Subnormality and Theory of L-subgroups

Naseem Ajmal[1], Ifffat Jahan[2,*], Bijan Davvaz[3]

[1] *Department of Mathematics, Zakir Husain Delhi College, University of Delhi, Delhi, India*
[2] *Department of Mathematics, Ramjas College, University of Delhi, Delhi, India*
[3] *Department of Mathematics, Yazd University, Yazd, Iran*

Abstract. The main focus in this work is to establish that L-group theory, which uses the language of functions instead of formal set theoretic language, is capable of capturing most of the refined ideas and concepts of classical group theory. We demonstrate this by extending the notion of subnormality to the L-setting and investigating its properties. We develop a mechanism to tackle the join problem of subnormal L-subgroups. The conjugate L-subgroup as is defined in our previous paper [4] has been used to formulate the concept of normal closure and normal closure series of an L-subgroup which, in turn, is used to define subnormal L-subgroups. Further, the concept of subnormal series has been introduced in L-setting and utilized to establish the subnormality of L-subgroups. Also, several results pertaining to the notion of subnormality have been established. Lastly, the level subset characterization of a subnormal L-subgroup is provided after developing a necessary mechanism. Finally, we establish that every subgroup of a nilpotent L-group is subnormal. In fact, it has been exhibited through this work that L-group theory presents a modernized approach to study classical group theory.

2020 Mathematics Subject Classifications: 11E57, 20N25, 20A10

Key Words and Phrases: L-subgroup, Normal L-subgroup, Nilpotent L-subgroup, Normal closure, Subnormal subgroup, Subnormal series

1. Introduction

The notion of subnormality in classical group theory is an important generalization of the notion of normality. This notion is introduced to repair the deficiency of the notion of normality that it fails to be a transitive relation. In fact, the relation of subnormality can be defined as the transitive closure of the relation of normality. The class of subnormal subgroups not only properly contains the class of normal subgroups but in the class of finitely generated groups this class coincides with the important subclass of nilpotent subgroups. Besides, this notion has many more pleasing properties which have been the focus of attention of algebraist for a long period of time. The work of Wielandt in the

year 1939 on the join of two subnormal subgroups in a finite group provided an impetus for the development of group theory. The question of finding necessary and sufficient conditions for the join of subnormal subgroups to be subnormal is one of the most important problems in the area of group theory even today. Here in the Part I of this work, we introduce and study the notion of subnormality in L-group theory with a special attention of approaching the join problem of subnormal L-subgroups. This study is carried out in detail in Part II of this work and we establish necessary and sufficient conditions for the subnormality of the join of two subnormal L-subgroups of an arbitrary group. In the end of this paper, we present the main result of Part II without proof.

In this paper, we present a systematic and successful development of L-group theory which has been made in papers [2–7, 10]. The important concepts of nilpoent L-subgroups, solvable L-subgroups have been introduced and studied and their inter-relationship is established in [3, 6]. Moreover, the concept of normalizer of an L-subgroup in an L-group is formulated in [2]. It has been proved in [10] that a nilpotent L-subgroup of an L-group satisfies the normalizer condition.

As an application and motivation, here we mention that if we replace the lattice L, in our work by the closed unit interval $[0, 1]$, then we retrieve the corresponding version of fuzzy group theory. Moreover, as an application of this theory we also mention that if we replace the lattice L by the two elements set $\{0, 1\}$, then the results of classical group theory follow as simple corollaries of the corresponding results of L-group theory. Moreover, this development of L-group theory is beyond the purview of metatheorem, contrary to the development of fuzzy group theory.

The development of fuzzy group theory could not be sustained due to some inherent problems and the emergence of metatheorem formulated by Head [9]. The hurdles faced by the researchers in this development are removed in L-group theory by considering the parent structure as an L-group rather than an ordinary group. Many higher concepts of classical group theory such as supersolvability, nilpotency and subnormality could not be studied in the framework of fuzzy group theory. The above mentioned modification allowed the formation of normalizer of an L-subgroup of an L-group similar to that of classical group theory.The question of formulation of the notion of subnormality in L-group theory leads to the question of formation of normal closure of an L-subgroup in an L-group. The concept of normal closure of an L-subgroup in an L-group should be defined in such a manner that the normal closure series arising by this normal closure should be compatible with a subnormal series arising by the given L-subgroup likewise its classical counterparts. For this purpose, we define the conjugate of an L-subset by an L-subset of an L-group instead of a crisp point. Then, the L-subgroup generated by the conjugate of an L-subgroup by the given parent L-group is called the normal closure of the given L-subgroup. This notion satisfies most of the desirable properties related to this concept. For example, the compatibility of this notion with that of commutator L-subgroup of an L-group exists like its counterpart in classical group theory. Then, this notion is utilized

in the formation of normal closure series of an L-subgroup of an L-group which leads to the definition of subnormal L-subgroup. The notion of subnormal series for an L-subgroup is introduced which exihibits the desired relationship with the normal closure series. In the end, we establish the inter-connection of subnormality of an L-subgroup with the subnormality in the usual group theoretic sense of the level subsets of the given L-subgroup by providing a result which is called the level subset characterization. Finally, by an application of this result we establish that every subgroup of a nilpotent L-subgroup is subnormal.

Although so much has been said and done in the area of L-group theory, but still researchers from all over the world are involved in investigating various types of structures in the fuzzy environments. A significant attempt made in this direction is due to A. Razzaque and A. Razaq [12] wherein they have defined and studied a class of fuzzy structure which properly contains the classes of intuitionistic fuzzy subgroups and Pythagorean fuzzy subgroups as its subclasses.

2. Preliminaries

Throughout this paper $L = \langle L, \leq, \vee, \wedge \rangle$ denotes a complete and completely distributive lattice where '$\leq$' denotes the partial ordering of L, the join (sup) and the meet (inf) of the elements of L are denoted by '$\vee$' and '$\wedge$' respectively. Also, we write 1 and 0 for maximal and minimal elements of L respectively. The definition of a completely distributive lattice is well known in the literature and can be found in any standard text on the subject.

If $\{J_i : i \in I\}$ is any family of subsets of a complete lattice L, let F denote the set of choice functions for J_i, i.e. functions $f : I \to \prod_{i \in I} J_i$ such that $f(i) \in J_i$ for each $i \in I$. Then, we say that L is a completely distributive lattice, if

$$\bigwedge \left\{ \bigvee_{i \in I} J_i \right\} = \bigvee_{f \in F} \left\{ \bigwedge_{i \in I} f(i) \right\}.$$

The above law is known as the complete distributive law. Moreover, a lattice L is said to be infinitely meet distributive if for every subset $\{b_\beta : \beta \in B\}$ of L, we have:

$$a \wedge \left\{ \bigvee_{\beta \in B} b_\beta \right\} = \bigvee_{\beta \in B} \{a \wedge b_\beta\},$$

provided L is join complete. The above law is known as the infinitely meet distributive law. The definition of infinitely join distributive lattice is dual to the above definition. A complete lattice which satisfies infinitely meet distributive law is known as a complete Heyting algebra or a frame.Therefore, a completely distributive lattice is always a complete Heyting algebra. Further, we introduce some basic definitions and results which are used in the sequel. For details we refer to [1, 3, 8, 14? , 15] .

An L-subset of X is a function from X into L. The set of L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the tip of μ is defined as $\bigvee_{x \in X} \{\mu(x)\}$.

If $\mu, \nu \in L^X$, then we say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted by $\mu \subseteq \nu$. For a family $\{\mu_i : i \in I\}$ of L-subsets of X, where I is a non-empty index set, the union $\bigcup_{i \in I} \mu_i$ and the intersection $\bigcap_{i \in I} \mu_i$ of $\{\mu_i : i \in I\}$ are respectively defined by:

$$\left(\bigcup_{i \in I} \mu_i \right)(x) = \bigvee_{i \in I} \{\mu_i(x)\} \quad \text{and} \quad \left(\bigcap_{i \in I} \mu_i \right)(x) = \bigwedge_{i \in I} \{\mu_i(x)\},$$

for each $x \in X$. Let f be a mapping from a set X to a set Y. If $\mu \in L^X$ and $\nu \in L^Y$, then the image $f(\mu)$ of μ under f and the preimage $f^{-1}(\nu)$ of ν under f are L-subsets of Y and X respectively, defined by

$$f(\mu)(y) = \bigvee_{x \in f^{-1}(y)} \{\mu(x)\} \quad \text{and} \quad f^{-1}(\nu)(x) = \nu(f(x)).$$

Here we point out that in the above definition if $f^{-1}(y) = \phi$, then $f(\mu)(y)$, being the least upper bound of the empty set, is zero. The set product $\mu \circ \nu$ of $\mu, \nu \in L^S$ where S is a groupoid, is an L-subset of S defined by

$$\mu \circ \nu(x) = \bigvee_{x=yz} \{\mu(y) \wedge \nu(z)\}.$$

If $\mu \in L^X$ and $a \in L$, then the notions of level subset μ_a and strong level subset $\mu_a^>$ of μ are, respectively, defined by:

$$\mu_a = \{x \in X : \mu(x) \geq a\} \text{ and } \mu_a^> = \{x \in X : \mu(x) > a\}.$$

Clearly, if $a \leq b$ for $a, b \in L$, then $\mu_b \subseteq \mu_a$ and $\mu_b^> \subseteq \mu_a^>$.

Proposition 1. *[15] Let $\mu, \nu \in L^X$ and $\mu \subseteq \nu$. Then,*

    (i) if $\mu \subseteq \nu$, then $\mu_a \subseteq \nu_a$ for each $a \in L$,

    (ii) if $\mu_a \subseteq \nu_a$ for each $a \in Im\mu$, then $\mu \subseteq \nu$,

    (iii) if $\mu \subseteq \nu$, then $\mu_a^> \subseteq \nu_a^>$ for each $a \in L$ provided L is a chain,

    (iV) if $\mu_a^> \subseteq \nu_a^>$ for each $a \in Im\nu$, then $\mu \subseteq \nu$ provided L is a chain.

    Throughout this paper G denotes an ordinary group with the identity element 'e' and I denotes a non-empty index set. For $A \subseteq X$, 1_A denotes the characteristic function of A in X.

Definition 1. *[15] Let $\mu \in L^G$. Then, μ is called an L-subgroup of G if for each $x, y \in G$*

    (i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,

    (ii) $\mu(x^{-1}) = \mu(x)$.

The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element 'e' of G.

Theorem 1. *Let $\mu \in L^G$ with tip a_0. Then,*

(i) $\mu \in L(G)$ if and only if μ_a is a subgroup of G for each $a \le a_0$,

(ii) $\mu \in L(G)$ if and only if $\mu_a^>$ is a subgroup of G for each $a < a_0$.

It is well known in the literature that the intersection of an arbitrary family of L-subgroups of a group is an L-subgroup.

Definition 2. *Let $\mu \in L^G$. Then, the L-subgroup of G generated by μ is defined as the smallest L-subgroup of G which contains μ. It is denoted by $\langle \mu \rangle$. That is*

$$\langle \mu \rangle = \bigcap_{i \in I} \{\mu_i \in L(G) : \mu \subseteq \mu_i\}.$$

Definition 3. *Let $\mu \in L(G)$. Then, μ is called an L-normal subgroup of G if for all $x, y \in G$, $\mu(xy) = \mu(yx)$.*

Definition 4. *If $\mu, \eta \in L^X$ and $\eta \subseteq \mu$, then we say that η is an L-subset of μ. The set of L-subsets of μ is denoted by L^μ.*

Definition 5. *If $\mu, \eta \in L(G)$ and $\eta \subseteq \mu$, then we say that η is an L-subgroup of μ. The set of L-subgroups of μ is denoted by $L(\mu)$.*

Hence onwards μ denotes an L-subgroup of G and we shall call the parent L-subgroup μ simply an L-group.

Theorem 2. *Let $\eta \in L^\mu$ with tip a_0. Then,*

(i) $\eta \in L(\mu)$ if and only if η_a is a subgroup of μ_a for each $a \le a_0$,

(ii) $\eta \in L(\mu)$ if and only if $\eta_a^>$ is a subgroup of $\mu_a^>$ for each $a < a_0$ provided L is a chain.

Below we extend a result from [1] to L-setting:

Proposition 2. *[?] Let $\eta, \theta \in L(\mu)$. Then,*

$$\eta \circ \theta \in L(\mu) \text{ if and only if } \eta \circ \theta = \theta \circ \eta.$$

Proposition 3. *Let $\eta, \theta \in L(\mu)$. Then, η and $\theta \subseteq \eta \circ \theta$ if and only if $\eta(e) = \theta(e)$.*

If $\eta \in L^\mu$ and $\langle \eta \rangle_\mu$ denotes the L-subgroup of μ generated by η, then it can be easily verified that $\langle \eta \rangle_\mu = \langle \eta \rangle$.

We recall the definition of a normal L-subgroup of an L-group.

Definition 6. *[3] Let $\eta \in L(\mu)$. Then, η is said to be a normal L-subgroup of μ, if for all $x, y \in G$*

$$\eta(yxy^{-1}) \geq \eta(x) \wedge \mu(y).$$

The set of normal L-subgroup of μ is denoted by $NL(\mu)$.

Theorem 3. *Let $\eta \in L(\mu)$. Then, $\eta \in NL(\mu)$ if and only if η_a is a normal subgroup of μ_a for each $a \leq \eta(e)$.*

Theorem 4. *Let $f : G \to K$ be a group homomorphism. If $\mu \in L(G)$ and $\nu \in L(K)$, then*

    (i) $f(\eta)$ is an L-subgroup of $f(\mu)$ for each $\eta \in L(\mu)$,

    (ii) $f^{-1}(\theta)$ is an L-subgroup of $f^{-1}(\nu)$ for each $\theta \in L(\nu)$.

Theorem 5. *[3] Let $\eta, \theta \in L(\mu)$. Then,*

    (i) $\eta \circ \theta \in L(\mu)$ if η or $\theta \in NL(\mu)$,

    (ii) $\eta \circ \theta \in NL(\mu)$ if η and $\theta \in NL(\mu)$.

Theorem 6. *[?] Let $\eta \in L^{\mu}$. Let $a_0 = \bigvee\limits_{x \in G} \{\eta(x)\}$ and define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \bigvee\limits_{a \leq a_0} \{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$.

Corollary 1. *Let $\eta \in L^{\mu}$. Then, $\hat{\eta}(e) = \bigvee\limits_{x \in G} \{\eta(x)\}$.*

Definition 7. *[15] Let $\eta \in L^{\mu}$. Then, η is said to have sup-property if for every non-empty subset A of G, there exists $a_0 \in A$ such that $\bigvee\limits_{a \in A} \{\eta(a)\} = \eta(a_0)$.*

Theorem 7. *[?] Let $\eta \in L^{\mu}$ and possesses sup-property. Then, define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \bigvee\limits_{a \in Im\eta} \{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$. Moreover, $\hat{\eta}$ possesses sup-property and $Im\,\hat{\eta} \subseteq Im\,\eta$.

Recall the following from [3]:

Definition 8. *Let $\eta, \theta \in L^{\mu}$. Then, the commutator of η and θ is an L-subset (η, θ) of G defined as follows:*

$$(\eta, \theta)(x) = \begin{cases} \vee\{\eta(y) \wedge \theta(z)\} & \text{if } x = [y, z] \text{ for some } y, z \in G, \\ \inf \eta \wedge \inf \theta & \text{if } x \neq [y, z] \text{ for any } y, z \in G. \end{cases}$$

Clearly, $(\eta, \theta) \subseteq \mu$. The commutator L-subgroup of η and θ is defined as $\langle (\eta, \theta) \rangle$ and we write $\langle (\eta, \theta) \rangle = [\eta, \theta]$. Note that $\inf(\eta, \theta) = \inf \eta \wedge \inf \theta$ and $[\eta, \theta] \in L(\mu)$.

Here recall that if η is non-constant and $\eta \neq \mu$, then η is said to be a proper L-subgroup of μ. Also, η is said to be a trivial L-subgroup of μ if its chain of level subgroups contains only $\{e\}$ and G. Clearly, η is a proper L-subgroup of μ if and only if η has distinct tip and tail, and $\eta \neq \mu$.

Definition 9. *Let η be a proper L-subgroup of μ. Then, we define an L-subgroup of μ contained in η, denoted by $\eta_{t_0}^{a_0}$, as follows:*

$$\eta_{t_0}^{a_0}(y) = \begin{cases} a_0, & if \ \ y = e, \\ t_0, & if \ \ y \neq e, \end{cases}$$

where $a_0 = \eta(e)$ and $t_0 = \inf \eta$. Here $\eta_{t_0}^{a_0}$, a trivial L-subgroup of μ, is called the trivial L-subgroup of η.

3. Subnormal L-subgroup

In order to introduce the concept of subnormality in the L-setting, we start with the definition of conjugate of an L-subset by an L-subset like its counterpart in classical group theory.Then, the notion of normal closure of an L-subgroup is defined and utilized to formulate the concept of subnormal L-subgroups.

Definition 10. *[4] Let $\eta, \theta \in L^\mu$. Define an L-subset $\theta\eta\theta^{-1}$ of G as follows:*

$$\theta\eta\theta^{-1}(x) = \bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \theta(z)\} \quad for \ each \ x \in G.$$

We call $\theta\eta\theta^{-1}$ the conjugate of η by θ. Clearly, $\theta\eta\theta^{-1} \subseteq \mu$. Hence the L-subgroup $\langle \theta\eta\theta^{-1} \rangle \in L(\mu)$ and is denoted by η^θ.

The following lemma is instrumental in the development of this paper:

Lemma 1. *Let $\eta, \theta \in L(\mu)$. Then,*

  (i) $\eta \subseteq \theta\eta\theta^{-1}$ provided $\eta(e) \leqslant \theta(e)$,

  (ii) $\bigvee_{x \in G} \{\theta\eta\theta^{-1}(x)\} = \theta\eta\theta^{-1}(e) = \eta(e) \wedge \theta(e)$,

  (iii) $\theta\eta\theta^{-1}(x) = \theta\eta\theta^{-1}(x^{-1})$ for each $x \in G$,

  (iv) $\theta\eta\theta^{-1}(gxg^{-1}) \geq \theta\eta\theta^{-1}(x) \wedge \theta(g)$ for each $x, g \in G$.

Proof. The proofs of (i)-(iii) are obvious. We only prove (iv). So, let $x, g \in G$. Then,

$$\theta\eta\theta^{-1}(x) \wedge \theta(g) = \left\{ \bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \theta(z)\} \right\} \wedge \theta(g)$$

$$= \bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \theta(z) \wedge \theta(g)\}$$

(as L is a completely distributive lattice)

$$\leq \bigvee_{\substack{x=zyz^{-1} \\ gxg^{-1}=(gz)y(gz)^{-1}}} \{\eta(y) \wedge \theta(gz)\}$$

$$= \theta\eta\theta^{-1}(gxg^{-1}).$$

Corollary 2. *Let $\eta \in L(\mu)$. Then, $\eta \subseteq \mu\eta\mu^{-1} \subseteq \mu$.*

Our next definition leads to the notion of normal closure of an L-subgroup of an L-group which in its essence is parallel to classical group theory.

Definition 11. *Let $\eta \in L(\mu)$. Then, the normal closure of η in μ is defined as the L-subgroup of μ generated by the conjugate '$\mu\eta\mu^{-1}$'. It is denoted by η^{μ}. Thus,*

$$\eta^{\mu} = \langle \mu\eta\mu^{-1} \rangle.$$

Note that by Corollary 2, the normal closure η^{μ} is an L-subgroup of μ containing η. In classical group theory, the normal closure of a subgroup of a group G is the smallest normal subgroup of G which contains the given subgroup. For the sake of completeness, recall the following from [2] :

Theorem 8. *Let $\eta \in L(\mu)$. Then, the normal closure η^{μ} is the least normal L-subgroup of μ containing η.*

Proof. As η^{μ} is an L-subgroup of μ containing η, we first establish that $\eta^{\mu} \in NL(\mu)$. For this purpose, let $x, g \in G$. Note that, by Lemma 1,

$$\bigvee_{y \in G} \mu\eta\mu^{-1}(y) = \eta(e).$$

Hence in view of Theorem 6,

$$\eta^{\mu}(x) \wedge \mu(g) = \left\{ \bigvee_{a \leq \eta(e)} \{a : x \in \langle(\mu\eta\mu^{-1})_a\rangle\} \right\} \wedge \mu(g)$$

$$= \bigvee_{a \leq \eta(e)} \{a \wedge \mu(g) : x \in \langle(\mu\eta\mu^{-1})_a\rangle\}.$$

(As L is a completely distributive lattice)

Next, claim that

$$\text{if } x \in \langle(\mu\eta\mu^{-1})_a\rangle, \text{ then } \eta^{\mu}(gxg^{-1}) \geq a \wedge \mu(g).$$

In order to prove the claim, let $x \in \langle (\mu\eta\mu^{-1})_a \rangle$. Then, $x = x_1 \, x_2 \, \cdots \, x_n$, where either x_i or $x_i^{-1} \in (\mu\eta\mu^{-1})_a$ for each i. Thus,

$$gxg^{-1} = (gx_1g^{-1})(gx_2g^{-1}) \cdots (gx_ng^{-1}).$$

By Lemma 1, for each i

$$\mu\eta\mu^{-1}(gx_ig^{-1}) \geq \mu\eta\mu^{-1}(x_i) \wedge \mu(g).$$

Again by Lemma 1, for each i

$$\mu\eta\mu^{-1}(x_i) = \mu\eta\mu^{-1}(x_i^{-1}).$$

As x_i or $x_i^{-1} \in (\mu\eta\mu^{-1})_a$, it follows that

$$\mu\eta\mu^{-1}(gx_ig^{-1}) \geq a \wedge \mu(g) \text{ for each } i.$$

Hence,

$$\bigwedge_{i=1}^{n} \{\mu\eta\mu^{-1}(gx_ig^{-1})\} \geq a \wedge \mu(g).$$

Therefore,

$$\eta^{\mu}(gxg^{-1}) \geq \bigwedge_{i=0}^{n} \{\eta^{\mu}(gx_ig^{-1})\}$$
$$\geq \bigwedge_{i=1}^{n} \{\mu\eta\mu^{-1}(gx_ig^{-1})\}$$
$$\geq a \wedge \mu(g).$$

This establishes the claim. Consequently,

$$\eta^{\mu}(x) \wedge \mu(g) = \bigvee_{a \leq \eta(e)} \{a \wedge \mu(g) : x \in \langle (\mu\eta\mu^{-1})_a \rangle\}$$
$$\leq \bigvee_{a \leq \eta(e)} \{\eta^{\mu}(gxg^{-1}) : x \in \langle (\mu\eta\mu^{-1})_a \rangle\}$$
$$= \eta^{\mu}(gxg^{-1}).$$

Lastly to prove that η^{μ} is the least normal L-subgroup of μ containing η, let λ be a normal L-subgroup of μ containing η and $x \in G$. Then,

$$\mu\eta\mu^{-1}(x) = \bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \mu(z)\}$$
$$\leq \bigvee_{x=zyz^{-1}} \{\lambda(y) \wedge \mu(z)\} \text{ (as } \eta \subseteq \lambda)$$

$$\leq \bigvee_{x=zyz^{-1}} \{\lambda(zyz^{-1})\} \quad (\text{as } \eta \in NL(\mu))$$
$$= \lambda(x).$$

Thus $\mu\eta\mu^{-1} \subseteq \lambda$ and as λ is an L-subgroup of μ, we have $\eta^{\mu} = \langle\mu\eta\mu^{-1}\rangle \subseteq \lambda$. This proves our result.

Corollary 3. *Let $\eta \in L(\mu)$. Then, $\eta \in NL(\mu)$ if and only if $\eta^{\mu} = \eta$.*

Moreover,

Proposition 4. *Let $\eta \in L(\mu)$. Then, $\eta^{\mu}(e) = \eta(e)$.*

Next, we provide an example of the 'normal closure' of an L-subgroup of an L-group:

Example 1. *Let $D_8 = \{\langle x, y\rangle : x^2 = e = y^8; xy = y^{-1}x\}$ be the dihedral group of degree 8. If*

$$D_4 = \{\langle x, y^2\rangle : x^2 = e = (y^2)^4; xy^{-2} = y^{-2}x\},$$

is a dihedral subgroup of D_8, then define the following L-subsets of D_8:

$$\mu(z) = \begin{cases} \dfrac{1}{2} & \text{if } z \in D_4, \\[2mm] \dfrac{1}{4} & \text{if } z \in D_8 \sim D_4; \end{cases}$$

$$\eta(z) = \begin{cases} \dfrac{1}{3} & \text{if } z \in \langle x\rangle, \\[2mm] \dfrac{1}{6} & \text{if } z \in D_4 \sim \langle x\rangle, \\[2mm] \dfrac{1}{9} & \text{if } z \in D_8 \sim D_4. \end{cases}$$

Here $A \sim B$ means usual set difference and $\langle x\rangle$ denotes subgroup of D_8 generated by 'x'. Clearly $\eta \subseteq \mu$, $\eta \neq \mu$ and $\eta, \mu \in L(G)$. It can be verified easily that the conjugate '$\mu\eta\mu^{-1}$' is defined by the following level subsets:

$$(\mu\eta\mu^{-1})_{\frac{1}{3}} = \{e, x, xy^4\},$$
$$(\mu\eta\mu^{-1})_{\frac{1}{4}} = \{e, x, xy^2, xy^4, xy^6\},$$
$$(\mu\eta\mu^{-1})_{\frac{1}{6}} = \{e, x, xy^2, xy^4, xy^6, y^2, y^4, y^6\} \text{ and } (\mu\eta\mu^{-1})_{\frac{1}{9}} = D_8.$$

Consequently, if $K_4 = \{e, x, y^4, xy^4\}$ is a Klein-4 subgroup of D_8, then by applying the construction of Theorem 6 on the L-subset 'η' of μ and after doing necessary calculations,

we get L-subgroup '$\eta^\mu = \langle \mu\eta\mu^{-1}\rangle$' as follows:

$$
\eta^\mu(z) = \begin{cases} \dfrac{1}{3} & \text{if } z \in K_4, \\[2mm] \dfrac{1}{4} & \text{if } z \in D_4 \sim K_4, \\[2mm] \dfrac{1}{9} & \text{if } z \in D_8 \sim D_4. \end{cases}
$$

In the following, we discuss some properties of conjugate of L-subsets where the L-subsets in question are L-subgroups of a given parent L-group. The significance of such properties have already been shown in classical group theory for establishing certain properties of ith normal closure of a subgroup of a group.

Theorem 9. *Let $\eta, \theta, \gamma \in L(\mu)$. Let $\theta \subseteq \gamma$ and $\eta(e) = \theta(e)$. Then,*
(i) $(\eta^\gamma)^\theta = \eta^\gamma$,
(ii) $(\eta^\theta)^\gamma = \eta^\gamma$,
(iii) $\eta^{\theta \circ \eta} = \eta^\theta$.

Proof. (i) In view of Lemma 1 and Corollary 1 and using the fact $\theta \subseteq \gamma$, we have

$$
\eta^\gamma(e) = \eta(e) \wedge \gamma(e) = \theta(e).
$$

Hence by Lemma 1, $\eta^\gamma \subseteq (\eta^\gamma)^\theta$. In order to prove the reverse inclusion, firstly we shall show that

$$
\theta\eta^\gamma\theta^{-1} \subseteq \eta^\gamma.
$$

So let $x \in G$. Then,

$$
\theta\eta^\gamma\theta^{-1}(x) = \bigvee_{x = y_i z_i y_i^{-1}} \{\eta^\gamma(z_i) \wedge \theta(y_i)\}.
$$

Here, in view of Lemma 1 and Corollary 1, we observe that

$$
\bigvee_{x \in G} \{\gamma\eta\gamma^{-1}(x)\} = \eta^\gamma(e) = \theta(e) = \eta(e).
$$

Thus $\gamma\eta\gamma^{-1}(e) = \eta(e)$. Therefore, in view of Theorem 6, as $\eta^\gamma = \langle \gamma\eta\gamma^{-1} \rangle$, we have

$$
\eta^\gamma(z_i) = \bigvee_{a_j^i \le \eta(e)} \{a_j^i : z_i \in \langle (\gamma\eta\gamma^{-1})_{a_j^i} \rangle\}.
$$

Therefore,

$$
\theta\eta^\gamma\theta^{-1}(x) = \bigvee_{x = y_i z_i y_i^{-1}} \{\eta^\gamma(z_i) \wedge \theta(y_i)\}
$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{ \bigvee_{a_j^i \le \eta(e)} \{a_j^i : z_i \in \langle (\gamma\eta\gamma^{-1})_{a_j^i} \rangle \} \wedge \theta(y_i) \right\}$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{ \bigvee_{a_j^i \le \eta(e)} \{a_j^i \wedge \theta(y_i) : z_i \in \langle (\gamma\eta\gamma^{-}1)_{a_j^i} \rangle \} \right\}$$

(as L is a completely distributive lattice)

$$= \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ a_j^i \le \eta(e)}} \{a_j^i \wedge \theta(y_i) : z_i \in \langle (\gamma\eta\gamma^{-1})_{a_j^i} \rangle \}$$

$$\le \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ a_j^i \le \eta(e)}} \{a_j^i \wedge \gamma(y_i) : z_i \in \langle (\gamma\eta\gamma^{-}1)_{a_j^i} \rangle \}. \quad (\text{As } \theta \subseteq \gamma)$$

Now, let $y_i \in G$ be such that $x = y_i z_i y_i^{-1}$ for some $z_i \in \langle (\gamma\eta\gamma^{-1})_{a_j^i} \rangle$ and $a_j^i \le \eta(e)$. We write $c_{a_j^i, y_i} = a_j^i \wedge \gamma(y_i)$. Then, $c_{a_j^i, y_i} \le \eta(e)$. We shall establish that

$$y_i z_i y_i^{-1} \in \langle (\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}} \rangle.$$

As $z_i \in \langle (\gamma\eta\gamma^{-1})_{a_j^i} \rangle$, we have, $z_i = z_1^i z_2^i ... z_n^i$ where either z_k^i or $(z_k^i)^{-1} \in (\gamma\eta\gamma^{-1})_{a_j^i}$ for $k = 1, 2, ..., n$. Thus,

$$y_i z_i y_i^{-1} = (y_i z_1^i y_i^{-1})(y_i z_2^i y_i^{-1})...(y_i z_n^i y_i^{-1}).$$

By Lemma 1, for each k

$$\gamma\eta\gamma^{-1}(y_i z_k^i y_i^{-1}) \ge \gamma\eta\gamma^{-1}(z_k^i) \wedge \gamma(y_i).$$

Moreover, again by Lemma 1, $\gamma\eta\gamma^{-1}(z_k^i) = \gamma\eta\gamma^{-1}((z_k^i)^{-1})$ and as z_k^i or $(z_k^i)^{-1} \in (\gamma\eta\gamma^{-1})_{a_j^i}$, it follows that

$$\gamma\eta\gamma^{-1}(y_i z_k^i y_i^{-1}) \ge a_j^i \wedge \gamma(y_i) = c_{a_j^i, y_i}.$$

Hence $y_i z_k^i y_i^{-1} \in (\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}}$. Again, in view of Lemma 1,

$$\gamma\eta\gamma^{-1}(y_i(z_k^i)^{-1} y_i^{-1}) = \gamma\eta\gamma^{-1}(y_i z_k^i y_i^{-1}).$$

Then, using the above arguments, we have $y_i(z_k^i)^{-1} y_i^{-1} \in (\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}}$. Consequently, $y_i(z_k^i)^{-1} y_i^{-1}$ and $y_i z_k^i y_i^{-1} \in (\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}}$. This implies

$$y_i z_i y_i^{-1} \in \langle (\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}} \rangle.$$

Thus our claim is established. Therefore, we have

$$\theta\eta^\gamma\theta^{-1}(x) \le \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ a_j^i \le \eta(e)}} \{a_j^i \wedge \gamma(y_i) : z_i \in \langle(\gamma\eta\gamma^-1)_{a_j^i}\rangle\}$$

$$\le \bigvee_{\substack{x=y_i z_i y_i^{-1}) \\ c_{a_j^i, y_i} \le \eta(e)}} \{c_{a_j^i, y_i} : y_i z_i y_i^{-1} \in \langle(\gamma\eta\gamma^{-1})_{c_{a_j^i, y_i}}\rangle\}$$

$$\le \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ b \le \eta(e)}} \{b : y_i z_i y_i^{-1} \in \langle(\gamma\eta\gamma^{-1})_b\rangle\}$$

$$\le \bigvee_{d \le \eta(e)} \{d : x \in \langle(\gamma\eta\gamma^{-1})_d\rangle\}$$

$$= \langle\gamma\eta\gamma^{-1}\rangle(x) \text{ (by Theorem 6)}$$

$$= \eta^\gamma(x).$$

Consequently, $\theta\eta^\gamma\theta^{-1} \subseteq \eta^\gamma$. As $\eta^\gamma \in L(\mu)$, it follows that

$$\langle\theta\eta^\gamma\theta^{-1}\rangle = (\eta^\gamma)^\theta \subseteq \eta^\gamma.$$

This establishes the desired equality.

(ii) As $\theta \subseteq \gamma$, by the definition of a conjugate L-subgroup, we have $\eta^\theta \subseteq \eta^\gamma$ and hence $(\eta^\theta)^\gamma \subseteq (\eta^\gamma)^\gamma$. By part(i), $(\eta^\gamma)^\gamma = \eta^\gamma$. Thus, $(\eta^\theta)^\gamma \subseteq \eta^\gamma$. In order to prove the reverse inclusion, note that $\eta(e) = \theta(e)$ so that, by Lemma 1, $\eta \subseteq \eta^\theta$. Therefore, we have $\eta^\gamma \subseteq (\eta^\theta)^\gamma$. This establishes the desired equality.

(iii) As $\eta(e) = \theta(e)$, by Proposition 3, we have $\theta \subseteq \theta \circ \eta$. Thus by the definition of a conjugate L-subset, we get $\eta^\theta \subseteq \eta^{\theta\circ\eta}$. Now to prove the reverse inclusion, we take $\lambda = \theta \circ \eta$. Firstly, we shall show that

$$\lambda\eta\lambda^{-1} \subseteq \eta^\theta.$$

So let $x \in G$ and consider

$$\lambda\eta\lambda^{-1}(x) = \bigvee_{x=y_i z_i y_i^{-1}} \{\eta(z_i) \wedge \lambda(y_i)\}$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{\eta(z_i) \wedge \left\{\bigvee_{y=u_j^i v_j^i} \{\theta(u_j^i) \wedge \eta(v_j^i)\}\right\}\right\}$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{\bigvee_{y_i=u_j^i v_j^i} \{\eta(z_i) \wedge \{\theta(u_j^i) \wedge \eta(v_j^i)\}\}\right\}$$

(as L is a completely distributive lattice)

$$= \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ y_i = u_j^i v_j^i}} \{\eta(z_i) \wedge \theta(u_j^i) \wedge \eta(v_j^i)\}$$

$$\leq \bigvee_{x=u_j^i(v_j^i z_i (v_j^i)^{-1})(u_j^i)^{-1}} \{\eta(v_j^i z_i (v_j^i)^{-1}) \wedge \theta(u_j^i)\}$$

$$\leq \theta\eta\theta^{-1}(x)$$

$$\leq \eta^\theta(x).$$

This establishes the claim. As $\eta^\theta \in L(\mu)$, it follows that

$$\eta^{\theta \circ \eta} = \langle \lambda \eta \lambda^{-1} \rangle \subseteq \eta^\theta.$$

This establishes the desired equality.

Theorem 10. *Let* $\eta, \theta, \gamma \in L(\mu)$ *be such that* $\gamma(e) = \eta(e) = \theta(e)$. *Then,* $(\eta^\theta)^\gamma = \eta^{\gamma \circ \theta}$.

Proof. As $\gamma(e) = \theta(e)$, by Proposition 3, $\gamma \subseteq \gamma \circ \theta$. Hence,

$$(\eta^\theta)^\gamma \subseteq (\eta^\theta)^{\gamma \circ \theta}.$$

By Theorem 9, we have $(\eta^\theta)^{\gamma \circ \theta} = \eta^{\gamma \circ \theta}$. Therefore, $(\eta^\theta)^\gamma \subseteq \eta^{\gamma \circ \theta}$. For the reverse inclusion, we take $\lambda = \gamma \circ \theta$. We shall establish that

$$\lambda \eta \lambda^{-1} \subseteq \gamma \eta^\theta \gamma^{-1}.$$

So let $x \in G$ and consider

$$\lambda \eta \lambda^{-1}(x) = \bigvee_{x=y_i z_i y_i^{-1}} \{\eta(z_i) \wedge \lambda(y_i)\}$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{ \eta(z_i) \wedge \left\{ \bigvee_{y_i = u_j^i v_j^i} \{\gamma(u_j^i) \wedge \theta(v_j^i)\} \right\} \right\}$$

$$= \bigvee_{x=y_i z_i y_i^{-1}} \left\{ \bigvee_{y_i = u_j^i v_j^i} \{\eta(z_i) \wedge \{\gamma(u_j^i) \wedge \theta(v_j^i)\}\} \right\}$$

$$\text{(as } L \text{ is a completely distributive lattice)}$$

$$= \bigvee_{\substack{x=y_i z_i y_i^{-1} \\ y_i = u_j^i v_j^i}} \{\eta(z_i) \wedge \gamma(u_j^i) \wedge \theta(v_j^i)\}$$

$$\leq \bigvee_{x=u_j^i(v_j^i z_i (v_j^i)^{-1})(u_j^i)^{-1}} \{\theta\eta\theta^{-1}(v_j^i z_i (v_j^i)^{-1}) \wedge \gamma(u_j^i)\}$$

$$\text{(by the definition of a conjugate } L\text{-subset)}$$

$$\leq \bigvee_{x=u_j^i(v_j^i z_i(v_j^i)^{-1})(u_j^i)^{-1}} \{\eta^\theta(v_j^i z_i(v_j^i)^{-1}) \wedge \gamma(u_j^i)\}$$

$$\leq \gamma\eta^\theta\gamma^{-1}(x).$$

This establishes the claim and hence

$$\eta^{\gamma\circ\theta} = \langle\lambda\eta\lambda^{-1}\rangle \subseteq \langle\gamma\eta^\theta\gamma^{-1}\rangle = (\eta^\theta)^\gamma.$$

This proves the result.

Theorem 11. *Let $\eta,\theta \in L(\mu)$ and $\eta^\theta = \eta$. Then, $\eta \circ \theta \in L(\mu)$.*

Proof. Firstly we establish that

$$\eta^\theta \circ \theta = \theta \circ \eta^\theta.$$

Note that, by Lemma 1, we have $\bigvee_{x\in G} \theta\eta\theta^{-1}(x) = \eta(e)\wedge\theta(e)$. Let $x \in G$ and $a_0 = \eta(e)\wedge\theta(e)$. Then,

$$\eta^\theta \circ \theta(x) = \bigvee_{x=u_iv_i} \{\eta^\theta(u_i) \wedge \theta(v_i)\}$$

$$= \bigvee_{x=u_iv_i} \left\{ \left\{ \bigvee_{a_j^i \leq a_0} \{a_j^i : u_i \in \langle(\theta\eta\theta^{-1})_{a_j^i}\rangle\} \right\} \wedge \theta(v_i) \right\}$$

$$= \bigvee_{x=u_iv_i} \left\{ \left\{ \bigvee_{a_j^i \leq a_0} \{a_j^i \wedge \theta(v_i) : u_i \in \langle(\theta\eta\theta^{-1})_{a_j^i}\rangle\} \right\} \wedge \theta(v_i) \right\}.$$

(As L is a completely distributive lattice)

Let $a_j^i \leq a_0$ and $u_i \in \langle(\theta\eta\theta^{-1})_{a_j^i}\rangle$. It can be verified, as in Theorem 9, that

$$v_i^{-1}u_iv_i \in \langle(\theta\eta\theta^{-1})_{a_j^i\wedge\theta(v_i)}\rangle.$$

Thus,

$$\eta^\theta \circ \theta(x) = \bigvee_{x=u_iv_i} \left\{ \left\{ \bigvee_{a_j^i \leq a_0} \{a_j^i \wedge \theta(v_i) : u_i \in \langle(\theta\eta\theta^{-1})_{a_j^i}\rangle\} \right\} \wedge \theta(v_i) \right\}$$

$$\leq \bigvee_{x=u_iv_i} \left\{ \left\{ \bigvee_{c \leq a_0} \{c : v_i^{-1}u_iv_i \in \langle(\theta\eta\theta^{-1})_c\rangle\} \right\} \wedge \theta(v_i) \right\}$$

$$= \bigvee_{x=v_i(v_i^{-1}u_iv_i)} \{\eta^\theta(v_i^{-1}u_iv_i) \wedge \theta(v_i)\} \text{ (by Theorem 6)}$$

$$\leq \theta \circ \eta^\theta(x).$$

Similarly, we can prove that $\theta \circ \eta^\theta \subseteq \eta^\theta \circ \theta$. This implies $\theta \circ \eta^\theta = \eta^\theta \circ \theta$. As $\eta^\theta = \eta$, we have $\eta \circ \theta = \theta \circ \eta$. Therefore, by Proposition 2, $\eta \circ \theta \in L(\mu)$.

In order to introduce the notion of subnormality of an L-subgroup of an L-group, firstly we define a descending series. For $\eta \in L(\mu)$, define a series of L-subgroups of μ inductively as follows:

$$\eta_0 = \mu, \quad \eta_1 = \eta^\mu, \quad \eta_2 = \eta^{\eta_1}, \ldots, \eta_i = \eta^{\eta_{i-1}} \ldots$$

By Theorem 8, η_1 is the smallest normal L-subgroup of μ containing η and η_2 is the smallest normal L-subgroup of η_1 containing η and so on. Thus, we have

$$\eta \subseteq \cdots \lhd \eta_{i+1} \lhd \eta_i \lhd \cdots \lhd \eta_1 \lhd \eta_0 = \mu.$$

This inductively defined series is known as the normal closure series of η in μ and we call η_i the ith normal closure of η in μ. In view of Proposition 4 and using the Principle of Mathematical Induction, we have:

Proposition 5. *Let $\eta \in L(\mu)$. Then, $\eta_i(e) = \eta(e)$ for each i.*

Theorem 12. *Let $\eta, \theta \in L(\mu)$ such that $\eta(e) = \theta(e)$. Let η_i be the ith normal closure of η in μ and $\eta_i^\theta = \eta_i$. Then, $\eta_{i+1} \in NL(\eta_i \circ \theta)$.*

Proof. In order to establish the result, firstly we show that

$$\eta_{i+1} \subseteq \eta_i \circ \theta.$$

By Proposition 5, for each i, we have $\eta_i(e) = \eta(e)$. And since $\eta(e) = \theta(e)$, we have $\eta_i(e) = \theta(e)$ for each i. Thus in view of Proposition 3, $\eta_i \subseteq \eta_i \circ \theta$ and hence by the definition of ith normal closure of η in μ, we obtain

$$\eta_{i+1} \subseteq \eta_i \subseteq \eta_i \circ \theta.$$

Moreover as $\eta_i^\theta = \eta_i$, by Theorem 11, $\eta_i \circ \theta \in L(\mu)$. Thus, $\eta_{i+1} \in L(\eta_i \circ \theta)$. Here note that

$$\begin{aligned}
\theta \eta_{i+1} \theta^{-1}(e) &= \eta_{i+1}(e) \wedge \theta(e) \text{ (by Lemma 1)} \\
&= \eta(e) \wedge \theta(e) \text{ (as } \eta_i(e) = \eta(e)) \\
&= \eta(e) \quad \text{(as } \eta(e) = \theta(e)) \\
&= \eta_{i+1}(e).
\end{aligned}$$

So by Lemma 1, $\eta_{i+1} \subseteq \theta \eta_{i+1} \theta^{-1}$. Now to prove that $\eta_{i+1} \in NL(\eta_i \circ \theta)$, let $x, g \in G$ and consider

$$\eta_{i+1}(x) \wedge \eta_i \circ \theta(g) \leq \theta \eta_{i+1} \theta^{-1}(x) \wedge \eta_i \circ \theta(g)$$

$$= \left\{ \bigvee_{x = y_j z_j y_j^{-1}} \{\eta_{i+1}(z_j) \wedge \theta(y_j)\} \right\} \wedge \left\{ \bigvee_{g = u_k v_k} \{\eta_i(u_k) \wedge \theta(v_k)\} \right\}$$

$$= \bigvee_{x=y_j z_j y_j^{-1}} \left\{ \{\eta_{i+1}(z_j) \wedge \theta(y_j)\} \wedge \left\{ \bigvee_{g=u_k v_k} \{\eta_i(u_k) \wedge \theta(v_k)\} \right\} \right\}$$

(as L is a completely distributive lattice)

$$= \bigvee_{x=y_j z_j y_j^{-1}} \left\{ \bigvee_{g=u_k v_k} \{\eta_{i+1}(z_j) \wedge \theta(y_j)\} \wedge \{\eta_i(u_k) \wedge \theta(v_k)\} \right\}$$

(as L is a completely distributive lattice)

$$= \bigvee_{\substack{x=y_j z_j y_j^{-1} \\ g=u_k v_k}} \{\eta_{i+1}(z_j) \wedge \theta(y_j) \wedge \eta_i(u_k) \wedge \theta(v_k)\}\}$$

$$\leq \bigvee_{\substack{x=y_j z_j y_j^{-1} \\ g=u_k v_k}} \{\eta_{i+1}(z_j) \wedge \theta(v_k y_j) \wedge \eta_i(u_k)\} \quad (\text{as } \theta \in L(\mu))$$

$$= \bigvee_{gxg^{-1}=u_k v_k y_j z_j y_j^{-1} v_k^{-1} u_k^{-1}} \{\eta_{i+1}(z_j) \wedge \theta(v_k y_j) \wedge \eta_i(u_k)\}$$

$$\leq \bigvee_{gxg^{-1}=u_k(v_k y_j z_j y_j^{-1} v_k^{-1})u_k^{-1}} \left\{ \theta \eta_{i+1} \theta^{-1}(v_k y_j z_j y_j^{-1} v_k^{-1}) \wedge \eta_i(u_k) \right\}$$

$$\leq \bigvee_{gxg^{-1}=u_k(v_k y_j z_j y_j^{-1} v_k^{-1})u_k^{-1}} \left\{ \eta_{i+1}^{\theta}(v_k y_j z_j y_j^{-1} v_k^{-1}) \wedge \eta_i(u_k) \right\}$$

$$= \bigvee_{gxg^{-1}=u_k(v_k y_j z_j y_j^{-1} v_k^{-1})u_k^{-1}} \left\{ \eta_{i+1}(v_k y_j z_j v_k^{-1} y_j^{-1}) \wedge \eta_i(u_k) \right\} \quad (\text{as } \eta_{i+1}^{\theta} = \eta_{i+1})$$

$$\leq \bigvee_{gxg^{-1}=u_k(v_k y_j z_j y_j^{-1} v_k^{-1})u_k^{-1}} \left\{ \eta_{i+1}(u_k y_j v_k z_j v_k^{-1} y_j^{-1} u_k^{-1}) \right\} \quad (\text{as } \eta_{i+1} \in NL(\eta_i))$$

$$= \eta_{i+1}(gxg^{-1}).$$

Now, define the notion of subnormal L-subgroup of an L-group as follows:

Definition 12. *Let $\eta \in L(\mu)$ and η_i be the ith normal closure of η in μ. If there exists a non negative integer m such that*

$$\eta_m = \eta \lhd \eta_{m-1} \lhd \cdots \lhd \eta_0 = \mu,$$

then η is known as a subnormal L-subgroup of μ with defect m. We shall denote a subnormal L-subgroup η of μ with defect m by $\eta \overset{m}{\lhd} \mu$. If η is a subnormal L-subgroup of μ, then we shall also write 'η is subnormal in μ.'

Remark 1. *Obviously m equals 0 if $\eta = \mu$ and $m = 1$ if $\eta \in NL(\mu)$ and $\eta \neq \mu$.*

The following example illustrates the notion of subnormal L-subgroup of an L-group:

Example 2. *Recall that in Example 1, the normal closure 'η^{μ}' of η in μ is an L-subgroup of μ defined as follows:*

$$
\eta_1(z) = \eta^{\mu}(z) = \begin{cases} \dfrac{1}{3} & \text{if } z \in K_4, \\[2mm] \dfrac{1}{4} & \text{if } z \in D_4 \sim K_4, \\[2mm] \dfrac{1}{9} & \text{if } z \in D_8 \sim D_4. \end{cases}
$$

Similarly, one can verify that the conjugate '$\eta_1\eta(\eta_1)^{-1}$' is defined by the following level subsets:

$$
(\eta_1\eta(\eta_1)^{-1})_{\frac{1}{3}} = \{e, x, xy^4\},
$$
$$
(\eta_1\eta(\eta_1)^{-1})_{\frac{1}{6}} = \{e, x, xy^2, xy^4, xy^6, y^2, y^4, y^6\},
$$
$$
(\eta_1\eta(\eta_1)^{-1})_{\frac{1}{9}} = D_8 .
$$

So in view of Theorem 6, the normal closure of η in η_1 has the following definition:

$$
\eta_2(z) = \langle \eta_1\eta(\eta_1)^{-1} \rangle(z) = \begin{cases} \dfrac{1}{3} & \text{if } z \in K_4, \\[2mm] \dfrac{1}{6} & \text{if } z \in D_4 \sim K_4, \\[2mm] \dfrac{1}{9} & \text{if } z \in D_8 \sim D_4. \end{cases}
$$

Moreover, one can verify that the conjugate '$\eta_2\eta(\eta_2)^{-1}$' is defined by the following level subsets:

$$
(\eta_2\eta(\eta_2)^{-1})_{\frac{1}{3}} = \langle x \rangle, \quad (\eta_2\eta(\eta_2)^{-1})_{\frac{1}{6}} = D_4, \; and \; (\eta_2\eta(\eta_2)^{-1})_{\frac{1}{9}} = D_8.
$$

As $\eta_2\eta(\eta_2)^{-1} \in L(\mu)$, we have $\eta_3 = \eta^{\eta_2} = \eta_2\eta(\eta_2)^{-1} = \eta$. Consequently, we have

$$
\eta_3 = \eta \lhd \eta_2 \lhd \eta_1 \lhd \eta_0 = \mu.
$$

Hence η is a subnormal L-subgroup of μ with defect 3.

Next, we provide the definition of a subnormal seies for an L-subgroup:

Definition 13. *Let $\eta \in L(\mu)$. A finite series $\theta_0 = \mu, \theta_1, \theta_2, ..., \theta_m = \eta$ of L-subgroups of μ such that*

$$
\eta = \theta_m \lhd \theta_{m-1} \lhd ... \lhd \theta_0 = \mu
$$

is said to be a subnormal series of η.

We shall describe the notion of a subnormal L-subgroup through the notion of above defined subnormal series like their classical counterparts. The following lemma, is a step in this direction:

Lemma 2. *Let $\eta \in L(\mu)$ and*

$$\eta \subseteq \cdots \triangleleft \eta_{i+1} \triangleleft \eta_i \triangleleft \cdots \triangleleft \eta_1 \triangleleft \eta_0 = \mu$$

be the normal closure series of η. If there exists a descending series $\gamma_0 = \mu, \gamma_1, ..., \gamma_i...$ of L-subgroups of μ such that

$$\eta \subseteq ... \triangleleft \gamma_{i+1} \triangleleft ... \gamma_1 \triangleleft \gamma_0 = \mu,$$

then $\eta_i \subseteq \gamma_i$.

Proof. We shall prove the result by induction on i. For $i = 1, \gamma_1$ is a normal L-subgroup of μ containing η. But by Theorem 8, $\eta_1 = \eta^\mu$ is the smallest normal subgroup of μ containing η. Hence

$$\eta_1 \subseteq \gamma_1.$$

Now, we set the induction hypothesis as $\eta_k \subseteq \gamma_k$. Thus, in view of the definition of a conjugate L-subset, we have

$$\eta_{k+1} = \eta^{\eta_k} \subseteq \eta^{\gamma_k}. \tag{1}$$

Note that as $\eta \subseteq \gamma_k$, by Theorem 8, η^{γ_k} is the smallest normal L-subgroup of γ_k containing η. Moreover, γ_{k+1} is also a normal L-subgroup of γ_k containing η. This implies

$$\eta^{\gamma_k} \subseteq \gamma_{k+1}.$$

Therefore, by using (1), we have

$$\eta_{k+1} = \eta^{\eta_k} \subseteq \eta^{\gamma_k} \subseteq \gamma_{k+1}.$$

This proves the result completely.

The following result inter-connects the notions of subnormality and subnormal L-series of an L-subgroup:

Theorem 13. *Let $\eta \in L(\mu)$. Then, η is a subnormal L-subgroup of μ having defect m if and only if η has a subnormal series*

$$\eta = \gamma_m \triangleleft ... \triangleleft \gamma_{i+1} \triangleleft ... \gamma_1 \triangleleft \gamma_0 = \mu,$$

of length m and m is the smallest length of such a subnormal series.

Proof. If η is a subnormal L-subgroup of μ, then the normal closure series is the required subnormal series. Conversely, suppose that η has a subnormal series say

$$\eta = \theta_m \triangleleft \theta_{m-1} \triangleleft ... \triangleleft \theta_0 = \mu.$$

Then, by Lemma 2, $\eta_m \subseteq \theta_m$. As $\eta = \theta_m$, we have

$$\eta \subseteq \eta_m \subseteq \theta_m = \eta.$$

Hence, $\eta_m = \eta$. This proves that η is a subnormal L-subgroup of μ.

The following results are established by using the above theorem:

Theorem 14. *Let η be a subnormal L-subgroup of μ with defect m.*

(i) Let $\theta \in L(\mu)$. Then, $\eta \cap \theta$ is a subnormal L-subgroup of θ with defect c where $c \leq m$. In particular, η is a subnormal L-subgroup of λ where $\lambda \in L(\mu)$ such that $\eta \subseteq \lambda$ with defect c where $c \leq m$,

(ii) Let $\theta \in NL(\mu)$. Then, $\eta \circ \theta$ is a subnormal L-subgroup of μ with defect c where $c \leq m$.

It can be seen easily that the intersection of any finite set of subnormal L-subgroups is again subnormal. More generally :

Theorem 15. *Let $\{\theta_i : i \in I\}$ be a family of subnormal L-subgroups such that defect of θ_i is m_i where $m_i \leq m$. Then, $\underset{i \in I}{\cap} \theta_i$ is a subnormal L-subgroup of μ with defect c where $c \leq m$.*

Next result determines the transitivity of the notion of subnormality.

Theorem 16. *Let $\eta, \theta \in L(\mu)$ such that η is a subnormal L-subgroup of θ with defect m and θ is a subnormal L-subgroup of μ with defect n. Then, η is a subnormal L-subgroup of μ with defect m+n.*

The following theorem establishes that the subnormality in L-setting is also preserved under the action of a homomorphism and its inverse image:

Theorem 17. *Let $\eta \in L(\mu)$ and $f : G \to K$ be a group homomorphism. Then,*

(i) if η is a subnormal L-subgroup of μ with defect n, then $f(\eta)$ is a subnormal L-subgroup of $f(\mu)$ with defect m where $m \leq n$,

(ii) if η is a subnormal L-subgroup of μ with defect n, then $f^{-1}(\eta)$ is a subnormal L-subgroup of $f^{-1}(\mu)$ with defect m where $m \leq n$, provided that the group homomoirphism f is onto.

4. Subnormal L-subgroups and nilpotency

In this section, we characterize subnormal L-subgroups by the usual group theoretic subnormality of the level subsets of the given L-subgroups. We shall refer this as a level subset characterization of subnormality. Then, this characterization is used to establish that when the lattice L is an upper well ordered chain, then every L-subgroup of a nilpotent L-group is subnormal. For this purpose, we need to develop a nececcary mechanism. So we start with a characterization of the notion of sup-property which lends itself more easily for applications.

Definition 14. *A non-empty subset X of a lattice L is said to be a supstar subset of L if every non-empty subset A of X contains its supremum. That is, if $\sup A = a_0$, then $a_0 \in A$.*

Clearly, a subset of a supstar subset is again supstar.

Proposition 6. *Let $\eta \in L^\mu$. Then, η has sup-property if and only if $Im\eta$ is a supstar subset of L.*

The above characterization also allows a generalization of the concept of sup-property to an arbitrary family of L-subsets and hence widens the scope of its applications.

Definition 15. *Let $\{\eta_i\}_{i \in I} \subseteq L^\mu$. Then, $\{\eta_i\}_{i \in I}$ is said to be a supstar family if $\underset{i \in I}{\cup} Im\eta_i$ is a supstar subset of L. As a particular case, we say that two L-subsets θ and η are jointly supstar if $Im\theta \cup Im\eta$ is a supstar subset of L.*

In view of the above definition and Proposition 6, we have the following:

Proposition 7. *Each member of a supstar family of L-subsets satisfies sup-property.*

Lemma 3. *Let $\eta \in L^\mu$ and η has sup-property. If $a_0 = \underset{x \in G}{\vee}\{\eta(x)\}$, then $\langle \eta_b \rangle = \langle \eta \rangle_b$ for each $b \leq a_0$.*

Lemma 4. *Let $\eta \in L(\mu)$ be such that μ and η are jointly supstar.. Then, $Im\eta \cup Im\mu$ is a chain.*

Next, we prove:

Lemma 5. *Let $\eta \in L(\mu)$ be such that μ and η are jointly supstar. Then,*

$$Im\ \eta^\mu \subseteq Im\ (\mu\eta\mu^{-1}) \subseteq Im\mu \cup Im\eta.$$

Proof.
Let $a \in Im(\mu\eta\mu^{-1})$. So, there exists $x \in G$ such that $a = (\mu\eta\mu^{-1})(x)$. Now, define the following subset of $G \times G$:

$$C(x) = \{(v, u) \in G \times G : x = uvu^{-1}\}.$$

Thus,

$$a = (\mu\eta\mu^{-1})(x) = \bigvee_{(y,z) \in C(x)} \{\eta(y) \wedge \mu(z)\}. \tag{1}$$

Now, for any $(y, z) \in C(x)$,

$$\{\eta(y), \mu(z)\} \subseteq Im\eta \cup Im\mu.$$

As μ and η are jointly supstar, by Lemma 4, $Im\eta \cup Im\eta$ is a chain. Hence $\eta(y) \wedge \mu(z) = \eta(y)$ or $\mu(z)$ which is an element of $Im\eta \cup Im\mu$. As $Im\eta \cup Im\mu$ is supstar, the subset

$\{\eta(y) \wedge \mu(z) : (y, z) \in C(x)\} \subseteq Im\eta \cup Im\mu$ contains its supremum a and hence $a = (\mu\eta\mu^{-1})(x) \in Im\mu \cup Im\eta$. Therefore,

$$Im\ (\mu\eta\mu^{-1}) \subseteq Im\mu \cup Im\eta.$$

Further, as a subset of a supstar subset is again supstar, it follows that $Im(\mu\eta\mu^{-1})$ is also supstar. Hence by Proposition 6, $\mu\eta\mu^{-1}$ possesses sup-property. Hence in view of Theorem 7, it follows that

$$Im\eta^{\mu} = Im\langle\mu\eta\mu^{-1}\rangle \subseteq Im\ (\mu\eta\mu^{-1}) \subseteq Im\mu \cup Im\eta.$$

More generally we have:

Lemma 6. *Let $\eta \in L(\mu)$ be such that μ and η are jointly supstar. Then, for each i*

$$Im\ \eta_{i+1} \subseteq Im\ (\eta_i\eta\eta_i^{-1}) \subseteq Im\mu \cup Im\eta,$$

where η_i is the ith normal closure of η in μ.

Corollary 4. *Let $\eta \in L(\mu)$ be such that μ and η are jointly supstar. Then for each i, η and η_i are jointly supstar.*

Remark 2. *Note that if $\eta \in L(\mu)$ and μ and η be jointly supstar, then the L-subset $\eta_i\eta\eta_i^{-1}$ possesses sup-property for each i.*

Below, we discuss the level subset of the normal closure of η in μ.

Lemma 7. *Let $\eta \in L(\mu)$. Then,*

(i) *$(\eta^{\mu})_a = (\eta_a)^{\mu_a}$ for each $a \leq \eta(e)$ provided μ and η are supstar,*

(ii) *$(\eta^{\mu})_a^{>} = (\eta_a^{>})^{\mu_a^{>}}$ provided L is a chain.*

Proof. (i) Let $a \leq \eta(e)$. We show that

$$(\mu\eta\mu^{-1})_a \subseteq \mu_a\eta_a(\mu_a)^{-1}. \tag{1}$$

So, let $x \in (\mu\eta\mu^{-1})_a$. Then,

$$\mu\eta\mu^{-1}(x) = \bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \mu(z)\} \geq a.$$

As μ and η are jointly supstar, it can be verified, as in Lemma 5, that there exists y_0 and $z_0 \in G$ such that $x = z_0y_0z_0^{-1}$ and

$$\bigvee_{x=zyz^{-1}} \{\eta(y) \wedge \mu(z)\} = \eta(y_0) \wedge \mu(z_0).$$

This implies
$$\mu\eta\mu^{-1}(x) = \eta(y_0) \wedge \mu(z_0) \geq a.$$

Therefore, $\eta(y_0)$ and $\mu(z_0) \geq a$. So, we have $y_0 \in \eta_a$ and $z_0 \in \mu_a$. Hence,
$$x = z_0 y_0 z_0^{-1} \in \mu_a \eta_a (\mu_a)^{-1}.$$

This establishes (1). In order to prove the reverse inclusion, let $x \in \mu_a \eta_a(\mu_a)^{-1}$. Then, there exist $y_0 \in \eta_a$ and $z_0 \in \mu_a$ such that $x = z_0 y_0 (z_0)^{-1}$. Therefore,

$$\mu\eta\mu^{-1}(x) = \bigvee_{x = zyz^{-1}} \{\eta(y) \wedge \mu(z)\}$$
$$\geq \eta(y_0) \wedge \mu(z_0)$$
$$\geq a.$$

Hence $x \in (\mu\eta\mu^{-1})_a$. Thus,
$$\mu_a \eta_a (\mu_a)^{-1} \subseteq (\mu\eta\mu^{-1})_a.$$

Therefore,
$$\mu_a \eta_a (\mu_a)^{-1} = (\mu\eta\mu^{-1})_a.$$

Further, as μ and η are jointly supstar, by Remark 2 the L-subset $\mu\eta\mu^{-1}$ possesses sup-property. Hence, by Lemma 3, we have

$$(\eta^\mu)_a = \langle \mu\eta\mu^{-1} \rangle_a = \langle (\mu\eta\mu^{-1})_a \rangle = \langle \mu_a \eta_a (\mu_a)^{-1} \rangle = (\eta_a)^{\mu_a}.$$

This proves (i).

(ii) It can be prove by using the arguments of part(i).

Let η_i be the *ith* normal closure of η in μ. Then, more generally we have :

Lemma 8. *Let $\eta \in L(\mu)$. Then,*

(i) $(\eta_i)_a = (\eta_a)_i$ *for each $a \leq \eta(e)$ provided μ and η are supstar,*

(ii) $(\eta_i)_a^{\geq} = (\eta_a^{\geq})_i$ *provided L is a chain.*

Theorem 18. *Let $\eta \in L(\mu)$ be such that η and μ are jointly supstar. Then, η is subnormal having defect at most n if and only if each level subset η_a is subnormal having defect atmost n where $a \leq \eta(e)$.*

Proof. **(Condition is necessary.)** Since η is subnormal in μ having defect atmost n, by the definition, there exists a positive integer $m \leq n$ such that

$$\mu = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_m = \eta, \tag{1}$$

where η_i is the *ith* normal closure of η in μ. Let $a \leq \eta(e)$. As $\eta \in L(\mu)$, by Theorem 2, η_a is a subgroup of μ_a. We show that η_a is subnormal in μ_a. Note that, in view of (1) we have

$$\mu_a = (\eta_0)_a \supseteq (\eta_1)_a \supseteq \cdots \supseteq (\eta_m)_a = \eta_a.$$

Further, as η and μ are jointly supstar, by Lemma 8

$$(\eta_a)_i = (\eta_i)_a \text{ for all } i.$$

Thus, we have

$$\mu_a = (\eta_a)_0 \supseteq (\eta_a)_1 \supseteq \cdots \supseteq (\eta_a)_m = \eta_a.$$

Consequently, the normal closure series of η_a in μ_a has the length atmost m. As $m \leq n$, η_a is subnormal in μ_a having defect atmost n.

(Condition is Sufficient.) In order to show that η is subnormal having defect atmost n, we construct the normal closure series of η in μ and show that its length is atmost n. Now, by the hypothesis, each non-empty level subgroup of η is subnormal having defect atmost n. Let m_{a_j} be the defect of the level subgroup η_{a_j} in μ_{a_j} for any $a_j \leq \eta(e)$. Then, the normal closure series of η_{a_j} in μ_{a_j} is given by

$$\mu_{a_j} = (\eta_{a_j})_0 \supseteq (\eta_{a_j})_1 \supseteq ... \supseteq (\eta_{a_j})_{m_{a_j}} = \eta_{a_j} \text{ for all } a_j \leq \eta(e), \tag{2}$$

where $(\eta_{a_j})_i$ is the ith normal closure of η_{a_j} in μ_{a_j}. Next, let $C = Im\eta \cup \{a \in Im\mu : a \leq \eta(e)\}$. Then, by Lemma 4, C is a chain. Let $S = \{m_{a_j} \leq n : a_j \in C\}$. Then, $\sup S \leq n$. We write $m = \sup S \leq n$. Now, we shall construct the normal closure series of η in μ having the length m. In view of (2), we have

$$\mu_{a_j} = (\eta_{a_j})_0 \supseteq (\eta_{a_j})_1 \supseteq ... \supseteq (\eta_{a_j})_{m_{a_j}} = \eta_{a_j} \text{ for all } a_j \in C.$$

Further, as $m_{a_j} \leq m$ for each $a_j \in C$, we insert $m - m_{a_j}$ times the level subgroup η_{a_j} in the normal closure series of η_{a_j} in μ_{a_j}. Consequently, the normal closure series of η_{a_j} in μ_{a_j} aquire the length m. Obviously, for all $l = 1, 2, ..., m - m_{a_j}$

$$(\eta_{a_j})_{m_{a_j}+l} = \eta_{a_j}.$$

Thus

$$\mu_{a_j} = (\eta_{a_j})_0 \supseteq (\eta_{a_j})_1 \supseteq ... \supseteq (\eta_{a_j})_{m_{a_j}} = \eta_{a_j} \text{ for all } a_j \in C. \tag{3}$$

In view of Lemma 8, we have

$$\mu_{a_j} = (\eta_0)_{a_j} \supseteq (\eta_1)_{a_j} \supseteq ... \supseteq (\eta_m)_{a_j} = \eta_{a_j} \text{ for all } a_j \in C.$$

Let $a_j, a_k \in C$ and $a_j \geq a_k$. Hence for each fixed i, it follows that

$$\begin{aligned}
(\eta_{a_j})_i &= (\eta_{a_j})^{(\eta_{a_j})_{i-1}} \text{ (by the definition of classical } ith \text{ normal closure)} \\
&= \langle (\eta_{a_j})_{i-1} \eta_{a_j} ((\eta_{a_j})_{i-1})^{-1} \rangle \\
&\subseteq \langle (\eta_{a_k})_{i-1} \eta_{a_k} ((\eta_{a_k})_{i-1})^{-1} \rangle \\
&= (\eta_{a_k})_i.
\end{aligned} \tag{4}$$

Thus for a fixed i the *ith* normal closures of the level subsets η_{a_j} for each $a_j \in C$ constitutes a chain. In view of the fact that the level subsets η_{a_j} are ordinary subgroups,

$$\eta_{a_j}^{\eta_{a_j}} = \eta_{a_j},$$

insertion of the level subgroup η_{a_j} in the chain of its *ith* normal closures does not affect the above containment. In view of Lemma 8 and by (4), we obtain

$$(\eta_i)_{a_j} \subseteq (\eta_i)_{a_k}.$$

In view of (3), for each i, we have

$$(\eta_{i-1})_{a_j} \supseteq (\eta_i)_{a_j} \text{ for each } a_j \in C.$$

Now we claim that

$$(\eta_{i-1})_{a_j} \supseteq (\eta_i)_{a_j} \text{ for each } a_j \in Im\ \mu \cup Im\ \eta.$$

Here observe that, in view of Proposition 5, the tip of η_i, for each i, is $\eta(e)$. Hence

$$(\eta_i)_{a_j} = \phi \text{ for all } a_j \in \{a \in Im\ \mu : a \not\leq \eta(e)\}.$$

This establishes the claim. Further, by Lemma 6, for each i

$$Im\ \eta_i \subseteq Im\ \eta \cup Im\ \mu.$$

Thus, in view of Proposition 1, we have

$$\eta_{i-1} \supseteq \eta_i \text{ for each } i.$$

Consequently,

$$\mu = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_m = \eta.$$

Then, the normal closure series of η in μ has the length atmost $m \leq n$. Therefore, η is a subnormal L-subgroup of μ having defect atmost n.

Corollary 5. *Let G be a group and H be its subgroup. Then, H is a subnormal subgroup of G if and only if 1_H is a subnormal L-subgroup of 1_G.*

The strong level subset characterization of subnormal L-subgroup can be obtained easily by using the arguments of Theorem 18. So, we state it without proof:

Theorem 19. *Let L be chain and $\eta \in L(\mu)$. Then, η is subnormal having defect at most n if and only if each strong level subset $\eta_a^>$ is subnormal having defect atmost n where $a < \eta(e)$.*

Recall the following result from classical group theory [11, 13]:

Theorem 20. *Every subgroup of a nilpotent group is subnormal.*

Now, we shall deal with this result in L-group theory provided L is an upper well ordered chain. A chain is said to be upper well ordered if every non-empty subset of the given chain has a greatest element. Clearly, every subset of an upper well ordred chain is a supstar subset. Consequently, each L-subset for an upper well ordered chain L satisfies sup-property.

Proposition 8. *Let L be an upper well ordered chain and $\eta, \theta \in L^\mu$. Then, η and θ are jointly supstar if and only if η and θ possess sup-property.*

Below, we recall the definition of descending central chain of an L-subgroup η of μ from [3]:

Take $\gamma_0(\eta) = \eta$, $\gamma_1(\eta) = [\gamma_0(\eta), \eta]$. And in general, for each i, we define $\gamma_i(\eta) = [\gamma_{i-1}(\eta), \eta]$.

Proposition 9. *Let $\eta \in L(\mu)$. Then for each $i, \gamma_i(\eta) \subseteq \gamma_{i-1}(\eta)$.*

Definition 16. *Let $\eta \in L(\mu)$. Then, the chain*

$$\eta = \gamma_0(\eta) \supseteq \gamma_1(\eta) \supseteq \cdots \supseteq \gamma_i(\eta) \supseteq \cdots$$

of L-subgroups of μ is called the descending central chain of η.

Definition 17. *Let $\eta \in L(\mu)$ with tip a_0 and tail t_0 and $a_0 \neq t_0$. If the descending central chain*

$$\eta = \gamma_0(\eta) \supseteq \gamma_1(\eta) \supseteq \cdots \supseteq \gamma_i(\eta) \supseteq \cdots$$

terminates finitely to the trivial L-subgroup $\eta_{t_0}^{a_0}$, then η is known as a nilpotent L-subgroup of μ. More precisely, η is said to be nilpotent of class c if c is the least non-negative integer such that $\gamma_c(\eta) = \eta_{t_0}^{a_0}$. In this case, the series

$$\eta = \gamma_0(\eta) \supseteq \gamma_1(\eta) \supseteq \cdots \supseteq \gamma_c(\eta) = \eta_{t_0}^{a_0},$$

is called the descending central series of η. If it is a nilpotent L-subgroup of μ, then we simply write η is nilpotent.

Theorem 21. *Let $\eta \in L(\mu)$ and possesses sup-property. Then, η is a nilpotent L-subgroup of μ of nilpotent length at most n if and only if each level subgroup η_a is a nilpotent subgroup of μ_a of nilpotent length at most n for each $a \not\leq \inf \eta$ and $a \leq \eta(e)$.*

Theorem 22. *Let $\eta \in L(\mu)$ and L be a chain. Then, η is a nilpotent L-subgroup of μ of nilpotent length at most n if and only if $\eta_a^>$ is a nilpotent subgroup of $\mu_a^>$ of nilpotent length at most n, where $\inf \eta \leq a < \eta(e)$.*

The notion of nilpotency of L-subgroup is exihibited in the following:

Example 3. *Let $G = Q_8$ and the evaluation lattice be given by the diagram :*
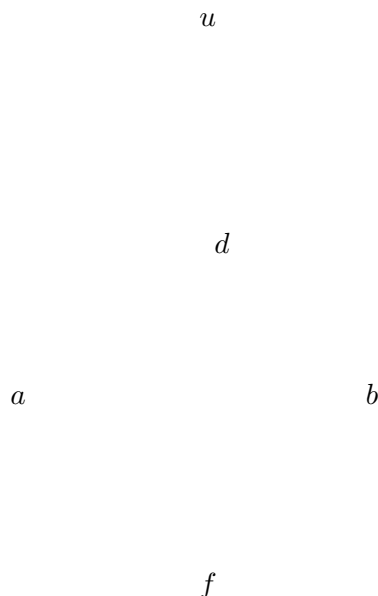
$$u$$

$$d$$

$$a \qquad\qquad b$$

$$f$$

Fig. 1

Consider the parent L-subgroup of G given by:

$$\mu(x) = \begin{cases} u & \text{if } x \in C, \\ d & \text{if } x \in G \setminus C. \end{cases}$$

Now define L-subset η of μ as given below:

$$\eta(x) = \begin{cases} u & \text{if } x \in C, \\ d & \text{if } x \in H_1 \setminus C, \\ a & \text{if } x \in H_2 \setminus C, \\ b & \text{if } x \in H_3 \setminus C; \end{cases}$$

where

$$C = \{\pm 1\}, \quad H_1 = \{\pm 1, \pm i\}, \quad H_2 = \{\pm 1, \pm j\}, \quad H_3 = \{\pm 1, \pm k\}.$$

Since the level subsets of η are normal subgroups of G, η is a normal L-subgroup of G and hence of μ. Now that η is a nilpotent L-subgroup of μ in view of Definition 17, we demonstrate this as follows:

Note that $G' = \{1, -1\}$. In order to obtain the members of descending central series of η, we set $\gamma_0(\eta) = \eta$ and consider the commutator (η, η)

$$(\eta, \eta)(x) = \begin{cases} u & \text{if } x = 1, \\ d & \text{if } x \in C \setminus \{1\}, \\ f & \text{if } x \in G \setminus C. \end{cases}$$

As the level subsets of (η, η) are subgroups of G,

$$\gamma_1(\eta) = [\eta, \eta] = (\eta, \eta).$$

Next, we calculate the commutator :

$$((\eta, \eta), \eta)(x) = \begin{cases} u & \text{if } x = 1, \\ f & \text{if } x \in G \setminus \{1\}. \end{cases}$$

Again, by the reasons as given above

$$\gamma_2(\eta) = [[\eta, \eta], \eta] = ((\eta, \eta), \eta).$$

Observe that $\gamma_2(\eta)$ is not only an L-subgroup, it is the trivial L-subgroup of η and so the descending central series terminates at $\gamma_2(\eta)$, i. e.

$$\eta = \gamma_0(\eta) \supseteq \gamma_1(\eta) \supseteq \gamma_2(\eta) = \eta_f^u.$$

Consequently, η is a nilpotent L-subgroup of μ having nilpotent length 2.

Theorem 23. *Let L be an upper well ordered chain. Let $\eta \in L(\mu)$ and η be nilpotent having the tip a_0 and the tail t_0. If $\theta \in L(\eta)$ having the tail t_0, then θ is a subnormal L-subgroup of η.*

Proof. Let $\theta \in L(\eta)$ having the tail t_0. We shall show that the level subgroup θ_a is a subnormal L- subgroup of η_a for each $a \leq \theta(e)$. If $a \leq t_0$, then obviously θ_a is a subnormal subgroup of η_a. If So, let $t_0 < a \leq \theta(e)$. As L is upper well ordered, it follows the L-subset η possesses sup-property. Now, as $t_0 < a \leq \theta(e) \leq \eta(e)$ and η is nilpotent, by Theorem 21, the level subgroup η_a is nilpotent. Thus θ_a is a subgroup of the nilpotent subgroup η_a, Hence by Theorem 20, θ_a is a subnormal subgroup of η_a. Again as L is an upper well ordered chain, by using the arguments as above, it follows that both the L-subgroups η and θ possess sup-property. Therefore, by Proposition 8, η and θ are joinly supstar. Consequently, by Theorem 18, θ is a subnormal L-subgroup of η.

In the part II of this paper, we develop a mechanism in order to tackle the join problem for subnormal L-subgroup.

Theorem 24. *Let η and θ be subnormal L-subgroups of μ . Let $\eta(e) = \theta(e)$ and $\eta \circ \theta \in L(\mu)$. Then, the following are equivalent:*
    (i) $\eta \circ \theta$ is subnormal in μ,
    (ii) η^{θ} is subnormal in μ,
    (iii) $[\eta, \theta]$ is subnormal in μ,

5. Conclusion

The Theory of L-groups is a very rich generalization of classical group theory. Here we study the group theoretic properties of posets of subgroups of a group or in particular chains of subgroups of a group rather than properties of a single subgroup. As an application of this work and the motivation for the development of L-group theory, we mention that if we replace the lattice L, in our work by the closed unit interval $[0,1]$, then we retrieve the corresponding version of fuzzy group theory. Moreover, as an application of this theory we also mention that if we replace the lattice L by the two elements set $\{0,1\}$, then the results of classical group theory follow as simple corollaries of the corresponding results of L group theory. This way, L-group theory provides us a new language and a new tool for the study of the classical group theory. The classical group theory has been founded on abstract sets and therefore the language used for its development is formal set theory. On the other hand, L-group theory expresses itself through the language of functions. The functions which are lattice valued. Therefore the approach adopted in the studies of L-group theory can be looked upon as a modernization of the approach of classical group theory.

Acknowledgements

References

[1] N Ajmal. Set product and fuzzy subgroups. In R Lowen and M Roubens, editors, *Proc. of the Fourth IFSA World Congress.*, pages 3–7, Belgium, 1991. University of Antwerp, Antwerp.

[2] N Ajmal and I Jahan. An L-point characterization of normality and normalizer of an L-subgroup of an L-group. *Fuzzy Information and Engineering*, 6(1):147–166, 2014.

[3] N Ajmal and I Jahan. Nilpotency and theory of L subgroups of an L-group. *Fuzzy Information and Engineering*, 6(1):1–17, 2014.

[4] N Ajmal and I Jahan. Normal Closure of an L-subgroup of an L-group. *The Journal of Fuzzy Mathematics*, 22(1):115–126, 2014.

[5] N Ajmal and I Jahan. Generated L-subgroup of an L-group. *Iranian Journal of Fuzzy Systems*, 12(1):129–136, 2015.

[6] N Ajmal and I Jahan. Solvable L-subgroup of an L-group. *Iranian Journal of Fuzzy Systems*, 12(3):151–161, 2015.

[7] N Ajmal, I Jahan, and B Davvaz. Nilpotent L-subgroups and the set product of L-subsets. *European Journal of Pure and Applied Mathematics*, 10(2):255–271, 2017.

[8] J A Goguen. *L*-fuzzy sets. *J. Math. Anal. Appl.*, 18(1):145–17, 1967.

[9] T Head. A metatheorem for deriving fuzzy theorems from crisp versions. *Fuzzy Sets and System*, 73(3):349–358, 1995.

[10] I Jahan, B Davvaz, and N Ajmal. Nilpotent *l*-subgroups satisfy the normalizer condition. *Journal of Intelligent and Fuzzy Systems*, 33(3):1841–1854, 2017.

[11] J C Lennox and S Stonehewer. *Subnormal Subgroups of Groups.* Oxford University Press, New York, 1987.

[12] A Razzaque and A Razaq. On *q*-Rung Orthopair Fuzzy Subgroups. *Journal of Function Spaces*, 2022(1):1–9, 2022.

[13] D J S Robinson. *A course in the Theory of Groups.* Springer-Verlag, New York, 1980.

[14] A Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.*, 35(3):512–517, 1971.

[15] Y Yu, S C Cheng, and J N Mordeson. *Elements of algebra(Lecture Notes in Fuzzy Mathematics and Computer Science).* Center for Research in Fuzzy Mathematics and Computer Science, Creighton University, USA, 1994.