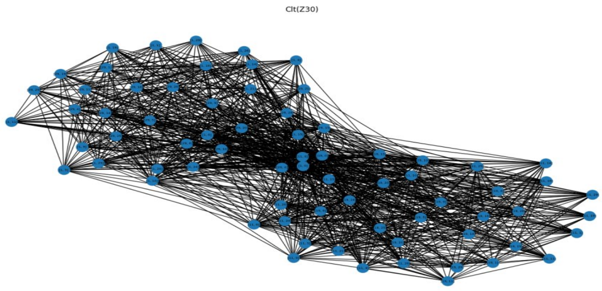
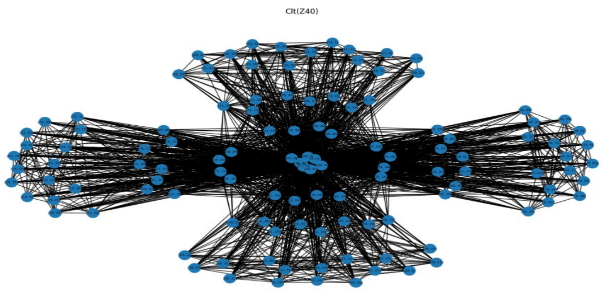
Table 1: The graphs $Cl_t(Z_n)$.

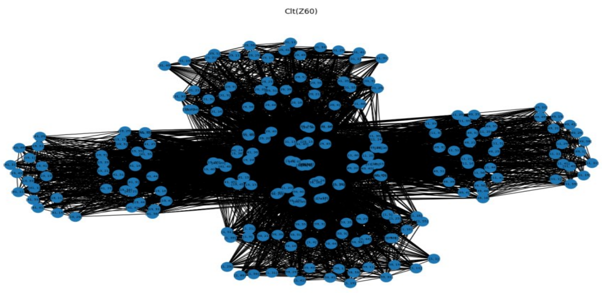
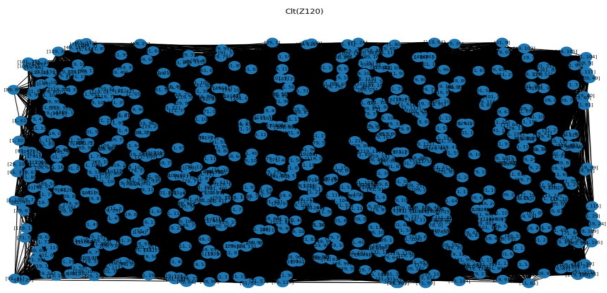
Ring	Graph
Z_2	
	$H(Cl_t(Z_2);x) = 2 + x$, $W(Cl_t(Z_2)) = 1$, $ad(Cl_t(Z_2)) = 1 D(Cl_t(Z_2)) = 1$

Ring	Graph
Z_3	
	$H(Cl_t(Z_3);x) = 6 + 13x + 2x^2$, $W(Cl_t(Z_3)) = 17$, $ad(Cl_t(Z_3)) = 4.333, D(Cl_t(Z_3)) = 1.1333$
Z_4	
	$H(Cl_t(Z_4);x) = 6 + 13x + 2x^2$, $W(Cl_t(Z_4)) = 17$, $ad(Cl_t(Z_4)) = 4.333, D(Cl_t(Z_4)) = 1.1333$
Z_5	
	$H(Cl_t(Z_5);x) = 6 + 13x + 2x^2$, $W(Cl_t(Z_5)) = 17$, $ad(Cl_t(Z_5)) = 4.333, D(Cl_t(Z_5)) = 1.1333$

Ring	Graph
Z_6	
	$H(Cl_t(Z_6);x) = 12 + 30x + 16x^2$, $W(Cl_t(Z_6)) = 82$, $ad(Cl_t(Z_6)) = 5, D(Cl_t(Z_6)) = 1.242$
Z_8	
	$H(Cl_t(Z_8);x) = 20 + 102x + 88x^2$, $W(Cl_t(Z_8)) = 278$, $ad(Cl_t(Z_8)) = 10.2, D(Cl_t(Z_8)) = 1.46$
Z_{10}	
	$(Cl_t(Z_10);x) = 12 + 50x + 16x^2$, $W(Cl_t(Z_10)) = 82$, $ad(Cl_t(Z_10)) = 5, D(Cl_t(Z_10)) = 1.242$

Ring	Graph
Z_{12}	
	$H(Cl_t(Z_12);x) = 36 + 310x + 320x^2$, $W(Cl_t(Z_12)) = 950$, $ad(Cl_t(Z_12)) = 17.222, D(Cl_t(Z_12)) = 1.50$
Z_{15}	
	$H(Cl_t(Z_15);x) = 36 + 310x + 320x^2$ $W(Cl_t(Z_15)) = 950$, $ad(Cl_t(Z_15)) = 17.222, D(Cl_t(Z_15)) = 1.50$
Z_{20}	
	$H(Cl_t(Z_20);x) = 36 + 310x + 320x^2$, $W(Cl_t(Z_20)) = 950$, $ad(Cl_t(Z_20)) = 17.222, D(Cl_t(Z_20)) = 1.50$

Ring	Graph
Z_{24}	
	$H(Cl_t(Z_24);x) = 120 + 2156x + 4984x^2$, $W(Cl_t(Z_24)) = 12124$, $ad(Cl_t(Z_24)) = 35.9333, D(Cl_t(Z_24)) = 1.69$
Z_{30}	
	$H(Cl_t(Z_30);x) = 72 + 1096x + 1460x^2$, $W(Cl_t(Z_30)) = 4016$, $ad(Cl_t(Z_30)) = 30.444, D(Cl_t(Z_30)) = 1.57$
Z_{40}	
	$H(Cl_t(Z_40);x) = 120 + 2156x + 4984x^2$, $W(Cl_t(Z_40)) = 12124$, $ad(Cl_t(Z_40)) = 35.9333, D(Cl_t(Z_40)) = 1.69$

Ring	Graph
Z_{60}	
	$H(Cl_t(Z_60);x) = 216 + 6412x + 16808x^2$ $W(Cl_t(Z_60)) = 40028$, $ad(Cl_t(Z_60)) = 59.370, D(Cl_t(Z_60)) = 1.72$
Z_{120}	
	$H(Cl_t(Z_120);x) = 720{+}43192x{+}215648x^2$, $W(Cl_t(Z_120)) = 474488$, $ad(Cl_t(Z_120)) = 119.97, D(Cl_t(Z_60)) = 1.83$

Theorem 3.2. *The graph $Cl_t(R)$ is connected for any invo-t-clean ring and has a diameter less than and equal 2.*

Proof. *Since $(u, 0)$ belong to $V(Cl_t(R))$, then $(u, 0)$ is adjacent to all vertices in a graph $Cl_t(R)$. Hence then $Cl_t(R)$ is connected. Now to prove that $Cl_t(R)$ has diameter less and equal 2. Let (u_1, t_1),$(u_2, t_2) \in V(Cl_t(R))$, then we have the following two cases:*
Case1: If $t_1 = 0$, then $d((u_1, t_1), (u_2, t_2)) = 1$.
Case2: If $u_1 + u_2 \neq 0$ and $t_1 \cdot t_2 \neq 0$, $u_i \in U_2(R)$, $t_i \in Tri(R)$, $i = 1, 2$,then $d((u_1, t_1), (u_2, t_2)) = 2$. ■

Theorem 3.3. *The graph $Cl_t(R)$ has girth $g(Cl_t(R))$ equal three for all $|Cl_t(R)| \geq 3$.*

Proof. *Since R is a ring with identity, there are two vertices $(1, 0)$ and $(-1, 0)$ belong to $V(Cl_t(R))$. Since $|Cl_t(R)| \geq 3$, there is another vertex (u, t), $(u, t) \neq (1, 0), (-1, 0)$ such that $(1, 0)$ and $(-1, 0)$ are adjacent to (u, t) by Theorem 3.2, also since $(1, 0)$ and $(-1, 0)$ are adjacent. Hence then $g(Cl_t(R)) = 3$.* ■

4. Conclusion

Through this generalization, 14 rings were obtained in Z_n for invo-t-clean rings, but it was 7 rings when it be invo-clean. Also through this generalization we obtained a representation of it in graph theory with a study of some properties.

References

[1] F.H. Abdulqadr. Maximal ideal graph of commutative rings. *Iraqi Journal of Science*, pages 2070–2076, 2020.

[2] Nahid Ashrafi and Ebrahim Nasibi. r-clean rings. *arXiv preprint arXiv:1104.2167*, pages 1–7, 2011.

[3] Mohammed Authman, Husam Q Mohammad, and Nazar H Shuker. Vertex and region colorings of planar idempotent divisor graphs of commutative rings. *Iraqi Journal For Computer Science and Mathematics*, 3(1):71–82, 2022.

[4] Istvan Beck. Coloring of commutative rings. *Journal of algebra*, 116(1):208–226, 1988.

[5] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs & digraphs.* CRC Press, Taylor & Francis Group, 2016.

[6] Peter V Danchev. Invo-clean unital rings. *Communications of the Korean Mathematical Society*, 32(1):19–27, 2017.

[7] I Gutman. Some properties of the wiener polynomial; graph theory notes of new york; xxv, 1993.

[8] Mohammad Habibi, Ece Yetkın Çelıkel, and Cihat Abdıoğlu. Clean graph of a ring. *Journal of Algebra and Its Applications*, 20(09):2150156, 2021.

[9] W Keith Nicholson. Lifting idempotents and exchange rings. *Transactions of the American Mathematical Society*, 229:269–278, 1977.

[10] Avinash Patil, Anil Khairnar, and PS Momale. Zero-divisor graph of a ring with respect to an automorphism. *Soft Computing*, 26(5):2107–2119, 2022.

[11] John Von Neumann. On regular rings. *Proceedings of the National Academy of Sciences*, 22(12):707–713, 1936.

[12] Harry Wiener. Structural determination of paraffin boiling points. *Journal of the American chemical society*, 69(1):17–20, 1947.