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The axisymmetric migration of an aerosol particle embedded in a Brinkmann medium of a couple stress fluid with slip regime

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Abstract. This study examined the relationship between stresses and couple stresses within the context of the couple stress fluid theory. The application of the effective medium approach, based on the Brinkman equation, is also discussed under the effect of slip and spin slip on the surface of the solid sphere immersed in porous couple stress fluid. The motion is generated by moving a solid sphere at a constant speed. Using Stokesian assumptions, the equation of motion ignores nonlinear terms. The surface conditions for slip and spin slip of couple stress fluids have been applied. Tables and graphs are used to represent the normalized drag force exerted by the fluid flow on the solid sphere. According to the numerical results, the drag force increases monotonically with permeability and couple stress. As a result, the results obtained are consistent with those found in the literature for viscous fluids, and the special cases are determined.

2020 Mathematics Subject Classifications: 76-XX, 76DXX

Key Words and Phrases: Couple stress fluid, Slip and Spin slip, Permeability

1. Introduction

Recent investigations pertaining to the theory of couple stress fluid manner are extremely valuable, and such analyses provide sufficient insight into the behavior of rheological complex fluids, such as polymeric suspensions containing long-chain molecules, liquid crystals, lubricants, and human blood. According to couple stress fluid theory, particle sizes are considered when describing a non-Newtonian fluid. A theory of microcontinuum obtained by Stokes [1] is considered to take into account the particle size effects in order to capture the significance of couple stresses. As a generalization of the traditional theory of fluids, Stokes microcontinuum allows for polar impacts, such as the existence of couple

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stresses, body couples, and antisymmetric stress tensors. For example, in pipe Poiseuille flow, even after all the variables have been nondimensionalized in the usual way, the velocity profile is a function of the pipe radius Stokes[2].

Thus, couple stresses may be important under lubrication conditions, where thin films are usually formed. Additionally, the theory of couple stress fluid flow introduces nonlinear terms in the relationship between shear stresses and velocity gradients. Accordingly, the lubricant should be considered non-Newtonian and described by two constants, shear viscosity, and couple stress. Therefore, Elsharkawy and Guedouar [3] developed a reverse solution for finite journal bearings lubricated with couple stress fluids to evaluate eccentricity proportion and couple stress parameters for an experimentally estimated pressure diffusion. In recent years, the interaction between two rigid spheres in an incompressible fluid flow has become increasingly important [4–8] and [9]. Furthermore, oscillatory flows allow one to observe and calculate multiple elastic effects of dilute polymers in a convenient manner. As a result of free stream oscillations, important physical problems arise, such as [10, 11].

Further, slip flow is more practical than no-slip flow (slide surface, flow through wetted surface, brick wall). It plays a significant role to illustrate the macroscopic influences in the study of fluid-solid surface interactions when fluid slips at a rigid wall. It is assumed that there is no slip in most investigations of peristaltic flow, however, wall slip is considered in a few types of research. Moreover, Saad [12] solved multiple problems such as the unstable plane Couette flow of an incompressible couple stress fluid with the slip conditions at both of the plates are assumed to be of different parameters and couple stresses at the boundaries are not present, the application of a time-dependent pressure gradient under the assumptions of the slip conditions and the zero couple stresses at the two boundaries, and the unsteady Poiseuille flow of an incompressible couple stress fluid inside an infinite cylinder. Moreover, Saad and Ashmawy [13] investigated the advective flow of an incompressible couple stress fluid between two parallel plates with slip boundary conditions put on the two plates and vanishing couple stress conditions at the interfaces.

The fluid dynamical process that takes place in many body organs, including vascular beds, lungs, kidneys, and tumorous vessels, must also be explained adequately, which makes study of porosity absolutely essential. The porosity of the media also plays a crucial role in many biomechanical investigations in determining the fluid transport. Parallel flows of fluids with various viscosities and densities through porous media are a common feature of technical procedures. Similar parallel flows can be seen in packed bed reactors used in the chemical industry, petroleum production engineering, sedimentation, diluted polymers, suspensions, journal bearing lubrication, and many other processes. To model flow across porous media, Brinkman [14] established a momentum equation with a Newtonian viscous drag term and a Darcy drag term, which together balance the pressure gradient. When the permeability of the medium approaches 0, the Brinkman equation becomes the Darcy equation, and when the permeability approaches infinity, it becomes the Stokes equation. The equation has been widely used to assess porous media with high porosity, such as fiberglass wool.

Sharma and Thakur [15] assumed a layer of electrically conducting couple stress fluid warmed from beneath in a permeable medium in the existence of the magnetic field. Alsaedi et al. [16] analyzed the peristaltic motion of a couple stress fluids occupying the porous medium through large wavelength and low Reynolds number. Opanuga et al. [17] studied the second law analysis of hydromagnetic couple stress fluid via a channel supplied with non Darcian permeable medium beneath the heat of the fluid exchanges with the ambient following Newtonian law. The thermal fluctuation in a layer of couple stress nanofluid saturated permeable medium was studied by Chand et al [18]. Yadav et al [19] introduced the collaborative impact of temperature-reliant viscosity and interior heat generation on the impression of convective movement in couple stress fluids saturated in a thin porous layer by utilizing the linear stability principles. There are multi applications on porous medium and magnetic field such as [20–28].

Thus, the objective of the current study is to examine the regular translational movement of a solid sphere immersed in an incompressible couple stress fluid with a slip regime and slip-spin while also getting the fundamental relationships analytically. The solution of the equation of momentum has been created analytically in terms of a spherical coordinate system. The assessed boundary conditions are then satisfied on the solid sphere surface. The drag force of the fluid flow on the rigid sphere is then calculated numerically and represented graphically for various values of the physical relevant parameters and also two special cases, Case I: perfect slip and spin-slip, and Case II: no-slip and no-spin slip.

2. Mathematical formulation of the problem

Fig. 1 shows the axisymmetric migration of an aerosol particle embedded in an unbounded incompressible Brinkmann medium of couple stress fluids under the impact of slip regime. An aerosol particle is a spherical particle of radius a moves along z axis with an uniform flow U. In the permeable region, the flow is governed by Brinkman's equation. Also, under the assumptions of low Reynolds number, the inertial terms in the fluid momentum are neglected.The Stokes microcontinuum theory [1] is considered the simplest theory of fluids, which allows the polar effects such as the presence of couple stresses, body couples and nonsymmetric tensors. The modified governing equations of momentum and the continuity of an incompressible fluid with couple stress in the presence of Brinkman's medium are given by [6] and [24]:

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\nabla p + \mu \nabla \wedge \nabla \wedge \vec{q} + \eta \nabla \wedge \nabla \wedge \nabla \wedge \nabla \wedge \vec{q} + \frac{\mu}{K} \vec{q} = 0. \tag{2}$$

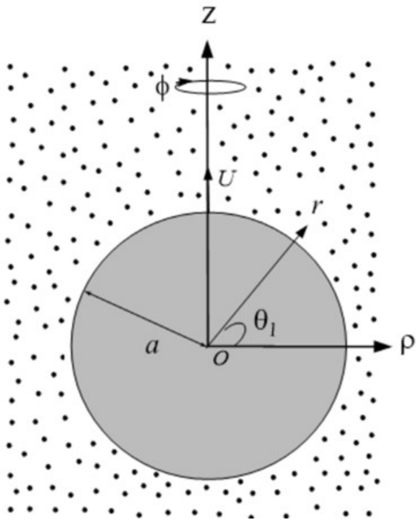

Figure 1: The physical representation of the problem.

where $\vec{q}$ is the volume averaged velocity parallel to the wall,p is the pore average pressure, μ represents the viscosity of the fluid, η is the first couple stress viscosity coefficient, and $\eta^{'}$ is the second couple stress viscosity coefficient. If the couple stress coefficient η is taken zero, then the equation of motion (2) reduces to the classical Navier–Stokes's equation. In addition, K is the Darcy permeability of the permeable medium and also is a scalar for isotropic porous medium. Otherwise, K a second order tensor [25] and the Brinkman equation of couple stress fluid reduces to the Darcy equation when $K \to 0$ and to the Stokes equation of couple stress fluid when $K \to \infty$.
The material constants are [17]:

$$\mu \geq 0, \quad \eta \geq 0 \quad \eta - \eta^{'} \geq 0. \tag{3}$$

The components of velocity field are of the form:

$$\vec{q} = -\nabla \wedge \left(\frac{\psi \vec{e}_\phi}{r \sin\theta} \right). \tag{4}$$

The constitutive equations for the couple stress fluids in dyadic form are:

(i) The stress tensor, $\mathbf{T}$ and the strain tensor,Δ:

$$T = -p\mathbf{I} + 2\mu E + \frac{1}{2}\mathbf{I} \wedge \nabla \, . \, \mathbf{M}, \tag{5}$$

$$E = \frac{1}{2}\left[\nabla\vec{q} + (\nabla\vec{q})^T\right]. \tag{6}$$

(ii) The couple stress tensor, $\mathbf{M}$

$$\mathbf{M} = mI + 4\eta\nabla\vec{\omega} + 4\eta^{'}(\nabla\vec{\omega})^T, \tag{7}$$

where I is the unit dyadic, $(.)^T$ denotes for transpose, and ω is the vorticity vector which it is defined in terms of the stream function ψ by:

$$\vec{\omega} = \tfrac{1}{2}\nabla \wedge \vec{q} = \tfrac{1}{2}\frac{1}{r\sin\theta}E^2\psi\ \vec{e}_\phi. \tag{8}$$

3. Axisymmetric expressions of the stresses and couple stresses for the couple stress fluids theory

The strain tensor in terms of the velocity components are obtained by the following from, Appendix (A.1):

$$\left.\begin{aligned}
\vec{\omega} =& \tfrac{1}{2}E_{rr} = \frac{\partial q_r}{\partial r},\ E_{r\theta} = E_{\theta r} = \frac{1}{2}\Big[\frac{1}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\Big],\\
E_{\theta\theta} =& \Big[\frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{q_\theta}{r}\Big], E_{\theta\phi} = E_{\phi\theta} = 0,\\
E_{r\phi} =& E_{\phi r} = 0,\ E_{\phi\phi} = \Big[\frac{q_r}{r} + \frac{\cot\theta q_\theta}{r} + \frac{q_\theta}{r}\Big].
\end{aligned}\right\} \tag{9}$$

The couple stresses in terms of the vorticity are calculated by from Appendix (A.2):

$$\left.\begin{aligned}
M_{rr} =& m,\ M_{r\theta},\ M_{r\phi} = 4\eta\frac{\partial \omega_\phi}{\partial r} - 4\eta'\frac{\omega_\phi}{r},\\
M_{\theta r} =& 0,\ M_{\theta\theta} = m,\ M_{\theta\phi} = 4\eta\frac{1}{r}\frac{\partial \omega_\phi}{\partial \theta} - 4\eta'\frac{1}{r}\cot\theta\ \omega_\phi,\\
M_{\phi r} =& -4\eta\frac{1}{r}\omega_\phi + 4\eta'\frac{\partial \omega_\phi}{\partial r},\ M_{\phi\theta} = -4\eta\frac{1}{r}\cot\theta\omega_\phi + 4\eta'\frac{1}{r}\frac{\partial \omega_\phi}{\partial \theta},\ M_{\phi\phi} = m,
\end{aligned}\right\} \tag{10}$$

where $m = \frac{1}{3}tr\mathbf{M} = \frac{1}{3}(M_{rr} + M_{\theta\theta} + M_{\phi\phi})$,is the trace of the couple stress and for any axisymmetric equation you may take, $m = 0$. From equations (9), we get the following term:

$$\nabla.\mathbf{M} = \tilde{e}_\phi\Big[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 M_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\ M_{\theta\phi}) + \frac{M_{\phi r}}{r} + \frac{\cot\theta}{r}M_{\phi\theta}\Big]$$

And so,

$$\mathbf{I}\wedge\nabla.\mathbf{M} = (-\tilde{e}_r\tilde{e}_\theta + \tilde{e}_\theta\tilde{e}_r)\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2M_{r\phi} + M_{\phi r}}{r} + \frac{\cot\theta}{r}(M_{\theta\phi} + M_{\phi\theta}) + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial \theta}\Big] \tag{11}$$

Therefore, the stresses are determined from equation (4):

$$\left.\begin{aligned}
T_{rr} =& -p + 2\mu E_{rr},\ T_{r\phi},\ T_{\theta\theta} = -p + 2\mu E_{\theta\theta},\\
T_{\theta\phi} =& 0,\ T_{\theta r} = m,\ T_{\phi\theta} = 0, T_{\phi\phi} = -p + 2\mu E_{\phi\phi},\\
T_{r\theta} =& 2\mu E_{r\theta} - \frac{1}{2}\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2M_{r\phi} + M_{\phi r}}{r} + \frac{\cot\theta}{r}(M_{\theta\phi} + M_{\phi\theta}) + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial \theta}\Big],\\
T_{\theta r} =& 2\mu E_{\theta r} + \frac{1}{2}\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2M_{r\phi} + M_{\phi r}}{r} + \frac{\cot\theta}{r}(M_{\theta\phi} + M_{\phi\theta}) + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial \theta}\Big].
\end{aligned}\right\} \tag{12}$$

For dimensionless governing equations, we used the following nondimensional variables:

$$\hat{\vec{q}} = \frac{\vec{q}}{U},\ \hat{p} = \frac{ap}{\mu U},\ \hat{\nabla} = a\nabla,\ \hat{T_{rr}} = \frac{aT_{rr}}{\mu U},\ \hat{\omega_\phi} = \frac{a\omega_\phi}{U}. \tag{13}$$

Substituting equation (12) in equation (2) and then dropping the tildes, we get:

$$\nabla p + \nabla \wedge \nabla \wedge \vec{q} + \frac{1}{\zeta^2}\nabla \wedge \nabla \wedge \nabla \wedge \nabla \wedge \vec{q} + K^2\vec{q} = 0. \tag{14}$$

where $k = \frac{a^2}{K}$, $\zeta^2 = \frac{\mu a^2}{\eta}$ are the permeability parameter and the length dependent parameter of the first couple stress fluid coefficient, respectively. If $\zeta \to \infty$ equation (13) represents the modified Stokes equation for nonpolar fluid.

4. Differential equation satisfied by the stream function, ψ

Further, the fluid movement is axisymmetric, so all the field functions are independent of ϕ and the spherical polar coordinate system is (r, θ, ϕ) where the origin located at the center of the sphere. Since the velocity vector $\vec{q}$ and the vorticity vector $\vec{\omega}$ are assumed as:

$$\vec{q} = q_r(r,\theta)\vec{e}_r + q_\theta(r,\theta)\vec{e}_\theta, \quad \vec{\omega} = \omega_\phi(r,\theta)\vec{e}_\phi. \tag{15}$$

From equation (1), the velocity components can be represented in terms of the stream function ψ from equation (3) as the following:

$$q_r = -\frac{1}{r^2\theta}\frac{\partial \psi}{\partial \theta}, \qquad q_\theta = \frac{1}{r\theta}\frac{\partial \psi}{\partial r}. \tag{16}$$

From equations (8)-(9) and (11) we obtained the following non-dimensional stresses:

$$T_{rr} = -p + 2\frac{\partial q_r}{\partial r}, \tag{17}$$

$$T_{r\theta} = \Big[\frac{1}{r} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\Big] - \frac{1}{2\zeta^2}\Big[\frac{\partial^2 \omega_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial \omega_\phi}{\partial r} - \frac{2\omega_\phi}{r^2}\Big], \tag{18}$$

$$M_{r\phi} = \frac{4}{\zeta^2}\frac{\partial \omega_\phi}{\partial r} - \frac{4}{\zeta'^2}\frac{\omega_{phi}}{r}, \tag{19}$$

where, $\zeta'^2 = \frac{\mu a^2}{\eta'}$ is the length dependent parameter on the second couple stress fluid coefficient. Thus, the governing equations have the form:

$$M_{r\phi} = \frac{4}{\zeta^2}\frac{\partial \omega_\phi}{\partial r} - \frac{4}{\zeta'^2}\frac{\omega_{phi}}{r}, \tag{20}$$

where, $\zeta'^2 = \frac{a^2\mu}{\eta'}$ is the length dependent parameter on the second couple stress fluid coefficient. Thus, the governing equations have the form:

$$\frac{\zeta^{-2}}{r^2\sin\theta}\frac{\partial}{\partial\theta}L_{-1}\Big(L_{-1} - \zeta^2\Big)\psi + \frac{\kappa^2}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} - \frac{\partial p}{\partial r} = 0, \tag{21}$$

$$\frac{-\zeta^{-2}}{r\sin\theta}\frac{\partial}{\partial r}L_{-1}\Big(L_{-1}-\zeta^{2}\Big)\psi-\frac{\kappa^{2}}{r\sin\theta}\frac{\partial\psi}{\partial r}-\frac{1}{r}\frac{\partial p}{\partial\theta}=0. \tag{22}$$

One can easily eliminate the pressure from equations (21) and (22) to get the following partial differential equation:

$$L_{-1}\big(L_{-1}-\alpha_1^2\big)\big(L_{-1}-\alpha_2^2\big)\psi=0, \tag{23}$$

where the axisymmetric Stokesian operator: $L_{-1}=\frac{\partial^2}{\partial r^2}+\frac{1-\xi^2}{r^2}\frac{\partial^2}{\partial\xi^2}$, $\xi=\cos\theta$, $\alpha_1^2\alpha_2^2=\zeta^2\kappa^2$, $\alpha_1^2+\alpha_2^2=\zeta^2$,

$$\left.\begin{aligned}\alpha_1^2&=\frac{\zeta^2+\zeta\sqrt{\zeta^2-4\kappa^2}}{2},\\ \alpha_2^2&=\frac{\zeta^2-\zeta\sqrt{\zeta^2-4\kappa^2}}{2}.\end{aligned}\right\} \tag{24}$$

5. The solution of the Problem

The regular solution of the sixth order linear partial differential equation (23) is obtained by using the method of separation of variables is given:

$$\psi=\tfrac{1}{2}\Big(Ar^{-1}+Br^{\frac{1}{2}}K_{\frac{3}{2}}(\alpha_1 r)+Cr^{\frac{1}{2}}K_{\frac{3}{2}}(\alpha_2 r)\Big)\sin^2\theta, \tag{25}$$

where the function $K_{\frac{3}{2}}(.)$ is the modified Bessel functions of the second kind of order $\frac{3}{2}$ where A,B,C are constants can be determined from the boundary conditions. Substituting equation (25) into equation (16), we get the velocity components:

$$q_r=\Big(Ar^{-3}+Br^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_1 r)+Cr^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_2 r)\Big)\cos\theta, \tag{26}$$

$$\begin{aligned}q_\theta=\tfrac{1}{2}\Big(-Ar^{-3}&-Br^{\frac{-3}{2}}\big[K_{\frac{3}{2}}(\alpha_1 r)+\alpha_1 rK_{\frac{1}{2}}(\alpha_1 r)\big]\\ &-Cr^{\frac{-3}{2}}\big[K_{\frac{3}{2}}(\alpha_2 r)+\alpha_2 rK_{\frac{1}{2}}(\alpha_2 r)\big]\Big)\sin\theta,\end{aligned} \tag{27}$$

Inserting the two expressions (26) and (27) into the relation (8) , we get:

$$\omega_\phi=\tfrac{1}{4}\Big(B\alpha_1^2 r^{\frac{-1}{2}}K_{\frac{3}{2}}(\alpha_1 r)+C\alpha_2^2 r^{\frac{-1}{2}}K_{\frac{3}{2}}(\alpha_2 r)\Big)\sin\theta. \tag{28}$$

The expression for pressure is:

$$p=-\frac{-1}{2r^2}\kappa^2A\cos\theta. \tag{29}$$

6. Boundary conditions

On the surface of the aerosol particle, $r = a$, we require three boundary conditions to find the unknowns A, B, and C. Recently, some researchers have studied the slip effect on the solid spheres (9),(26)-(28).

(i) The kinematic impenetrability boundary condition is:

$$q_r = U \cos\theta, \tag{30}$$

(ii) The dynamic slip boundary condition where the tangential velocity of the couple stress fluid relative to the solid at a point on its surface is proportional to the tangential stress prevailing at that point. The latter is known as the slip boundary condition introduced by Navier in 1823:

$$q_\theta = -U \sin\theta + \frac{1}{\beta_1} T_{r\theta}, \tag{31}$$

(iii) The spin slip couple stress condition is given as:

$$\omega_\phi = \frac{1}{\beta_2} M_{r\phi}, \tag{32}$$

where β_1 is the velocity slip coefficient and β_2 is the spin slip coefficient. Therefore,

- The slips coefficients depend on the nature of the solid and fluid surface.
- In the limiting case of $\beta_1 \to 0$, there is a perfect slip and the solid sphere acts like a spherical gas bubble if $\beta_1 \to \infty$, we get the standard no-slip condition.
- Also, for $\beta_2 \to 0$ we have the couple stress zero at the boundary which means that the mechanical interactions at the surface are equipollent to a force distribution only. On the other hand, when $\beta_2 \to \infty$ this is equivalent to prevent relative rotation of the fluid element at the surface of the aerosol.

The expressions for stress and couple stress components from Appendix (A.4) are :

$$\begin{aligned} T_{rr} = \Big(& \tfrac{1}{2} r^{-4}(\kappa^2 r^2 + 12)A + 2Br^{\frac{-5}{2}}\left[3K_{\frac{3}{2}}(\alpha_1 r) + \alpha_1 r K_{\frac{1}{2}}(\alpha_1 r)\right] \\ & + 2Cr^{\frac{-5}{2}}\left[3K_{\frac{3}{2}}(\alpha_2 r) + \alpha_2 r K_{\frac{1}{2}}(\alpha_2 r)\right]\Big) \cos\theta, \end{aligned} \tag{33}$$

$$\begin{aligned} T_{r\theta} = \Big(& 3r^{-4}A + \tfrac{1}{2}Br^{\frac{-5}{2}}\left[(\alpha_1^2 r^2 + 6 - \alpha_1^4 \zeta^{-2} r^2)K_{\frac{3}{2}}(\alpha_1 r) + 2\alpha_1 r K_{\frac{1}{2}}(\alpha_1 r)\right] \\ & + \tfrac{1}{2}Cr^{\frac{-5}{2}}\left[(\alpha_2^2 r^2 + 6 - \alpha_2^4 \zeta^{-2} r^2)K_{\frac{3}{2}}(\alpha_2 r) + 2\alpha_2 r K_{\frac{1}{2}}(\alpha_2 r)\right]\Big) \sin\theta, \end{aligned} \tag{34}$$

Therefore, relation (20) gives:

$$M_{r\phi} = -\Big(B\alpha_1^2 r^{\frac{-3}{2}}\left[(2\zeta^{-2}+\zeta^{-'2})K_{\frac{3}{2}}(\alpha_1 r)+\alpha_1 r\zeta^{-2}K_{\frac{1}{2}}(\alpha_1 r)\right] \\ +C\alpha_2^2 r^{\frac{-5}{2}}\left[(2\zeta^{-2}+\zeta^{-'2})K_{\frac{3}{2}}(\alpha_2 r)+\alpha_2 r\zeta^{-2}K_{\frac{1}{2}}(\alpha_2 r)\right]\Big)\sin\theta. \tag{35}$$

7. Drag force

The formula of drag force operating on the axisymmetric rigid particle along the axis of symmetry has been shown by h3:

$$F_z = 2\pi a^2\int_0^\pi \Big(T_{rr}\cos\theta - T_{r\theta}\sin\theta\Big)_{r=a}\sin\theta d\theta \tag{36}$$

Inserting the expressions (33) and (34) into (36), we arrive at the following:

$$\frac{F_z}{\pi a^2} = \frac{2}{3}\Big(\kappa^2 a^{-2}A + 2Ba^{\frac{-5}{2}}\left[\alpha_1^2 a^2\zeta^{-2}(\alpha_1^2-\zeta^2)K_{\frac{3}{2}}(\alpha_1 r)\right] \\ +2Ca^{\frac{-5}{2}}\left[\alpha_2^2 a^2\zeta^{-2}(\alpha_2^2-\zeta^2)K_{\frac{3}{2}}(\alpha_2 r)\Big)\right. \tag{37}$$

By applying the boundary conditions (30)-(32) , we get the unknown constant A , to obtain the following system of equations:

$$Aa^{\frac{-3}{2}} + BK_{\frac{3}{2}}(\alpha_1 a) + CK_{\frac{3}{2}}(\alpha_2 a) = -Ua^{\frac{3}{2}}, \tag{38}$$

$$A(1+6\beta_1^{-1}a^{-1}) \\ +Ba^{\frac{3}{2}}\Big[\big(1+\beta_1^{-1}a^{-1}(\alpha_1^2a^2+6-\alpha_1^4\zeta^{-2}a^2)\big)K_{\frac{3}{2}}(\alpha_1 a)+\alpha_1 a\big(1+2\beta_1^{-1}a^{-1}\big)K_{\frac{1}{2}}(\alpha_1 a)\Big] \\ +Ca^{\frac{3}{2}}\Big[\big(1+\beta_1^{-1}a^{-1}(\alpha_2^2a^2+6-\alpha_2^4\zeta^{-2}a^2)\big)K_{\frac{3}{2}}(\alpha_2 a)+\alpha_2 a\big(1+2\beta_1^{-1}a^{-1}\big)K_{\frac{1}{2}}(\alpha_2 a)\Big] \\ =2Ua^3, \tag{39}$$

$$B\alpha_1^2\Big[\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_1 a)+\alpha_1\zeta^{-2}\beta_2^{-1}K_{\frac{1}{2}}(\alpha_1 a)\Big] \\ +C\alpha_2^2\Big[\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_2 a)+\alpha_2\zeta^{-2}\beta_2^{-1}K_{\frac{1}{2}}(\alpha_2 a)\Big] = 0, \tag{40}$$

where,

$A = -Ua^3 - Ba^{\frac{3}{2}}K_{\frac{3}{2}}(\alpha_1 a) - Ca^{\frac{3}{2}}K_{\frac{3}{2}}(\alpha_2 a)$,

$B = 3Ua^3\alpha_2^2\Delta^{-1}(1+2\beta_1^{-1}a^{-}-1)\Big[\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_2 a)+\alpha_2\zeta^{-2}\beta_2^{-1}K_{\frac{1}{2}}(\alpha_2 a)\Big]$,

$C = -3Ua^3\alpha_1^2\Delta^{-1}(1+2\beta_1^{-1}a^{-1})[\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_1 a)+\alpha_1\zeta^{-2}\beta_2^{-1}K_{\frac{1}{2}}(\alpha_1 a)\Big]$,

$\Delta = \beta_1^{-1}\zeta^{-2}a^{\frac{1}{2}}(\alpha_1^2\alpha_2^4-\alpha_2^2\alpha_1^4)\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_1 a)K_{\frac{3}{2}}(\alpha_2 a)$
$+a^2(1+2\beta_1^{-1}a^{-1})\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)\big(\alpha_1\alpha_2^2K_{\frac{3}{2}}(\alpha_2 a)K_{\frac{1}{2}}(\alpha_1 a)-\alpha_2\alpha_1^2K_{\frac{3}{2}}(\alpha_1 a)K_{\frac{1}{2}}(\alpha_2 a)\big)$
$+a^{\frac{1}{2}}\beta_1^{-1}\zeta^{-2}\beta_2^{-1}\big(\alpha_2^3(\alpha_1^2a^2-\alpha_1^4\zeta^{-2}a^2)K_{\frac{1}{2}}(\alpha_2 a)K_{\frac{3}{2}}(\alpha_1 a)-\alpha_1^3(\alpha_2^2a^2-\alpha_2^4\zeta^{-2}a^2)K_{\frac{1}{2}}(\alpha_1 a)K_{\frac{3}{2}}(\alpha_2 a)\big)$
$a^2\zeta^{-2}\beta_2^{-1}(\alpha_1\alpha_2^3-\alpha_1^3\alpha_2)(1+2\beta_1^{-1}a^{-1})K_{\frac{1}{2}}(\alpha_1 a)K_{\frac{1}{2}}(\alpha_2 a)$.

Solving the system in the equations (38)-(40) algebraically, we get the value of the desired coefficient A needed in the drag expression (37).

$$\begin{aligned}\frac{F_z}{\frac{2}{3}\pi a^2} =& -\kappa^2Ua+3U\alpha_2^2\Delta^{-1}a^{\frac{3}{2}}(1+2\beta_1^{-1}a^{-1})(2\alpha_1^2(\alpha_1^2\zeta^{-2}-1)-\kappa^2)\\ &\times\Big(\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_1 a)K_{\frac{3}{2}}(\alpha_2 a)+\alpha_2\zeta^{-2}\beta_2^{-1}K_{\frac{3}{2}}(\alpha_1 a)K_{\frac{1}{2}}(\alpha_2 a)\Big)\\ &-3U\alpha_1^2\Delta^{-1}a^{\frac{3}{2}}(1+2\beta_1^{-1}a^{-1})(2\alpha_2^2(\alpha_2^2\zeta^{-2}-1)-\kappa^2)\\ &\times\Big(\big(1+4\beta_2^{-1}a^{-1}(2\zeta^{-2}+\zeta'^2)\big)K_{\frac{3}{2}}(\alpha_1 a)K_{\frac{3}{2}}(\alpha_2 a)+\alpha_1\zeta^{-2}\beta_2^{-1}K_{\frac{3}{2}}(\alpha_2 a)K_{\frac{1}{2}}(\alpha_1 a)\Big)\end{aligned} \quad (41)$$

The non-dimensional drag force of a rigid sphere is taken with respect to the drag force of a sphere translates through unbounded clear fluid which is given by $F_0 = -\mu\pi Ua$.

8. hpecial cases

8.1. No-slip and no-spin, $\beta_1 = \beta_2 \to \infty$

The drag force acting on the rigid sphere moves through unbounded porous medium under the conditions,$q_r = U\cos\theta,\ q_\theta = -U\sin\theta,\ \omega_\phi = 0$, we obtained the following:

$$Aa^{-3}+Ba^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_1 a)+Ca^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_2 a) = -U, \quad (42)$$

$$Aa^{-3}+Ba^{\frac{-3}{2}}\Big[K_{\frac{3}{2}}(\alpha_1 a)+\alpha_1 aK_{\frac{1}{2}}(\alpha_1 a)\Big]+Ca^{\frac{-3}{2}}\Big[K_{\frac{3}{2}}(\alpha_2 a)+\alpha_2 aK_{\frac{1}{2}}(\alpha_2 a)\Big] = 2U, \quad (43)$$

$$B\alpha_1^2a^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_1 a)+C\alpha_2^2a^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_2 a) = 0. \quad (44)$$

The values of the constants A, B, C are given by:
$A = -\Big(a^3+3a\alpha_1^{-1}\alpha_2^{-1}(\alpha_1 a+\alpha_2 a+2)+3a\kappa^{-2}+3\zeta^{-2}\kappa^{-2}(\alpha_1+\alpha_2)\Big)$,
$B = -3\exp\alpha_1 aU\sqrt{\frac{2}{\pi\alpha_1}\frac{(1+\alpha_2 a)}{(\alpha_1-\alpha_2)}}$,
$C = -3\exp\alpha_2 aU\sqrt{\frac{2}{\pi\alpha_2}\frac{(1+\alpha_1 a)}{(\alpha_1-\alpha_2)}}$.

The drag force in this case is obtained as:

$$\frac{F_z}{F_0} = -\tfrac{2}{3}\zeta^{-'2}(\alpha_1+\alpha_2)(1+\alpha_1 a)(1+\alpha_2 a)$$
$$-\tfrac{1}{9}\zeta^{-2}\Big(\alpha_1^2\alpha_2^2 a^3 + 3a\alpha_1\alpha_2(\alpha_1 a+\alpha_2 a+2) + 3a\zeta^2 + 3\alpha_1 + 3\alpha_2\Big) \tag{45}$$

8.2. Perfect slip and perfect spin, $\beta_1 = \beta_2 = 0$

The drag force acting on the rigid sphere moves through unbounded porous medium under the conditions, $q_r = U\cos\theta,\ T_{r\theta}=0,\ M_{R\phi}=0$, we obtained the following:

$$Aa^{-3} + Ba^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_1 a) + Ca^{\frac{-3}{2}}K_{\frac{3}{2}}(\alpha_2 a) = -U, \tag{46}$$

$$3Aa^{-3} + \tfrac{1}{2}Ba^{\frac{-3}{2}}\Big[(\alpha_1^2 a^2 + 6 - \alpha_1^4\zeta^{-2}a^2)K_{\frac{3}{2}}(\alpha_1 a) + 2\alpha_1 a K_{\frac{1}{2}}(\alpha_1 a)\Big]$$
$$+\tfrac{1}{2}Ca^{\frac{-3}{2}}\Big[(\alpha_2^2 a^2 + 6 - \alpha_2^4\zeta^{-2}a^2)K_{\frac{3}{2}}(\alpha_2 a) + \alpha_2 a K_{\frac{1}{2}}(\alpha_2 a)\Big] = 0, \tag{47}$$

$$B\alpha_1^2 a^{\frac{-3}{2}}\Big[(2\zeta^{-2} + \zeta^{-'2})K_{\frac{3}{2}}(\alpha_1 a) + \alpha_1 a\zeta^{-2}K_{\frac{1}{2}}(\alpha_1 a)\Big]$$
$$+C\alpha_2^2 a^{\frac{-3}{2}}\Big[(2\zeta^{-2} + \zeta^{-'2})K_{\frac{3}{2}}(\alpha_2 a) + \alpha_2 a\zeta^{-2}K_{\frac{1}{2}}(\alpha_2 a)\Big] = 0. \tag{48}$$

The values of the constants A, B, C in this case are given by:

$$A = -Ua^3 + 10Ua\Delta^{-1}\zeta^2\alpha_2^2(\alpha_1 a+1)(\alpha_2 a+1)(\zeta^2 + (\alpha_2 a+2)\zeta'^2)$$
$$-10Ua\Delta^{-1}\zeta^2\alpha_1^2(\alpha_1 a+1)(\zeta^2(\alpha_1 a+1) + (\alpha_2 a^2 + 2\alpha_1 a+2)\zeta'^2),$$
$$B = -10Ua\Delta^{-1}\exp\alpha_1 a\sqrt{\tfrac{2}{\pi\alpha_1}}\zeta^2\alpha_1^2\alpha_2^2(\alpha_2 a+1)(\zeta^2 + (\alpha_2 a+2)\zeta'^2),$$
$$C = 10Ua\Delta^{-1}\exp\alpha_2 a\sqrt{\tfrac{2}{\pi\alpha_2}}\zeta^2\alpha_1^2\alpha_2^2(\zeta^2(\alpha_1 a+1) + (\alpha_1^2 a^2 + 2\alpha_1 a+2)\zeta'^2),\ \text{where,}$$
$$\Delta = \alpha_1^2\alpha_2^2\Big((\alpha_1 a - 2\alpha_2 a - 1)\zeta^4 + \big[(\alpha_2 a+2)(\alpha_2 a^2 - \alpha_1\alpha_2 a^2 + \alpha_1 a - 3\alpha_2 a - 1)\zeta'^2$$
$$+ (\alpha_1^2 - \alpha_2^2)(\alpha_1 a+1)(\alpha_2 a+1)\big]\zeta^2 + (\alpha_2 a+1)\big[\alpha_1^3\alpha_2 a^2 - \alpha_1^2\alpha_2^2 a^2 + 2\alpha_1^3 a + \alpha_1^2\alpha_2 a$$
$$-2\alpha_1\alpha_2^2 a + 2\alpha_1^2 - 2\alpha_2\big]\Big).$$

The drag force is obtained as:

$$\frac{F_z}{\pi U a^2} = \tfrac{2}{3}\Delta^{-1}\Big(\kappa^2 a^{-1}\Big[-a^2\Delta + 10\zeta^2\alpha_2^2(\alpha_1 a+1)(\alpha_2 a+1)(\zeta^2 + (\alpha_2 a+2)\zeta'^2)$$
$$-10\zeta^2\alpha_1^2 a(\alpha_1 a+1)(\zeta^2(\alpha_1 a+1) + (\zeta^2 a^2 + 2\alpha_2 a+2)\zeta'^2)\Big]$$
$$-20\alpha_1^2\alpha_2^2 a^2(\alpha_1^2 - \zeta^2)(\alpha_1 a+1)(\alpha_2 a+1)(\zeta^2 + (\alpha_2 a+2)\zeta'^2)$$
$$+20\alpha_1^2\alpha_2^2 a^2(\alpha_2^2 - \zeta^2)(\alpha_2 a+1)(\zeta^2(\alpha_1 a+1) + (\alpha_2 a^2 + 2\alpha_1 a+2)\zeta'^2)\Big). \tag{49}$$

9. Results and discussions

In the present section, we aim to represent numerically the normalized drag force F_z/F_0 acting on a solid sphere and distributions of velocity components for different values of the physical parameters appearing in the governing equations in Figs 2-11 and Tables 1-3. All the results of drag force had been calculated for real part and MATLAB program neglect the imaginary part during the calculations. Fig. 2 shows the normalized drag force for different values of the couple stress parameter $\zeta = 0.01, 1.0, 10.0, \infty$ with $\zeta^{'}$ and $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 0.001, 1.0, 10.0, \infty$ which increases with the increase of the permeability parameter starts from low at $\kappa \to 0$ for Stokes flow but at $\kappa \to \infty$ gives Darcy flow. On the other hand, the normalized drag force decreases with the increase of the couple stress parameter. Fig. 3 presents the normalized drag force for different values $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 0.0, 1.0, 10.0, \infty$ at the various values of the couple stress parameter ζ at $\zeta^{'}$ which increases with the increase of the slip parameters. Table 1 represents the effects of slip and spin slip parameters with different values of the couple stress parameter. Fig. 4-5 exposes the velocity distribution for partial slip and partial spin slip for various values of the permeability and $\zeta = 4.0, \zeta^{'} = 0.01, \frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 2.0$ the normal and tangential velocities reduce with both the increase of the sphere radius and the increase of the permeability parameter. Figs. 6-8 and Table 2 show the special cases of no-slip and no-spin slip where the normalized drag force begins slowly at Stokes flow and increases gradually to reach high values for Darcy flow. In addition, the normal velocity decreases with the increase of radius r and reduces with the increase of the permeability while the tangential velocity reverse its improves the improve of the permeability parameter and near to $r = 7.0$ it behaves like the normal velocity. Figs. 9 and Table 3 illustrate the special cases of perfect slip and perfect spin slip for the normalized drag force gives the same effect for the above cases. Figs 10-11 explain the distributions of the streamlines for $\zeta = 10.0, \zeta^{'} = 0.1$ for the three cases for the slip conditions with $\kappa = 0.0, 1.0, 3.0, 4.0, 6.0, 10.0$ and the couple stress parameter with the values of $\zeta^{'} = 0.1, \frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 10.0$. It is noticed that the streamlines move uniformly at $\kappa = 0.0$ which agrees with the results of viscous fluid in the literature but for the increasing of the permeability, it appears as a loop around the sphere until reaching the Darcy flow the molecules are crowded. Moreover, it also clarifies in the case of the increase of the couple stress parameter with the rising in the values of κ.

Table 1: Normalized drag force for slip and spin slip with $\zeta^{'} = 0.01$.

		F_z/F_0				
$\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu}$	κ	$\zeta = 0.1$	$\zeta = 0.1$	$\zeta = 6.0$	$\zeta = 10.0$	$\zeta \to \infty$
0.0	0.0	1.962039	1.000000	0.736977	0.709803	0.635865
	0.2	4.444950	2.980156	1.826095	1.765162	1.646248
	4.0	5.799853	4.888037	3.538837	3.348328	3.208901
	6.0	8.037672	7.369523	6.134427	5.787377	5.558577
	8.0	11.162395	10.626770	9.523765	9.147747	8.750814
	10.0	15.174927	14.720016	13.735482	13.353335	12.806985
1.0	0.0	2.913268	1.199978	0.84000	0.804879	0.715290
	2.0	6.439568	3.954626	2.238630	2.137213	1.946697
	4.0	7.806085	6.226607	4.176816	3.904061	3.656288
	6.0	10.052062	8.890593	6.973032	6.481510	6.104393
	8.0	13.183798	12.255479	10.521991	9.976724	9.366142
	10.0	17.202734	16.419035	14.860377	14.297513	13.473763
10.0	0.0	10.723054	1.714241	1.063293	1.007636	0.880511
	2.0	24.330440	8.748913	3.697992	3.379155	2.850210
	4.0	25.826824	14.363925	7.041395	6.316924	5.319628
	6.0	28.157423	19.195222	11.255353	9.956683	8.440525
	8.0	31.358927	24.030911	16.123972	14.486632	12.280256
	10.0	35.440013	29.241834	21.640403	19.783658	16.882901
∞	0.0	50.690467	1.961480	1.153477	1.088268	0.944690
	2.0	195.779204	14.324858	4.721100	4.186681	3.367234
	4.0	200.897054	29.018703	9.890541	8.590854	6.566838
	6.0	205.118892	43.957608	16.588341	14.128548	10.593281
	8.0	209.642450	58.890966	24.546021	20.842851	15.449547
	10.0	214.770348	73.744811	33.603516	28.680957	21.138434

Table 2: Normalized drag force for no-slip and no-spin slip with $\zeta^{'} = 0.01$.

	F_z/F_0				
κ	$\zeta = 0.1$	$\zeta = 0.1$	$\zeta = 6.0$	$\zeta = 10.0$	$\zeta \to \infty$
0.0	11.0	2.0	1.17	1.11	1.00
0.2	216.47	15.69	4.90	4.33	3.45
4.0	374.044	34.031	10.475	9.078	6.803
6.0	526.757	55.061	17.9566	$15,172$	11.049
8.0	679.436	78.512	27.0281	22.626	16.192
10.0	833.66443	104.23	37.6196	31.4652	22.2316
100.0	11841.04	34461.03	20884.00	19090.00	14672.00

Table 3: Normalized drag force for perfect slip and perfect spin slip with $\zeta^{'} = 0.01$.

	F_z/F_0				
κ	$\zeta = 0.1$	$\zeta = 0.1$	$\zeta = 6.0$	$\zeta = 10.0$	$\zeta \to \infty$
0.0	31.00	18.85	13.89	13.38	11.98
0.2	80.121	56.155	34.412	33.26	31.024
4.0	107.00	92.110	66.692	63.103	60.476
6.0	149.556	138.882	115.614	109.075	104.765
8.0	208.563	200.278	179.498	172.414	164.936
10.0	284.198	277.433	258.000	251.6856	241.392
100.0	21020.3	21018.7	21016.0	21014.1	20980.9

10. Conclusion

The present work investigated the basic relations of stresses and couple stresses also the solution of the problem for movements of a rigid sphere with a constant velocity embedded through an incompressible porous medium saturated with the couple stress fluid under the effects of slip and spin slip by using an analytical procedure. The normalized drag force represents numerically and graphically. On the other side, the velocity distributions are represented graphically. It is noticed that the normalized drag force increases with the effects of the slip and spin slip and improves with the increase of the permeability parameter at constant values of the remaining parameters. In addition, there are improvements of the streamline's distributions for the relevant parameters of the permeability and the couple stress parameters.

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Appendix

Expressions of stresses and couple stresses for the couple stress fluids theory.
(A.1): The strain tensor components are obtained by from equation (5)

$$E_{rr} = \frac{\partial q_r}{\partial r},$$

$$
\begin{aligned}
E_{r\theta} &= E_{\theta r} = \frac{1}{2}\Big[\frac{1}{r}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\Big],\\
E_{\theta\theta} &= \Big[\frac{1}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{q_r}{r}\Big],\\
E_{\theta\phi} &= E_{\phi\theta} = \frac{1}{2}\Big[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big(\frac{q_\phi}{\sin\theta}\Big) + \frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}\Big],\\
E_{r\phi} &= E_{\phi r} = \frac{1}{2}\Big[\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\Big(\frac{q_\phi}{r}\Big)\Big] = \frac{1}{2}\Big[\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + \frac{\partial q_\phi}{\partial r} - \frac{q_\phi}{r}\Big],\\
E_{\phi\phi} &= \Big[\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi} + \frac{\partial q_r}{\partial r} + \frac{q_\theta\cot\theta}{r}\Big].
\end{aligned}
$$

(A.2): The couple stresses are calculated by from equation (6)

$$
\begin{aligned}
M_{rr} &= m + 4(\eta+\eta')\frac{\partial\omega_r}{\partial r},\\
M_{r\theta} &= 4\Big[\eta\frac{\partial\omega_\theta}{\partial r} + \eta'\frac{1}{r}\Big(\frac{\partial\omega_r}{\partial\theta} - \omega_\theta\Big)\Big],\\
M_{r\phi} &= 4\Big[\eta\frac{\partial\omega_\phi}{\partial r} + \eta'\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial\omega_r}{\partial\phi} - \omega_\phi\Big)\Big],\\
M_{\theta r} &= 4\Big[\eta\frac{1}{r}\Big(\frac{\partial\omega_r}{\partial\theta} - \omega_\theta\Big) + \eta'\frac{\partial\omega_\theta}{\partial r}\Big],\\
M_{\theta\theta} &= m + 4(\eta+\eta')\frac{1}{r}\Big(\frac{\partial\omega_\theta}{\partial\theta} + \omega_r\Big),\\
M_{\theta\phi} &= 4\Big[\eta\frac{1}{r}\frac{\partial\omega_\phi}{\partial\theta} + \eta'\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial\omega_\theta}{\partial\phi} - \cot\theta\omega_\phi\Big)\Big],\\
M_{\phi r} &= 4\Big[\eta\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial\omega_\theta}{\partial\phi} - \cot\theta\omega_\phi\Big) + \eta'\frac{1}{r}\frac{\partial\omega_\phi}{\partial\theta}\Big],\\
M_{\phi\phi} &= m + 4(\eta+\eta')\frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial\omega_\phi}{\partial\phi} + \omega_r + \cot\theta\omega_\theta\Big].
\end{aligned}
$$

(A.3): Used relations in the analysis

$$
\begin{aligned}
\mathbf{I}\times\nabla.\mathbf{M} &= (-\vec{e}_\theta\vec{e}_\phi + \vec{e}_\phi\vec{e}_\theta)\Big[\frac{\partial M_{rr}}{\partial r} + \frac{2}{r}M_{rr} + \frac{\cot\theta}{r}M_{\theta r} + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi r}}{\partial\phi} - \frac{M_{\theta\theta r}}{r} - \frac{M_{\phi\phi}}{r}\Big],\\
&= (\vec{e}_r\vec{e}_\phi - \vec{e}_\phi\vec{e}_r)\Big[\frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r}M_{r\theta} + \frac{\cot\theta}{r}M_{\theta\theta} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\theta}}{\partial\phi} + \frac{M_{\theta r}}{r} + \frac{\cot\theta}{r}M_{\phi\phi}\Big],\\
&+ (-\vec{e}_r\vec{e}_\theta + \vec{e}_\theta\vec{e}_r)\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2}{r}M_{r\phi} + \frac{\cot\theta}{r}M_{\theta\phi} + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\phi}}{\partial\phi} + \frac{M_{\phi r}}{r} + \frac{\cot\theta}{r}M_{\phi\theta}\Big].
\end{aligned}
$$

(A.4): The stresses are

$$T_{rr} = -P + 2\mu E_{rr}, \tag{50}$$

$$T_{r\theta} = 2\mu E_{r\theta} - \frac{1}{2}\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2}{r}M_{r\phi} + \frac{\cot\theta}{r}M_{\theta\phi} + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\phi}}{\partial\phi} + \frac{M_{\phi r}}{r} + \frac{\cot\theta}{r}M_{\phi\theta}\Big], \tag{51}$$

$$T_{r\phi} = 2\mu E_{r\phi} + \frac{1}{2}\Big[\frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r}M_{r\theta} + \frac{\cot\theta}{r}M_{\theta\theta} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\theta}}{\partial\phi} + \frac{M_{\theta r}}{r} + \frac{\cot\theta}{r}M_{\phi\phi}\Big], \tag{52}$$

$$T_{\theta r} = 2\mu E_{\theta r} + \frac{1}{2}\Big[\frac{\partial M_{r\phi}}{\partial r} + \frac{2}{r}M_{r\phi} + \frac{\cot\theta}{r}M_{\theta\phi} + \frac{1}{r}\frac{\partial M_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\phi}}{\partial\phi} + \frac{M_{\phi r}}{r} + \frac{\cot\theta}{r}M_{\phi\theta}\Big], \tag{53}$$

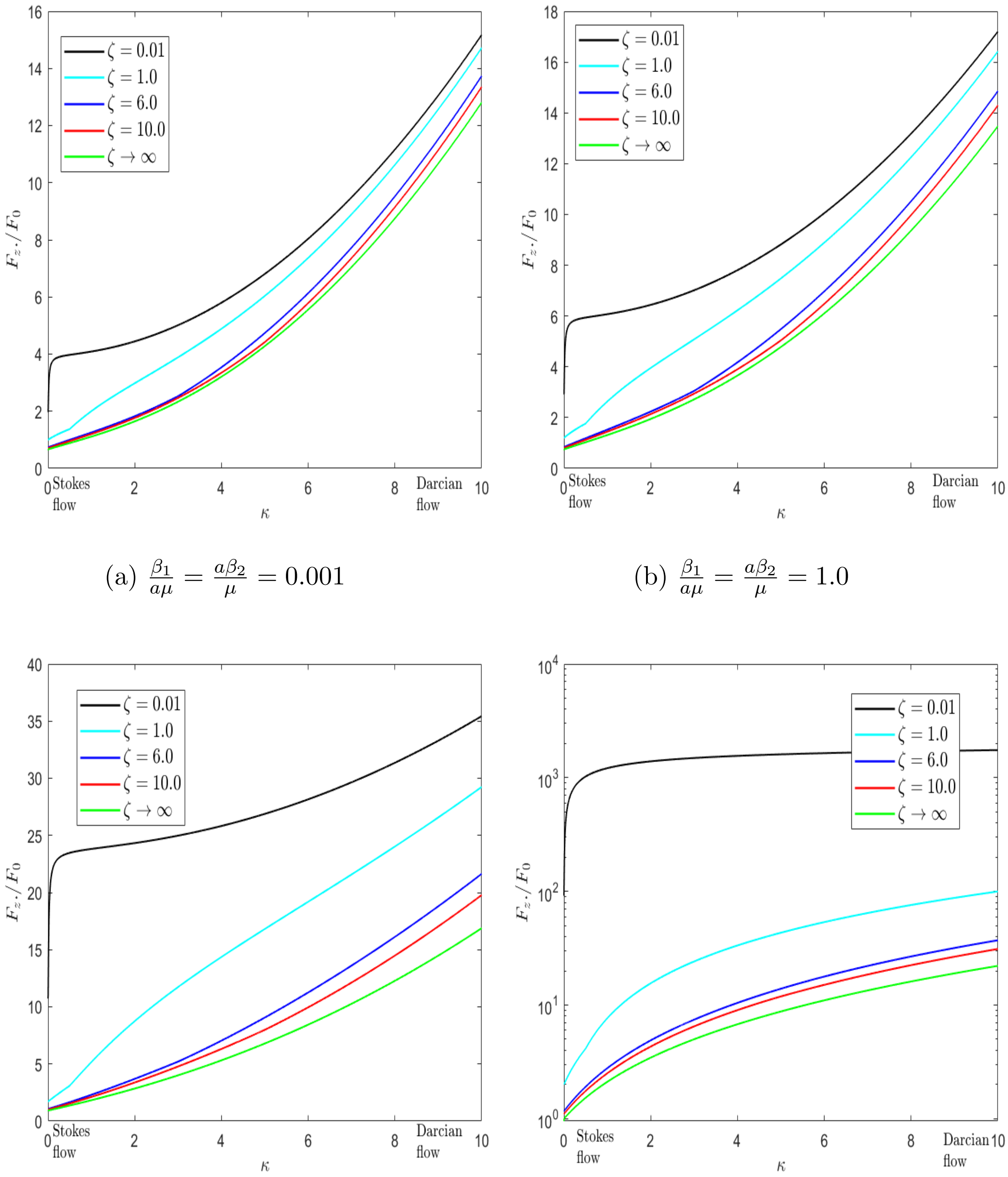
$$T_{\theta\theta} = -p + 2\mu E_{\theta\theta}, \tag{54}$$

$$T_{\theta\phi} = 2\mu E_{\theta\phi} - \frac{1}{2}\Big[\frac{\partial M_{rr}}{\partial r} + \frac{2}{r}M_{rr} + \frac{\cot\theta}{r}M_{\theta r} + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi r}}{\partial\phi} - \frac{M_{\theta\theta r}}{r} - \frac{M_{\phi\phi}}{r}\Big], \tag{55}$$

$$T_{\phi r} = 2\mu E_{\theta r} - \frac{1}{2}\Big[\frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r}M_{r\theta} + \frac{\cot\theta}{r}M_{\theta\theta} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi\theta}}{\partial\phi} + \frac{M_{\theta r}}{r} + \frac{\cot\theta}{r}M_{\phi\phi}\Big], \tag{56}$$

$$T_{\phi\theta} = 2\mu E_{\theta\theta} + \frac{1}{2}\Big[\frac{\partial M_{rr}}{\partial r} + \frac{2}{r}M_{rr} + \frac{\cot\theta}{r}M_{\theta r} + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial M_{\phi r}}{\partial\phi} - \frac{M_{\theta\theta r}}{r} - \frac{M_{\phi\phi}}{r}\Big], \tag{57}$$

$$T_{\theta\theta} = -p + 2\mu E_{\phi\phi}. \tag{58}$$



(a) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 0.001$

(b) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 1.0$

(c) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 10.0$

(d) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} \to \infty$

Figure 2: Normalized drag force exerted on a rigid sphere of a unit radius for various values of the slip and spin slip, permeability, and the couple stress parameter with $\zeta^{'} = 0.01$

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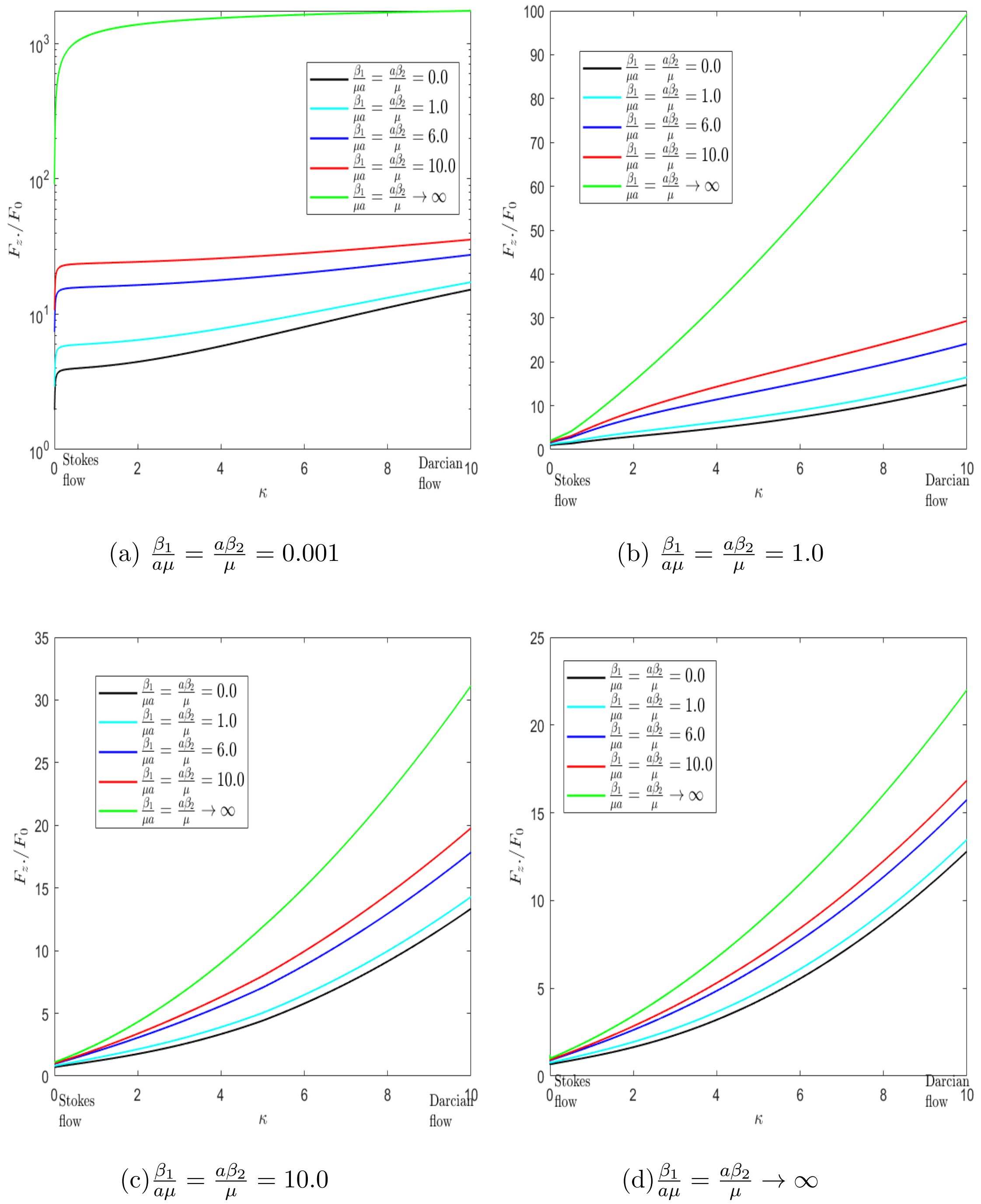

(a) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 0.001$

(b) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 1.0$

(c) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 10.0$

(d) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} \to \infty$

Figure 3: Normalized drag force exerted on a rigid sphere of a unit radius for various values of the spin slip, permeability, and the couple stress parameter with $\zeta' = 0.1$

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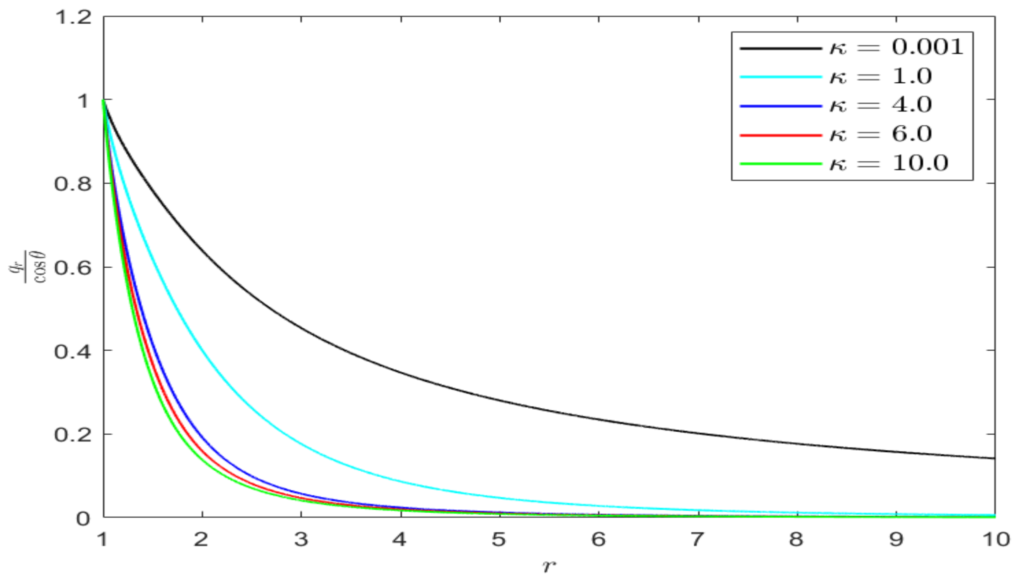


Figure 4: Normal velocity distribution for partial slip and partial spin slip for various values of the permeability and $\zeta = 3.0, \zeta^{'} = 0.01, \frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 2.0$

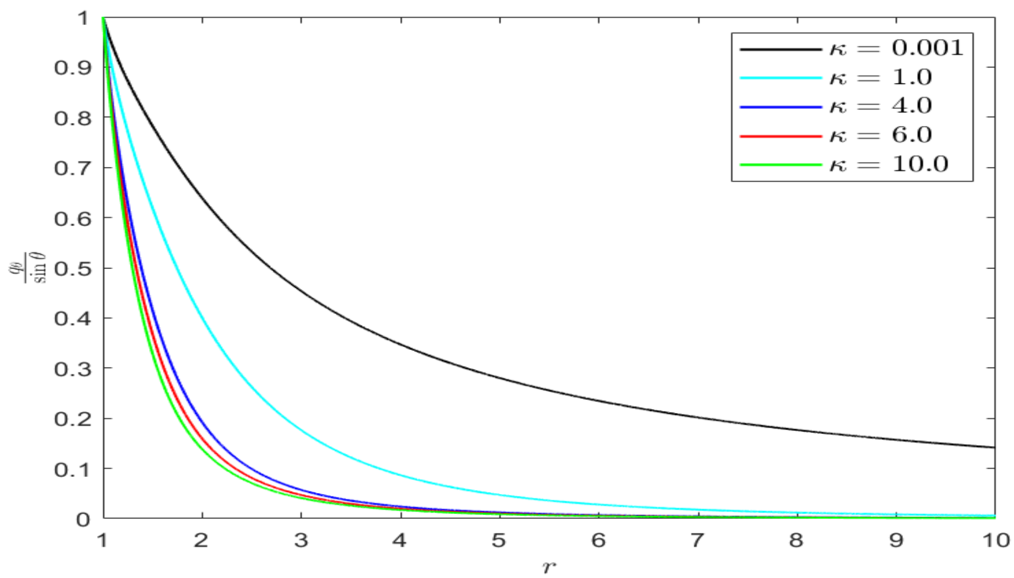


Figure 5: Tangential velocity distribution for partial slip and partial spin slip for various values of the permeability and $\zeta = 3.0, \zeta^{'} = 0.01, \frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 2.0$

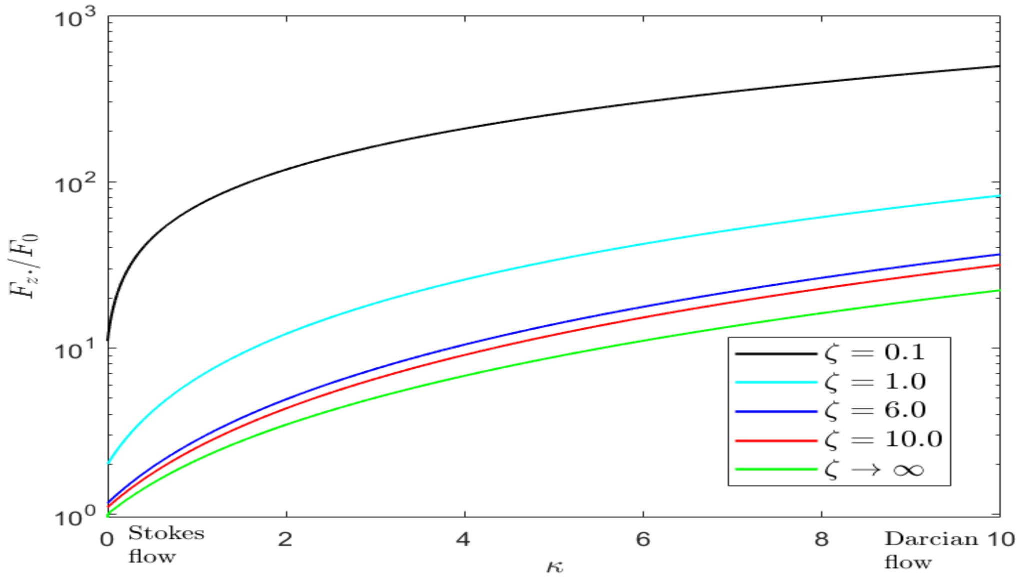


Figure 6: Normalized drag force exerted on a rigid sphere of a unit radius for no-slip and no-spin slip for various values of the permeability and the couple stress parameter.

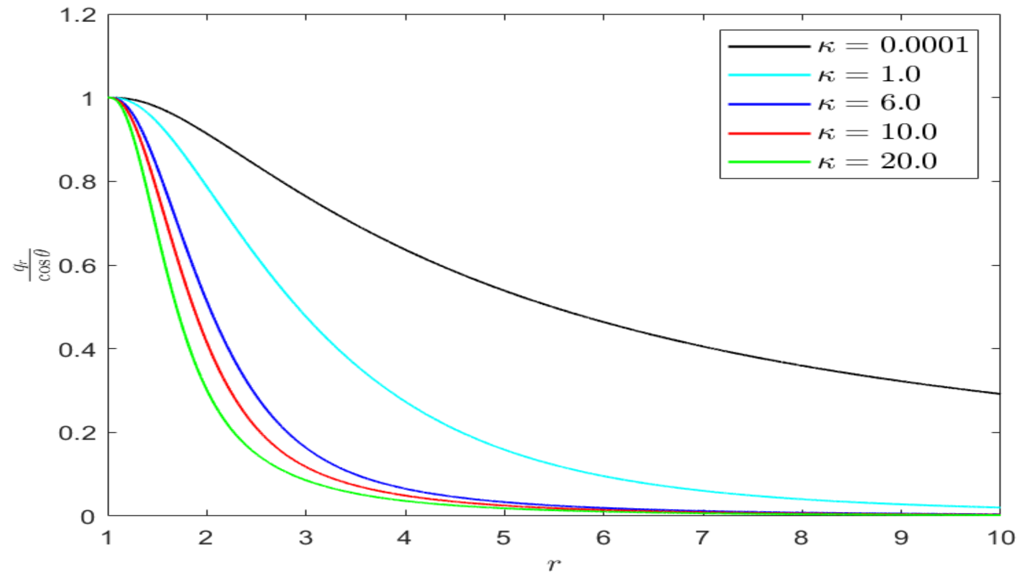


Figure 7: Normal velocity distribution for no-slip and no-spin slip for various values of the permeability and the couple stress parameter, $\zeta = 1.0$.

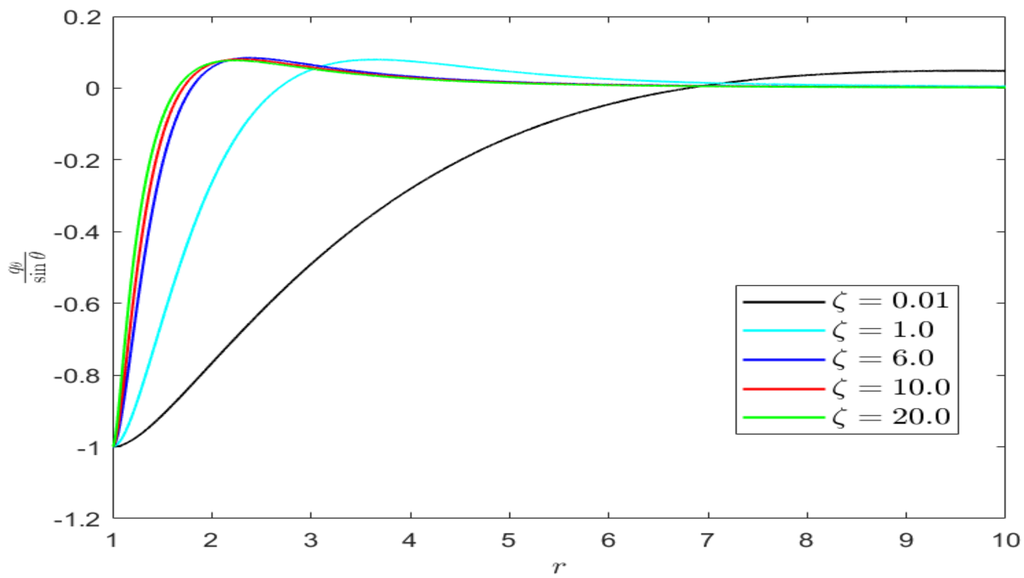


Figure 8: Tangential velocity distribution of a rigid sphere of a unit radius for no-slip and no-spin slip for various values of the couple stress parameter and the permeability, $\kappa = 2.0$.

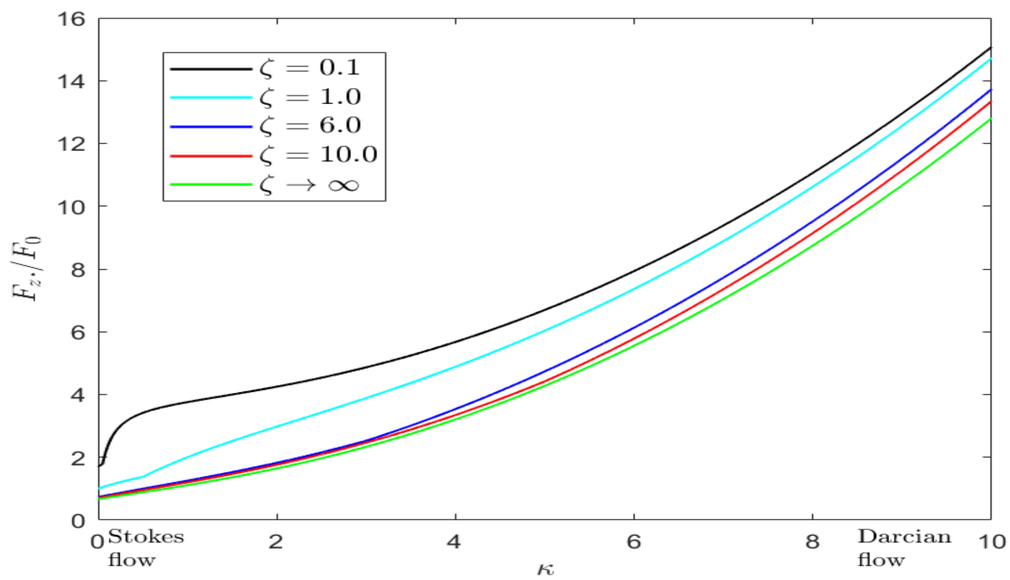
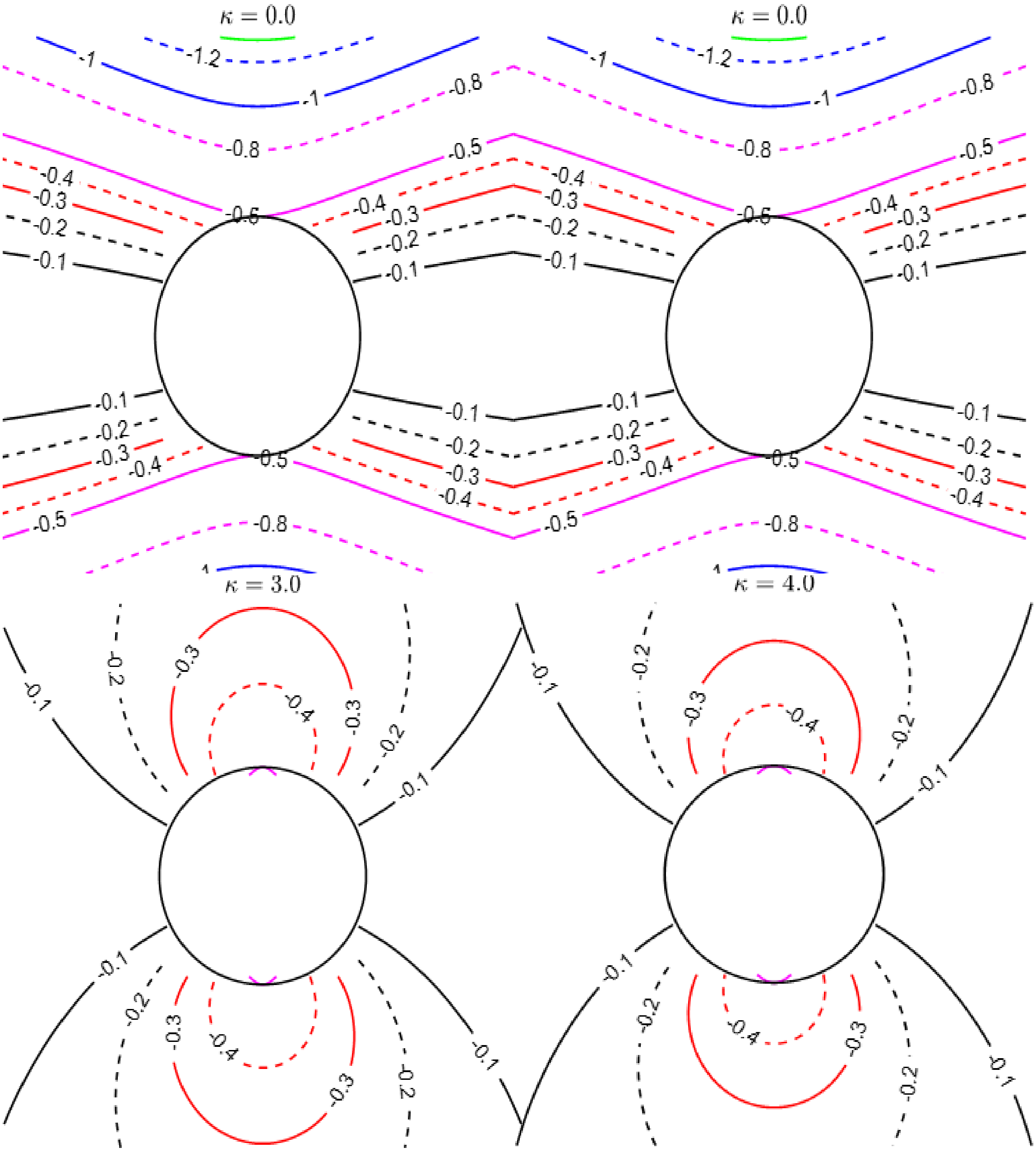
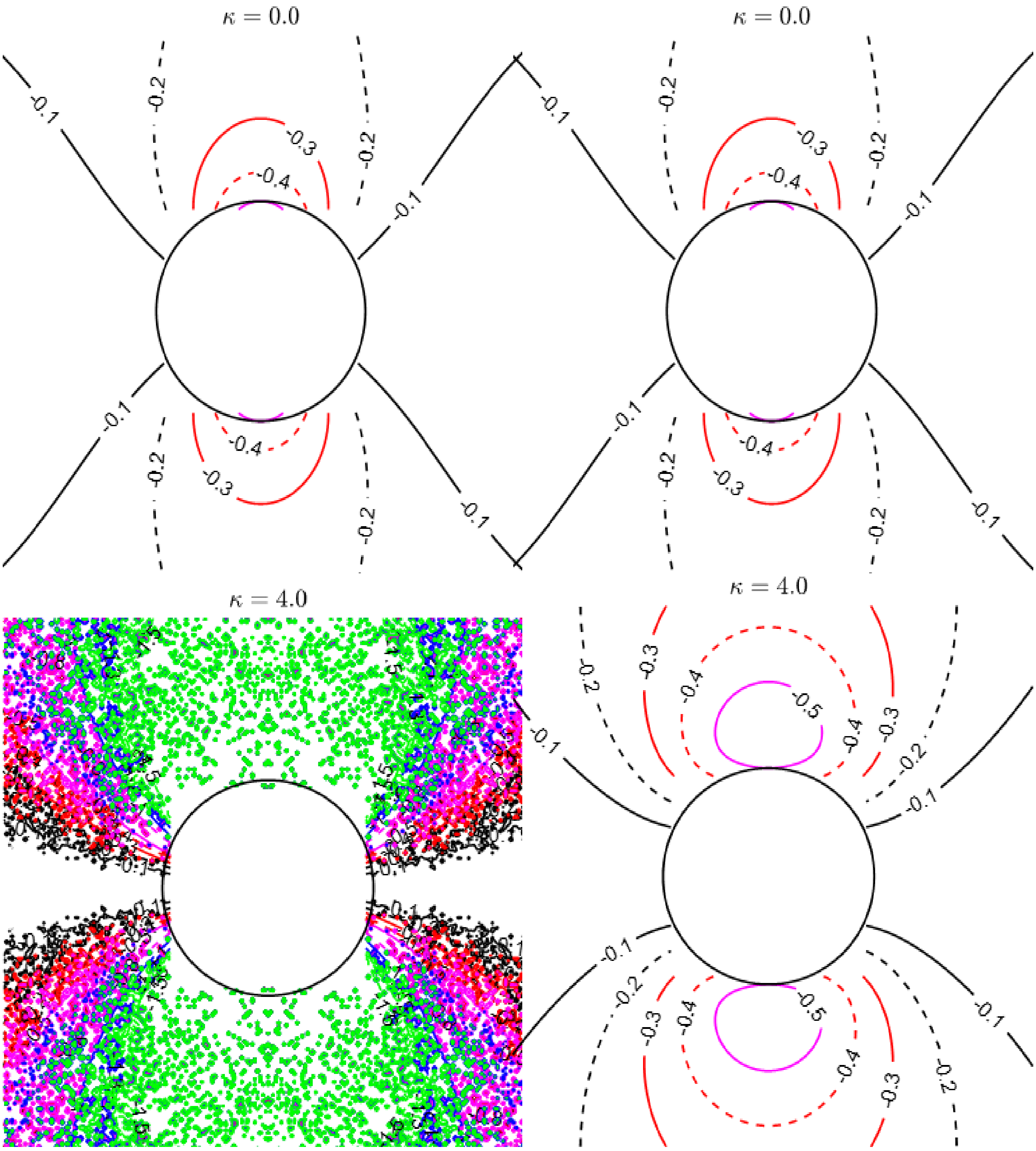


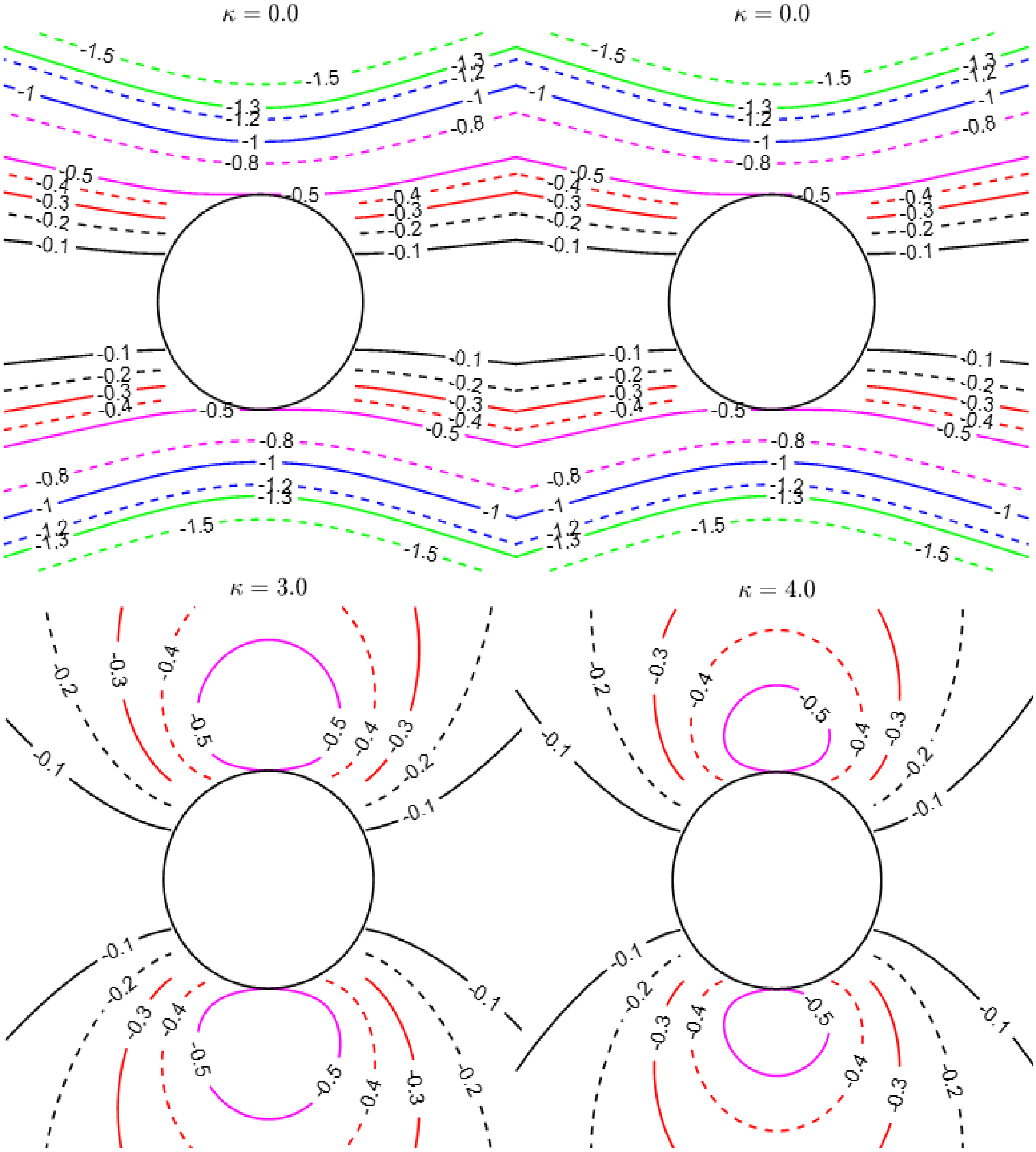
Figure 9: Normalized drag force exerted on a rigid sphere of a unit radius for perfect slip and perfect spin slip for various values of the permeability, the couple stress parameter and $\zeta^{'} = 0.01$.



(a) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 0.0.$

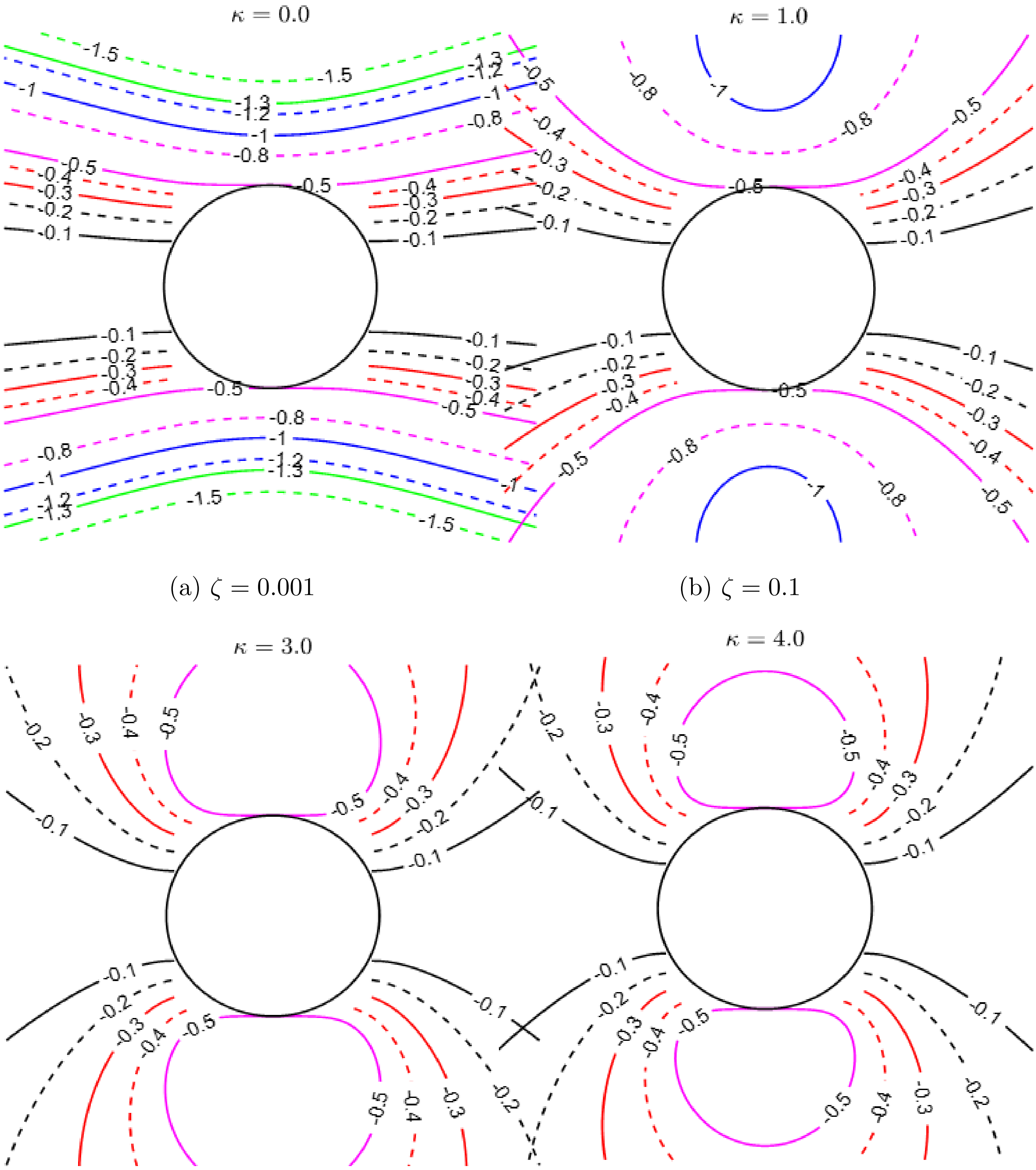


(b) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 1.0.$



(c) $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} \to \infty.$

Figure 10: htreamlines distribution for various parameters $\zeta = 10.0,\ \zeta' = 0.1.$



(a) $\zeta = 0.001$ (b) $\zeta = 0.1$

(c)$\zeta = 0.1$ (d)$\zeta = 10.0$

Figure 11: htreamlines distribution for various parameters $\frac{\beta_1}{a\mu} = \frac{a\beta_2}{\mu} = 10.0,\ \zeta^{'} = 0.1.$