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Global Asymptotical Stability of Delayed Impulsive Neural Networks without Lipschitz Neuron Activations

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Abstract. In this paper, based on the homeomorphism theory and Lyapunov functional method, we investigate global asymptotical stability for a novel class of delayed impulsive neural networks without Lipschitz neuron activations. Some sufficient conditions are derived which ensure the existence, uniqueness, and global asymptotical stability of the equilibrium point of neural networks. Finally, a numerical example is given to demonstrate the improvements of the paper.

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1. Introduction

In the design of neural networks, the model of neural networks is descried by the system of nonlinear ordinary differential equations or the system of nonlinear functional differential equations. In generality, these nonlinear systems possibly show complex dynamic behaviors, such as, periodic oscillatory, bifurcation, chaos, etc. However, in practical applications, especially for solving linear and quadratic programming problems in real time, it requires that networks have good convergent property. Under these good convergent property, the validity can be guaranteed during numeral solving. Due to these, stability analysis for neural networks with or without time delays has received a great of attention (see $[1-10]$). Recently, impulsive neural networks have been extensively studied in both theory and applications (see $[5-10]$). However, in the existing literatures, almost all results on the stability of neural networks are obtained under Lipschitz neuron activations $[1-6,9,10]$. When neuron activation functions do not satisfy Lipschitz conditions, people want to know whether the neural networks is stable. In practical engineering applications, people also need to present new neural

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networks. Therefore, developing a new class of neural networks without Lipschitz neuron activation functions and giving the conditions of the stability of new neural networks are very interesting and valuable.

In this paper, we investigate a general class of delayed neural networks with impulses where the neuron activations do not satisfy Lipschitz conditions. To the best of authors' knowledge, this is the first time to study the existence, uniqueness, and global asymptotical stability of equilibrium point for the neural networks developed by us.

Consider a general delayed neural networks with impulses:

$$\begin{cases} \dot{x}_i(t) & = & -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij})) + I_i, \quad t \neq t_k, \\ \triangle x_i(t_k) & = & x_i(t_k^+) - x_i(t_k^-) = J_{ik}(x_i(t_k)), \quad k = 1, 2, \cdots, \quad i = 1, \cdots, n, \end{cases} \tag{1}$$

where n denotes the number of the neurons; $x_i(t)$ is the state of the ith neuron at time t; $d_i > 0$ is the neural self-inhibitions of the ith neuron; $g_j(\cdot)$ represents the input-output activation of the jth neuron, $g_j(\cdot)$ is continuous and monotone nondecreasing; a_{ij} and b_{ij} denote the connection of the jth neuron to the ith neuron at time t and $t - \tau_{ij}$, respectively; I_i is the external bias on the ith neuron; $0 \leq \tau_{ij} \leq \tau$, τ is a positive constant; J_{ik} shows impulsive perturbation of the ith neuron at time t_k; $\triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-), k = 1, 2, \cdots$, are the impulses at moments t_k, and $0 < t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim\limits_{k\to\infty} t_k = \infty$.

The system (1) is supplemented with the initial conditions of the type $x(t) = \phi(t) = (\phi_1, \cdots, \phi_n)^T, -\tau \leq t \leq 0$ in which $\phi(t) \in C([-\tau, 0]; R^n)$ is a continuous function.
$C([-\tau, 0]; R^n)$ is a Banach space of continuous mapping which maps $[-\tau, 0]$ into R^n with a topology of uniform convergence.

For convenience, we introduce the following notations: Let matrix $Q = (q_{ij})_{n\times n}$, $R = (r_{ij})_{m\times n}$, Q^{-1} denotes the inverse of Q, $\|R\|_1$ and $\|R\|_\infty$ represent the first norm and infinity norm of matrix R, respectively. That is, $\|R\|_1 = \max\limits_{1\leq j\leq n} \sum\limits_{i=1}^{m} |r_{ij}|$, $\|R\|_\infty = \max\limits_{1\leq i\leq m} \sum\limits_{j=1}^{n} |r_{ij}|$. Symmetric matrix $S = (s_{ij})_{n\times n}$, $S > 0$ ($S \geq 0, S < 0, S \leq 0$) means that S is positive definite (positive semi-definite, negative definite, negative semi-definite). Given the vector $\psi = (\psi_1, \cdots, \psi_n)^T \in R^n$, $\|\psi\| = \max\limits_{1\leq i\leq n} |\psi_i|$. I denotes identical matrix. We will sometimes write $x(t)$ as x, $f(x(t))$ as $f(x)$.

Definition 1. *A function $x(t) : [-\tau, +\infty] \to R^n$ is said to be a solution of system (1) with initial conditions $x(t) = \phi(t), t \in [-\tau, 0]$, if the following conditions are satisfied:*
(1) $x(t)$ is piecewise continuous with first kind discontinuity at points $t_k, k = 1, 2, \cdots$. Moreover, $x(t)$ is right continuous at each discontinuity points;
(2) $x(t)$ satisfies system (1) for $t \geq 0$, and $x(s) = \phi(s)$ for $s \in [-\tau, 0]$.

Definition 2. *A constant vector $x^* = (x_1^*, \cdots, x_n^*)^T \in R^n$ is an equilibrium point of system (1.1)* *if and only if* x^* *is a solution of the following equations:*

$$-d_i x_i + \sum_{j=1}^{n} (a_{ij} + b_{ij}) g_j(x_j) + I_i = 0, \quad i = 1, \cdots, n, \tag{2}$$

and the impulsive jumps $J_{ik}(\cdot)$ *are assumed to satisfy* $J_{ik}(x_i^*) = 0, \quad k = 1, 2, \cdots, i = 1, \cdots, n.$

Lemma 1. *Continuous map* $H(x) : R^n \to R^n$ *is homeomorphic, if: (1)* $H(x)$ *is injective; (2)* $\lim_{\|x\|_p \to \infty} \|H(x)\|_p = \infty$.

2. Existence and Uniqueness of the Equilibrium Point

First, we definite the map $H(x)$ associated with (2) as follows:

$$H(x) = (H_1(x), \cdots, H_n(x))^T \tag{3}$$

where $H_i(x) = -d_i x_i + \sum_{j=1}^{n} (a_{ij} + b_{ij}) g_j(x_j) + I_i, \quad i = 1, \cdots, n.$

Theorem 1. *If there exist positive constants* $p_i > 0, i = 1, \cdots, n$*, such that*

$$p_i(-a_{ii} - |b_{ii}|) - \sum_{j=1, j \neq i}^{n} p_j(|a_{ji}| + |b_{ji}|) \geq 0, \quad i = 1, \cdots, n,$$

then Eq.(2) has a unique solution.

Proof. In order to complete the proof, we divide the proof into two steps.

Step 1. Let x and y be two different vectors in R^n, then we have

$$\begin{aligned}
&\sum_{i=1}^{n} p_i sign(x_i - y_i)(H_i(x) - H_i(y)) \\
\leq & -\sum_{i=1}^{n} p_i d_i |x_i - y_i| + \sum_{i=1}^{n} p_i (a_{ii} + b_{ii}) |g_i(x_i) - g_i(y_i)| \\
& + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} p_i (|a_{ij}| + |b_{ij}|) \cdot |g_j(x_j) - g_j(y_j)| \\
\leq & -\sum_{i=1}^{n} p_i d_i |x_i - y_i| + \sum_{i=1}^{n} p_i (a_{ii} + |b_{ii}|) |g_i(x_i) - g_i(y_i)| \\
& + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} p_j (|a_{ji}| + |b_{ji}|) \cdot |g_i(x_i) - g_i(y_i)| \\
\leq & -\sum_{i=1}^{n} p_i d_i |x_i - y_i| < 0
\end{aligned} \tag{4}$$

Moreover, there exists $k_0 \in \{1, \cdots, n\}$ such that $H_{k_0}(x) \neq H_{k_0}(y)$. That is, $H(x) \neq H(y)$ for all $x \neq y$.

Step 2. In (4), let $y = 0$, we get

$$\sum_{i=1}^{n} p_i(H_i(x) - H_i(0))sign(x_i - 0) \leq -\sum_{i=1}^{n} p_i d_i \left|x_i\right| \leq -p_{min} \sum_{i=1}^{n} \left|x_i\right| \tag{5}$$

where $p_{min} = \min\limits_{1 \leq i \leq n} \{p_i d_i\}$. From (5), it follows that

$$\begin{aligned} p_{min}\|x\|_1 &\leq \left|\sum_{i=1}^{n} p_i(H_i(x) - H_i(0))\right| \\ &\leq p_{max} \sum_{i=1}^{n} \left|(H_i(x) - H_i(0))\right| \\ &\leq p_{max}\|H(x) - H(0)\|_1 \\ &\leq p_{max}(\|H(x)\|_1 + \|H(0)\|_1) \end{aligned}$$

where $p_{max} = \max\{p_1, \cdots, p_n\}$. We obtain

$$\|H(x)\|_1 \geq \frac{p_{min}\|x\|_1 - p_{max}\|H(0)\|_1}{p_{max}}$$

from which it can be easily concluded that $\|H(x)\|_1 \to \infty$ as $\|x\|_1 \to \infty$. Hence, we have proved that $H(x)$ is a homeomorphism on R^n. That is, Eq.(2) has a unique solution.

Theorem 2. *If $\left|g_i(x_i)\right| \to \infty$ as $\left|x_i\right| \to \infty, i = 1, \cdots, n$, and there exists a positive constant r such that*

$$A + A^T + (\frac{1}{r}\|B\|_\infty + r\|B\|_1)I \leq 0$$

where $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, then Eq.(2) has a unique solution.

Proof. In order to complete the proof, we divide the proof into two steps.

Step 1. Let x and y be two different vectors in R^n, by $g_i(\cdot)$ is monotone nondecreasing, $x \neq y$ will imply two cases: (i) $x \neq y$ and $g(x) - g(y) \neq 0$; (ii) $x \neq y$ and $g(x) - g(y) = 0$.

First, consider the case (i) where $x \neq y$ and $g(x) - g(y) \neq 0$. In this case, there exists $h \in \{1, \cdots, n\}$ such that $(x_h - y_h)(g_h(x_h) - g_h(y_h)) > 0$ and $(x_i - y_i)(g_i(x_i) - g_i(y_i)) \geq 0$ for $i \neq h$. Moreover, we have

$$\begin{aligned} & 2(g(x) - g(y))^T \cdot (H(x) - H(y)) \\ \leq & -2\sum_{i=1}^{n} d_i(x_i - y_i)(g_i(x_i) - g_i(y_i)) + (g(x) - g(y))^T(A + A^T)(g(x) - g(y)) \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{r}\left|b_{ij}\right|(g_i(x_i) - g_i(y_i))^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} r\left|b_{ij}\right|(g_j(x_j) - g_j(y_j))^2 \end{aligned}$$

$$\begin{aligned}
&\le -2\sum_{i=1}^{n} d_i(x_i-y_i)(g_i(x_i)-g_i(y_i))\\
&\quad + (g(x)-g(y))^T[A+A^T+(\frac{1}{r}\|B\|_\infty + r\|B\|_1)I]\cdot(g(x)-g(y))\\
&\le -2\sum_{i=1}^{n} d_i(x_i-y_i)(g_i(x_i)-g_i(y_i))\\
&\le -2d_h(x_h-y_h)(g_h(x_h)-g_h(y_h))<0 \qquad (6)
\end{aligned}$$

Hence, $H_h(x)\neq H_h(y)$. That is, $H(x)\neq H(y)$ when $x\neq y$ and $g(x)\neq g(y)$.

Now consider the case (ii) where $x\neq y$ and $g(x)-g(y)=0$. In the case, we have

$$H(x)-H(y)=-D(x-y)\neq 0$$

where $D=diag(d_1,\cdots,d_n)$. Thus, $H(x)\neq H(y)$ for all $x\neq y$ and $g(x)=g(y)$. Hence, we have proved that $H(x)\neq H(y)$ when $x\neq y$.

Step 2. In (2.4), let $y=0$, we get

$$\begin{aligned}
2(g(x)-g(0))^T\cdot(H(x)-H(0)) &\le -2\sum_{i=1}^{n} d_i(x_i-0)(g_i(x_i)-g_i(0))\\
&\le -2dx^T(g(x)-g(0)) \qquad (7)
\end{aligned}$$

where $d=\min\{d_1,\cdots,d_n\}$. From (7) and g_i is monotone nondecreasing, it follows that

$$0\le d\sum_{i=1}^{n} x_i(g_i(x_i)-g_i(0))\le \sum_{i=1}^{n}\left|(g_i(x_i)-g_i(0))\cdot(H_i(x)-H_i(0))\right|. \qquad (8)$$

If $\lim\limits_{\|x\|\to\infty}\|H(x)\|\neq\infty$, then there exists a sequence $\{x^p\}$ such that $\lim\limits_{p\to\infty}\|x^p\|=\infty$ and for all p,

$$\|H(x^p)\|\le M_1,$$

where M_1 is a positive constant. Therefore, there exists a subsequence (for convenience, we also denote it as $\{x^p\}$) and a nonempty set $W\subset\{1,\cdots,n\}$, such that the follows hold:

(1) $\lim\limits_{p\to\infty}\left|x_i^p\right|=\infty$ for all $i\in W$;

(2) there exists a positive constant M_2 such that $\left|x_i^p\right|\le M_2$ for all p and $i\in\{1,\cdots,n\}\setminus W$;

(3) $\left\|H_i(x^p)\right\|\le M_1$ for all i and p.

Since $g_i(s)$ is continuous on $[-M_2,M_2]$, there exists a positive constant M_3 such that $\left|g_i(s)\right|\le M_3$ for all $s\in[-M_2,M_2]$ and $i\in\{1,\cdots,n\}\setminus W$. Thus, $\left|g_i(x_i^p)\right|\le M_3$ for all p and $i\in\{1,\cdots,n\}\setminus W$. Moreover, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|(g_i(x_i^p)-g_i(0))\cdot(H_i(x^p)-H_i(0))\right|\\
&=\sum_{i\in W}\left|(g_i(x_i^p)-g_i(0))\cdot(H_i(x^p)-H_i(0))\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\notin W}\left|(g_i(x_i^p)-g_i(0))\cdot(H_i(x^p)-H_i(0))\right| \\
&\le \sum_{i\in W}\left|g_i(x_i^p)-g_i(0)\right|\cdot(M_1+\left|H_i(0)\right|) \\
&+\sum_{i\notin W}(M_3+\left|g_i(0)\right|)\cdot(M_1+\left|H_i(0)\right|) \\
&\le M\sum_{i\in W}\left|g_i(x_i^p)-g_i(0)\right|+M
\end{aligned} \tag{9}
$$

where $M=\max\{\max_{1\le j\le n}\left\{M_1+\left|H_j(0)\right|\right\},\sum_{i\notin W}(M_3+\left|g_i(0)\right|)\cdot(M_1+\left|H_i(0)\right|)\}$. By $g_i(\cdot)$ is monotone nondecreasing, then we have

$$
\begin{aligned}
\sum_{i=1}^{n}x_i^p(g_i(x_i^p)-g_i(0)) &= \sum_{i\in W}x_i^p(g_i(x_i^p)-g_i(0))+\sum_{i\notin W}x_i^p(g_i(x_i^p)-g_i(0)) \\
&\ge \sum_{i\in W}\left|x_i^p\right|\cdot\left|(g_i(x_i^p)-g_i(0)\right|
\end{aligned} \tag{10}
$$

Substituting (9) and (10) into (8), we can obtain for all p

$$
\begin{aligned}
d\sum_{i\in W}\left|x_i^p\right|\cdot\left|(g_i(x_i^p)-g_i(0)\right| &\le d\sum_{i=1}^{n}x_i^p(g_i(x_i^p)-g_i(0)) \\
&\le \sum_{i=1}^{n}\left|(g_i(x_i^p)-g_i(0))\cdot(H_i(x_i^p)-H_i(0))\right| \\
&\le M\sum_{i\in W}\left|g_i(x_i^p)-g_i(0)\right|+M
\end{aligned}
$$

So we have

$$
\sum_{i\in W}(d\left|x_i^p\right|-M)\cdot\left|g_i(x_i^p)-g_i(0)\right|\le M \tag{11}
$$

Since $\lim_{p\to\infty}\left|x_i^p\right|=\infty$ for all $i\in W$ and $\lim_{\left|x_i\right|\to\infty}\left|g_i(x_i)\right|=\infty$, there exists a positive constant P such that for all $p>P$, $\left|(g_i(x_i^p)-g_i(0)\right|\ge 1$ and $\left|x_i^p\right|>\frac{2M}{d}$. Hence,

$$
\sum_{i\in W}(d\left|x_i^p\right|-M)\cdot\left|g_i(x_i^p)-g_i(0)\right|>M \tag{12}
$$

which is contradict to (11). So $H(x)$ is a homeomorphism on R^n. That is, Eq.(2) has a unique solution.

Theorem 3. *Under assumptions of Theorem 1 (or Theorem 2), system (1) has a unique equilibrium point* $x^*=(x_1^*,\cdots,x_n^*)^T$.

Proof. By Definition 2 and Theorem 1 (or Theorem 2), it is obvious that the constant vector x^* is the unique equilibrium point of system (1).

3. Global Asymptotic Stability of the Equilibrium Point

In this section, we aim to find some sufficient conditions ensuring the global asymptotic stability of the equilibrium point of system (1). The equilibrium point of system (1) is said to be globally asymptotically stale if it is locally stable in sense of Lyapunov and globally attractive, i.e., every solution of system (1) corresponding to an arbitrary given set of initial conditions satisfy $\lim_{t\to\infty} x_i(t) = x_i^*, i = 1, \cdots, n$. To prove the global asymptotic stability of the equilibrium point, we will employ the Lyapunov direct method. Namely, the equilibrium point x^* is stable and every solution is bounded if there exists a continuously differentiable Lyapunov function $V : R^n \to R$ which is positive definite and radially unbounded, i.e., $V(x^*) = 0$, $V(x) > 0$ for $x \neq 0$, $\lim_{\|x-x^*\|\to\infty} V(x) = \infty$, such that the time-derivative of V along the solution of system (1) is negative semi-definite. If, in addition, $\dot{V}(x)$ is negative definite, then the equilibrium point of system (1) is globally asymptotically stable.

Theorem 4. *Under assumptions of Theorem 1, further if the following conditions are satisfied*

$$J_{ik}(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*), \quad k = 1, 2, \cdots, \quad i = 1, \cdots, n,$$

where $x^* = (x_1^*, \cdots, x_n^*)^T$ *is the equilibrium point of system (1),* $0 < \gamma_{ik} < 2$*, then system (1) has a unique equilibrium point which is globally asymptotically stable.*

Proof. In order to complete the proof, we divide the proof into four steps.

Step 1. Consider the following system:

$$\begin{cases} \dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij})) + I_i, \quad t \in [0, t_1], \\ x_i(t) = \phi_i(t), \quad t \in [-\tau, 0], \quad i = 1, \cdots, n. \end{cases} \tag{13}$$

By g_j is a continuous function, $U_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij})) + I_i$ is continuous and local bounded. It is easy to obtain the existence of a solution of system (3.1) on $[0, t^*(\phi))$, where $t^*(\phi) \in (0, t_1)$ or $t^*(\phi) = t_1$, and $[0, t^*(\phi))$ is the maximal right-side existence interval of the solution of system (13). We denote this solution by $x(t, \phi)$, $x(t, \phi) = (x_1(t, \phi_1), \cdots, x_n(t, \phi_n))^T$.

Make a transformation $z(t) = x(t) - x^*$, system (13) is transformed into

$$\dot{z}_i(t) = -d_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t - \tau_{ij})), \quad t \in [0, t_1], \quad i = 1, \cdots, n, \tag{14}$$

where $f_i(z_i(t)) = g_i(z_i(t) + x_i^*) - g_i(x_i^*)$. Hence, $z(t, \tilde{\phi}) = x(t, \phi) - x^*$ is a solution of system (14) with initial conditions $z(t) = \phi(t) - x^*, t \in [-\tau, 0]$ on $[0, t^*(\phi))$.

Step 2. Consider the following Lyapunov functional

$$V(z(t)) = \sum_{i=1}^{n} p_i \{ \left| z_i(t) \right| + \sum_{j=1}^{n} \int_{t-\tau_{ij}}^{t} \left| b_{ij} \right| \cdot \left| f_j(z_j(\theta)) \right| \mathrm{d}\theta \}. \tag{15}$$

Obviously, $V(z)$ is positive definite and $\lim_{\|z\|\to\infty} V(z)=\infty$. Calculating the derivative of $V(z)$ along the solution $z(t,\tilde{\phi})$ of system (14) on $[0,t^*(\phi))$, then we can get

$$\begin{aligned}\dot{V}(z(t,\tilde{\phi})) \le & -\sum_{i=1}^{n}\{p_i d_i \left|z_i(t,\tilde{\phi})\right| - p_i(a_{ii}+\left|b_{ii}\right|)\left|f_i(z_i(t,\tilde{\phi}))\right| \\ & - \sum_{j=1,j\neq i}^{n} p_i(\left|a_{ij}\right|+\left|b_{ij}\right|)\cdot\left|f_j(z_j(t,\tilde{\phi}))\right|\} \\ = & -\sum_{i=1}^{n}\{p_i d_i \left|z_i(t,\tilde{\phi})\right| - p_i(a_{ii}+\left|b_{ii}\right|)\left|f_i(z_i(t,\tilde{\phi}))\right| \\ & - \sum_{j=1,j\neq i}^{n} p_j(\left|a_{ji}\right|+\left|b_{ji}\right|)\cdot\left|f_i(z_i(t,\tilde{\phi}))\right|\} \\ \le & -p_{min}\sum_{i=1}^{n}\left|z_i(t,\tilde{\phi})\right| < 0\end{aligned}$$

where $p_{min}=\min_{1\le i\le n}\{p_i d_i\}$. This implies $V(z(t,\tilde{\phi}))<V(z(0)), t\in[0,t^*(\phi))$. By (15), we can get

$$\sum_{i=1}^{n} p_i\left|z_i(t,\tilde{\phi})\right| < V(z(0)) \tag{16}$$

According to (16), it is easy to derive that $z_i(t,\tilde{\phi}), i=1,\cdots,n$, are bounded on $[0,t^*(\phi))$. By virtue of the continuous theorem of differential equations, we can conclude that system (14) has a solution on $[0,t_1]$, i.e., system (13) has a solution on $[0,t_1]$. We denote this solution of system (13) by $x^0(t)$.

Step 3. Consider the following system:

$$\left\{\begin{array}{lcl} \dot{x}_i(t) & = & -d_i x_i(t)+\sum\limits_{j=1}^{n} a_{ij}g_j(x_j(t))+\sum\limits_{j=1}^{n} b_{ij}g_j(x_j(t-\tau_{ij}))+I_i, \quad t\in[t_1,t_2], \\ x_i(t_1) & = & x_i^0(t_1)+J_{i1}(x_i^0(t_1)), \quad i=1,\cdots,n. \end{array}\right. \tag{17}$$

Arguing as in step 1 and step 2, system (17) has a solution $x^1(t)$ on $[t_1,t_2]$. As inductive step, we can derive that the following system:

$$\left\{\begin{array}{lcl} \dot{x}_i(t) & = & -d_i x_i(t)+\sum\limits_{j=1}^{n} a_{ij}g_j(x_j(t))+\sum\limits_{j=1}^{n} b_{ij}g_j(x_j(t-\tau_{ij}))+I_i, \quad t\in[t_m,t_{m+1}], \\ x_i(t_m) & = & x_i^{m-1}(t_m)+J_{im}(x_i^{m-1}(t_m)), \quad i=1,\cdots,n, \end{array}\right.$$

also has a solution $x^m(t)$ on $[t_m,t_{m+1}]$, $m=2,3,\cdots$.

Define

$$x(t,\phi)=\begin{cases} x^0(t), t\in[0,t_1],\\ x^1(t), t\in(t_1,t_2],\\ \cdots,\\ x^m(t), t\in(t_m,t_{m+1}],\\ \cdots,\end{cases}$$

then $x(t,\phi)$ is the solution of system (1) with initial conditions $x(s)=\phi(s), s\in[-\tau,0]$. This completes the proof of the existence of solutions of system (1).

Step 4. Assume that $x(t)$ is a solution of system (1), and x^* is the unique equilibrium point of system (1). Make a transformation $z(t)=x(t)-x^*$, then system (1) is transformed into the following system:

$$\begin{cases} \dot{z}_i(t) &= -d_iz_i(t)+\sum\limits_{j=1}^n a_{ij}f_j(z_j(t))+\sum\limits_{j=1}^n b_{ij}f_j(z_j(t-\tau_{ij})), \quad t\neq t_k,\\ \triangle z_i(t_k) &= J_{ik}(z_i(t_k))=-\gamma_{ik}z_i(t_k), \quad k=1,2,\cdots, \quad i=1,\cdots,n,\end{cases} \tag{18}$$

where $f_i(z_i(t))=g_i(z_i(t)+x_i^*)-g_i(x_i^*)$.

Consider Lyapunov functional $V(z(t))$, the $V(z(t))$ is the same as (15). Calculating the derivative of $V(z(t))$ along the solution $z(t)$ of system (18) for any t, $t\neq t_k, k=1,2,\cdots$. Arguing as in step 2, we have $\dot{V}(z(t))<0, t\neq t_k, k=1,2,\cdots$. Also,

$$\begin{aligned} V(z(t_k+0)) &= \sum_{i=1}^n p_i\Big\{\Big|z_i(t_k+0)\Big|+\sum_{j=1}^n\int_{(t_k+0)-\tau_{ij}}^{(t_k+0)}\Big|b_{ij}\Big|\cdot\Big|f_j(z_j(\theta))\Big|\mathrm{d}\theta\Big\}\\ &= \sum_{i=1}^n p_i\Big\{\Big|(1-\gamma_{ik})z_i(t_k)\Big|+\sum_{j=1}^n\int_{t_k-\tau_{ij}}^{t_k}\Big|b_{ij}\Big|\cdot\Big|f_j(z_j(\theta))\Big|\mathrm{d}\theta\Big\}\\ &< V(z(t_k)), \quad k=1,2,\cdots. \end{aligned}$$

Then we can easily follow that $\dot{V}(z(t))<0$ for $t>0$. Therefore, the equilibrium point x^* of system (1) is globally asymptotically stable. This proof is completed.

Theorem 5. *Under assumptions of Theorem 2, further if the following conditions are satisfied*

$$J_{ik}(x_i(t_k))=-\gamma_{ik}(x_i(t_k)-x_i^*), \quad k=1,2,\cdots, \quad i=1,\cdots,n,$$

where $x^*=(x_1^*,\cdots,x_n^*)^T$ *is the equilibrium point of system (1),* $0<\gamma_{ik}<1$, *then system (1) has a unique equilibrium point which is stable and every solution is bounded. If, in addition,* $g_i(s), i=1,\cdots,n$, *are strictly increasing functions, then the unique equilibrium point of system (1) is globally asymptotically stable.*

Proof. In order to complete the proof, we divide the proof into four steps: step A, step B, step C, step D.

Step A. It is the same as step 1 of Theorem 4, so we do not repeat it here.

Step B. Consider the following Lyapunov functional

$$V(z(t)) = 2\sum_{i=1}^{n}\int_{0}^{z_i(t)} f_i(s)\mathrm{d}s + r\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ji}}^{t}\left|b_{ji}\right|\cdot f_i^2(z_i(\theta))\mathrm{d}\theta. \tag{19}$$

Obviously, $V(z)$ is positive definite and $\lim_{\|z\|\to\infty} V(z) = \infty$. Calculating the derivative of $V(z)$ along the solution $z(t,\tilde{\phi})$ of system (14) on $[0, t^*(\phi))$, then we can get

$$\begin{aligned}
\dot{V}(z(t,\tilde{\phi})) = & -2\sum_{i=1}^{n} d_i f_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi}) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(z_i(t,\tilde{\phi}))\cdot f_j(z_j(t,\tilde{\phi}))\\
& + 2\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} f_i(z_i(t,\tilde{\phi}))\cdot f_j(z_j(t-\tau_{ij},\tilde{\phi})) + r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t,\tilde{\phi}))\\
& - r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t-\tau_{ji},\tilde{\phi}))\\
\le & -2\sum_{i=1}^{n} d_i f_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi}) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(z_i(t,\tilde{\phi}))\cdot f_j(z_j(t,\tilde{\phi}))\\
& + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|\left|f_i(z_i(t,\tilde{\phi}))\right|\cdot\left|f_j(z_j(t-\tau_{ij},\tilde{\phi}))\right| + r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t,\tilde{\phi}))\\
& - r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t-\tau_{ji},\tilde{\phi})).
\end{aligned} \tag{20}$$

Since

$$2\left|f_i(z_i(t,\tilde{\phi}))\right|\cdot\left|f_j(z_j(t-\tau_{ij},\tilde{\phi}))\right| \le \frac{1}{r} f_i^2(z_i(t,\tilde{\phi})) + r f_j^2(z_j(t-\tau_{ij},\tilde{\phi}))$$

then we get

$$\begin{aligned}
\dot{V}(z(t,\tilde{\phi})) \le & -2\sum_{i=1}^{n} d_i f_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi}) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(z_i(t,\tilde{\phi}))\cdot f_j(z_j(t,\tilde{\phi}))\\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{r}\left|b_{ij}\right| f_i^2(z_i(t,\tilde{\phi})) + \sum_{i=1}^{n}\sum_{j=1}^{n} r\left|b_{ij}\right| f_j^2(z_j(t-\tau_{ij},\tilde{\phi}))\\
& + r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t,\tilde{\phi})) - r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| f_i^2(z_i(t-\tau_{ji},\tilde{\phi}))\\
= & -2\sum_{i=1}^{n} d_i f_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi}) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(z_i(t,\tilde{\phi}))\cdot f_j(z_j(t,\tilde{\phi}))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{r}\left|b_{ij}\right|f_i^2(z_i(t,\tilde{\phi}))+r\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right|f_i^2(z_i(t,\tilde{\phi}))\\
\le{}&-2\sum_{i=1}^{n}d_if_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi})+f^T(z(t,\tilde{\phi}))\cdot(A+A^T)\cdot f(z(t,\tilde{\phi}))\\
&+\frac{1}{r}\|B\|_\infty f^T(z(t,\tilde{\phi}))f(z(t,\tilde{\phi}))+r\|B\|_1 f^T(z(t,\tilde{\phi}))f(z(t,\tilde{\phi}))\\
={}&-2\sum_{i=1}^{n}d_if_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi})+f^T(z(t,\tilde{\phi}))\cdot[A+A^T\\
&+(\frac{1}{r}\|B\|_\infty+r\|B\|_1)I]\cdot f(z(t,\tilde{\phi}))\\
\le{}&-2\sum_{i=1}^{n}d_if_i(z_i(t,\tilde{\phi}))z_i(t,\tilde{\phi})\le 0.
\end{aligned}
$$

The rest of step B and step C are similar as step 2 and step 3 of Theorem 4, respectively, so we also do not repeat them, then we can easily obtain the existence of solutions of system (1).

Step D. Consider Lyapunov functional $V(z(t))$, the $V(z(t))$ is the same as (19). Calculating the derivative of $V(z(t))$ along the solution $z(t)$ of system (3.6) for any t, $t\neq t_k, k=1,2,\cdots$. Arguing as in step B, we have $\dot{V}(z(t))\le 0, t\neq t_k, k=1,2,\cdots$. Also,

$$
\begin{aligned}
V(z(t_k+0))&=2\sum_{i=1}^{n}\int_0^{z_i(t_k+0)}f_i(s)\mathrm{d}s+r\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{(t_k+0)-\tau_{ji}}^{(t_k+0)}\left|b_{ji}\right|\cdot f_i^2(z_i(\theta))\mathrm{d}\theta\\
&=V(z(t_k))+2\sum_{i=1}^{n}\int_{z_i(t_k)}^{(1-\gamma_{ik})z_i(t_k)}f_i(s)\mathrm{d}s\le V(z(t_k)),\quad k=1,2,\cdots.
\end{aligned}
$$

Then we can easily follow that $\dot{V}(z(t))\le 0$ for $t>0$. Therefore, the equilibrium point x^* of system (1) is stable and every solution is bounded. If, in addition, $g_i(s), i=1,\cdots,n$, are strictly increasing functions, then $\dot{V}(z(t))<0$ is negative definite. Thus, the unique equilibrium point of system (1) is globally asymptotically stable.

Remark 1. *Compared to the existing literatures, our results have improved those results. The results of [5, 6, 9, 10] are obtained under Lipschitz neuron activations. At the same time, our results are easy to be checked.*

In order to show that the conditions we have obtained in this paper provide new sufficient criteria for system (1), we consider the following example.

Example 1. *Consider the following delayed impulsive two-neuron network model (1) descried by:*

$$
\dot{x}_1(t)=-\frac{2}{21}x_1(t)-\frac{10}{21}g_1(x_1(t))+\frac{2}{21}g_2(x_2(t-\tau_2))+I_1,
$$

$$\dot{x}_2(t) = -\frac{5}{21}x_2(t) + \frac{2}{21}g_1(x_1(t-\tau_1)) - \frac{6}{21}g_2(x_2(t)) + I_2,$$
$$\triangle x_1(t_k) = -0.2(x_1(t_k) - x_1^*),$$
$$\triangle x_2(t_k) = -0.3(x_2(t_k) - x_2^*), \quad t_k = kT, \quad k = 1, 2, \cdots. \tag{21}$$

where $T > 0$ is a positive constant, $g_1(\theta) = \arctan(\theta), g_2(\theta) = \theta^3$, $x^* = (x_1^*, x_2^*)^T$ is the equilibrium point of system (21).

Obviously, those results in [5, 6, 8, 9, 10] would fail when applying to this example. However, we select $p_1 = p_2 = 1$ (or by $A + A^T + (\|B\|_\infty + \|B\|_1)I = -diag(\frac{16}{21}, \frac{8}{21}) < 0$), from Theorem 4 (or Theorem 5), system (21) has a unique equilibrium point x^* which is globally asymptotically stable. Therefore, our results establish new criteria for the global asymptotic stability of delayed neural networks with impulses and improve those results in the existing literatures.

4. Conclusion

This paper has developed a class of delayed neural networks with impulses, where the neuron activations do not satisfy Lipschitz conditions. Some general sufficient conditions are derived for the global asymptotic stability of delayed neural networks with impulses. A comparison between our results and the previous results is also given, which shows that our results establish a new criteria for global asymptotic stability of delayed neural networks with impulses. At the same time, the criteria in this paper is also valuable in the design of neural networks which is used to solve efficiently classes of optimization problems arising in practical engineering applications.

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