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A Mathematical Framework for Modeling the Spread of HIV-disease within Two Different Age Classes

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Abstract. In this paper, we propose a mathematical model for the spread of HIV disease within two different age classes. We define a basic reproduction number R_0 that depends on the characteristics of the two age classes. We prove that if $\mathrm{R}_0 < 1$, then the disease is extinct in both age classes. In contrast, we prove that if $\mathrm{R}_0 > 1$, then the disease is endemic in both age classes.

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Key Words and Phrases: HIV-AIDS, epidemic model, basic reproduction number, asymptotic stability, global stability, numerical simulations

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1. Introduction

Acquired Immunodeficiency Syndrome (AIDS) is one the most deadly disease caused by a Human Immunodeficiency Virus (HIV). The virus destroys all the immune system, in particular the CD4 T-lymphocytes, and leaves individuals susceptible to any other infections. It multiplies within those lymphocytes and eventually destroys them. Once the lymphocytes are depleted, then the immune system stops functioning properly. As a result, the individual can catch any kind of disease that might kill him easily because of the failure of its immune system. However, there are drugs that can slow down the progression of the virus. HIV-AIDS is usually transmitted in three different ways, namely the sexual contact, blood transfusion, mother-to-child exchanges during pregnancy, childbirth and breastfeeding. Many mathematical models are used to study the impact of preventive control strategies on the spread of HIV-AIDS in given populations [1, 3, 5, 6, 9–14]. Some of these models have shown that a change in risky behavior is necessary to prevent the spread of HIV-AIDS, even in the presence of a treatment [2, 8].

In this paper, we study the spread of HIV-AIDS in the age structured populations. In fact, we consider two different age classes: A first class that corresponds to individuals aged

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25 years or less and a second class that corresponds to individuals aged over 25 years. We suppose that each age class is composed of susceptible and infected individuals. According to the Center of Disease Control (CDC), there are three main classes of HIV-AIDS infected individuals based on CD4 T-lymphocyte counts, see Table 1.

Table 1: Main classes of HIV-AIDS infected individuals

Stages of infection	CD4 T-lymphocytes/mm^3
Stage 1	> 500
Stage 2	$200 < \text{CD4 T} < 500$
Stage 3	< 200

The first stage of infection occurs between two and six weeks after HIV infection. The infected individual begins to produce antibodies that are detectable by HIV tests. The individual is then called HIV-positive. The second stage of infection is characterized by a reduction in the number of viral particles in the blood, marking the beginning of the clinical latency phase of the infection. Finally, the third stage of infection is characterized by the presence of major infections.

We aim to capture the spread of HIV-AIDS in a population divided into two different age classes by a system of ordinary differential equations.

The paper is organized as follows. In Section 2, we formulate a mathematical model for HIV-AIDS. The basic properties of the model are given in Section 3. In Section 4, the disease-free equilibrium point (DFE) and the basic reproduction number R_0 are calculated. In section 5, we prove the local extinction of the infected populations in both age classes when $R_0 < 1$. In Section 6, the global extinction of the disease in both age classes is studied and followed by some concluding results. Sections 7 and 8 deal with the persistence of the disease in both classes when $R_0 > 1$. In Section 9, some numerical results are presented. The main conclusions are recapped in Section 10.

2. Mathematical Model for HIV-AIDS

In this section, we formulate a mathematical model for HIV-AIDS. We divide the total population N into two age classes. The first age class is denoted by C_1 and the second by C_2. In each age class, there is one compartment of susceptible individuals and three compartments of infected individuals. In the class C_1, S_1 represents the compartment of susceptible individuals and I_1^1, I_1^2 and I_1^3 are the compartments of infected individuals at stage 1, 2 and 3 of infection, respectively. Similarly, in the class C_2, S_2 is the compartment of susceptible individuals and I_2^1, I_2^2 and I_2^3 represent the compartments of infected individuals at stage 1, 2 and 3 of infection, respectively.

The total population N can be expressed as the following sum:

$$N = \sum_{j=1}^{2} S_j + \sum_{j=1}^{3} I_1^j + \sum_{j=1}^{3} I_2^j.$$

The Spread of HIV-AIDS within two different age classes in the population is illustrated in Figure 1.

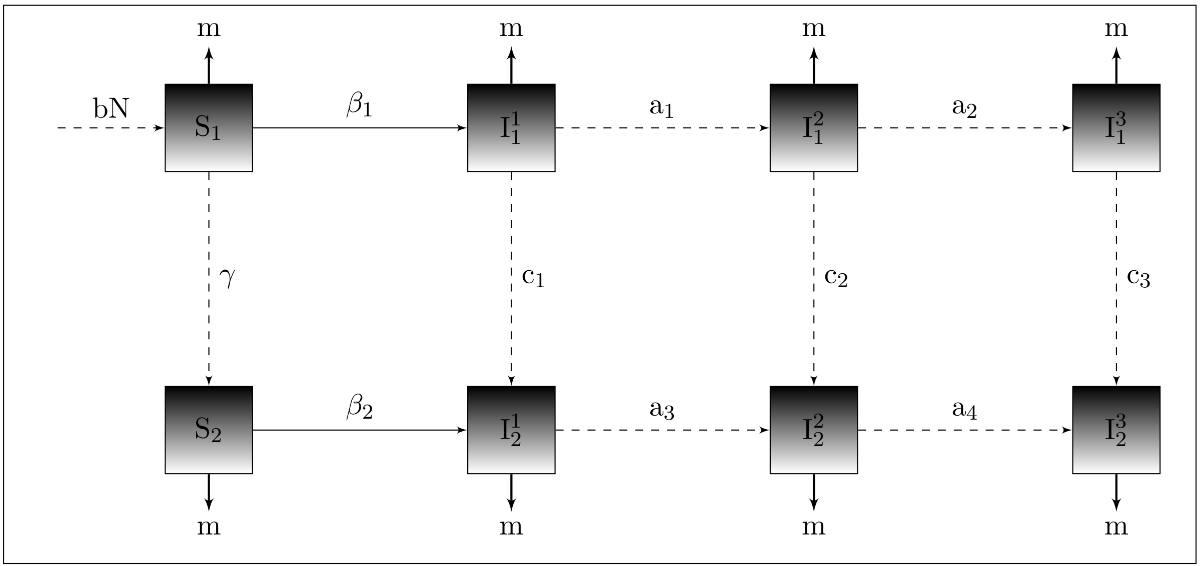


Figure 1: Flow diagram of HIV-AIDS transmission dynamics

Using the above representation, we formulate the corresponding dynamical model as follows:

$$\begin{cases} \dfrac{\mathrm{d}\ \mathrm{S}_1}{\mathrm{dt}} = \mathrm{bN} - \dfrac{\mathrm{S}_1}{\mathrm{N}}\left(\beta_1 \displaystyle\sum_{\mathrm{j}=1}^{3} \mathrm{I}_1^{\mathrm{j}} + \beta_2 \sum_{\mathrm{j}=1}^{3} \mathrm{I}_2^{\mathrm{j}}\right) - (\mathrm{m}+\gamma)\mathrm{S}_1 \\ \dfrac{\mathrm{d}\ \mathrm{I}_1^1}{\mathrm{dt}} = \dfrac{\mathrm{S}_1}{\mathrm{N}}\left(\beta_1 \displaystyle\sum_{\mathrm{j}=1}^{3} \mathrm{I}_1^{\mathrm{j}} + \beta_2 \sum_{\mathrm{j}=1}^{3} \mathrm{I}_2^{\mathrm{j}}\right) - (\mathrm{m} + \mathrm{a}_1 + \mathrm{c}_1)\mathrm{I}_1^1 \\ \dfrac{\mathrm{d}\ \mathrm{I}_1^2}{\mathrm{dt}} = \mathrm{a}_1\mathrm{I}_1^1 - (\mathrm{m} + \mathrm{a}_2 + \mathrm{c}_2)\mathrm{I}_1^2 \\ \dfrac{\mathrm{d}\ \mathrm{I}_1^3}{\mathrm{dt}} = \mathrm{a}_2\mathrm{I}_1^2 - (\mathrm{m} + \mathrm{c}_3)\mathrm{I}_1^3 \\ \dfrac{\mathrm{d}\ \mathrm{S}_2}{\mathrm{dt}} = \gamma\mathrm{S}_1 - \dfrac{\mathrm{S}_2}{\mathrm{N}}\left(\beta_1 \displaystyle\sum_{\mathrm{j}=1}^{3} \mathrm{I}_1^{\mathrm{j}} + \beta_2 \sum_{\mathrm{j}=1}^{3} \mathrm{I}_2^{\mathrm{j}}\right) - \mathrm{mS}_2 \\ \dfrac{\mathrm{d}\ \mathrm{I}_2^1}{\mathrm{dt}} = \dfrac{\mathrm{S}_2}{\mathrm{N}}\left(\beta_1 \displaystyle\sum_{\mathrm{j}=1}^{3} \mathrm{I}_1^{\mathrm{j}} + \beta_2 \sum_{\mathrm{j}=1}^{3} \mathrm{I}_2^{\mathrm{j}}\right) - (\mathrm{m} + \mathrm{a}_3)\mathrm{I}_2^1 + c_1\mathrm{I}_1^1 \\ \dfrac{\mathrm{d}\ \mathrm{I}_2^2}{\mathrm{dt}} = \mathrm{a}_3\mathrm{I}_2^1 - (\mathrm{m} + \mathrm{a}_4)\mathrm{I}_2^2 + \mathrm{c}_2\mathrm{I}_1^2 \\ \dfrac{\mathrm{d}\ \mathrm{I}_2^3}{\mathrm{dt}} = \mathrm{a}_4\mathrm{I}_2^2 - \mathrm{mI}_2^3 + \mathrm{c}_3\mathrm{I}_1^3 \end{cases} \tag{1}$$

The system (1) is completed with the following initial conditions:

$$\mathrm{S}_1 \geq 0, \quad \mathrm{I}_1^1 \geq 0, \quad \mathrm{I}_1^2 \geq 0, \quad \mathrm{I}_1^3 \geq 0, \quad \mathrm{S}_2 \geq 0, \quad \mathrm{I}_2^1 \geq 0, \quad \mathrm{I}_2^2 \geq 0, \quad \mathrm{I}_2^3 \geq 0. \tag{2}$$

Summing the equations of system (1), we obtain:

$$\frac{\mathrm{d}\ \mathrm{N}}{\mathrm{dt}} = (\mathrm{b} - \mathrm{m})\,\mathrm{N}.$$

The parameters of the model are reported in Table 2. They are all positive.

The parameter b represents the rate at which young begin sexual activity. The parameter m is the death rate. The parameters β_1 and β_2 are infection rates in C_1 and C_2, respectively. The rate at which susceptible individuals in C_1 get older and reach the age of becoming susceptible individuals in C_2 is given by γ. The parameters a_1 and a_2 are the rates at which infected individuals in the class C_1 move from stage 1 to stage 2 and from stage 2 to stage 3, respectively, within this class. Similarly, the parameters a_3 and

Table 2: Description of the model parameters

Parameters	Description
b, m	Recruitment rate, natural mortality rate
γ, c_1, c_2, c_3	Transfer rates from C_1 to C_2
a_1, a_2	Transfer rates within C_1
a_3, a_4	Transfer rates within C_2
β_1, β_2	Transmission rates

a_4 describe the rates at which infected individuals in the class C_2 move from stage 1 to stage 2 and from stage 2 to stage 3, respectively, within this class. The parameters c_k, $k = 1, 2, 3$, denote the rates at which infected individuals in the stage k of C_1 get older and reach the age of becoming infected individuals in the stage k of C_2.

For the mathematical analysis of the model, we introduce the following scalings:

$$s_1 = \frac{S_1}{N}, \quad i_1^1 = \frac{I_1^1}{N}, \quad i_1^2 = \frac{I_1^2}{N}, \quad i_1^3 = \frac{I_1^1}{N}, \quad s_2 = \frac{S_2}{N}, \quad i_2^1 = \frac{I_2^1}{N}, \quad i_2^2 = \frac{I_2^2}{N}, \quad i_2^3 = \frac{I_2^3}{N}. \tag{3}$$

From the equation 3, it holds:

$$\sum_{j=1}^{2} s_j + \sum_{j=1}^{3} i_1^j + \sum_{j=1}^{3} i_2^j = 1. \tag{4}$$

According to the new variables, the system (1) can be rewritten as follows:

$$\left\{\begin{aligned}
\frac{d\ s_1}{dt} &= b - s_1\left(\beta_1\sum_{j=1}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right) - \kappa_1 s_1\\
\frac{d\ i_1^1}{dt} &= s_1\left(\beta_1\sum_{j=1}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right) - \kappa_2 i_1^1\\
\frac{d\ i_1^2}{dt} &= a_1 i_1^1 - \kappa_3 i_1^2\\
\frac{d\ i_1^3}{dt} &= a_2 i_1^2 - \kappa_4 i_1^3\\
\frac{d\ s_2}{dt} &= \gamma s_1 - s_2\left(\beta_1\sum_{j=1}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right) - b s_2\\
\frac{d\ i_2^1}{dt} &= s_2\left(\beta_1\sum_{j=1}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right) - \kappa_5 i_2^1 + c_1 i_1^1\\
\frac{d\ i_2^2}{dt} &= a_3 i_2^1 - \kappa_6 i_2^2 + c_2 i_1^2\\
\frac{d\ i_2^3}{dt} &= a_4 i_2^2 - b\ i_2^3 + c_3 i_1^3
\end{aligned}\right. \tag{5}$$

where

$\kappa_1 = b+\gamma, \quad \kappa_2 = b+a_1+c_1, \quad \kappa_3 = b+a_2+c_2, \quad \kappa_4 = b+c_3, \quad \kappa_5 = b+a_3, \quad \kappa_6 = b+a_4.$

After normalization of the initial data, we obtain:

$$\sum_{j=1}^{2} s_j(0) + \sum_{j=1}^{3} i_1^j(0) + \sum_{j=1}^{3} i_2^j(0) = 1 \tag{6}$$

and

$$s_1(0) \geq 0,\ i_1^1(0) \geq 0,\ i_1^2(0) \geq 0,\ i_1^3(0) \geq 0,\ s_2(0) \geq 0,\ i_2^1(0) \geq 0,\ i_2^2(0) \geq 0,\ i_2^3(0) \geq 0. \tag{7}$$

The variables of the model (5) are reported in Table 3.

Table 3: Variables for the re-scaled HIV-AIDS model

Variables	Description
s_1	Proportion of susceptible individuals in class C_1
s_2	Proportion of susceptible individuals in class C_2
i_1^1	Proportion of individuals at stage 1 of infection in C_1
i_1^2	Proportion of individuals at stage 2 of infection in C_1
i_1^3	Proportion of individuals at stage 3 of infection in C_1
i_2^1	Proportion of individuals at stage 1 of infection in C_2
i_2^2	Proportion of individuals at stage 2 of infection in C_2
i_2^3	Proportion of individuals at stage 3 of infection in C_2

3. Basic Properties

Theorem 1.
The feasible region Γ defined by

$$\Gamma = \left\{ e = \left(s_1, \quad i_1^1, \quad i_1^2, \quad i_1^3, \quad s_2, \quad i_2^1, \quad i_2^2, \quad i_2^3\right) \in \mathbb{R}_+^8 : \quad 0 \leq \sum_{j=1}^{2} s_j + \sum_{j=1}^{3} i_1^j + \sum_{j=1}^{3} i_2^j \leq 1 \right\}$$

with the initial conditions

$$s_1(0) \geq 0\,,\ i_1^1(0) \geq 0\,,\ i_1^2(0) \geq 0\,,\ i_1^3(0) \geq 0\,,\ s_2(0) \geq 0\,,\ i_2^1(0) \geq 0\,,\ i_2^2(0) \geq 0\,,\ i_2^3(0) \geq 0$$

is a positively invariant set for the system (5).

Proof. **1. Positivity of Solutions**

We show by absurd that for all $t \geq 0$, $e(t) \geq 0$. Suppose that for a time $t' > 0$, we have $e(t') < 0$. The function e being continuous, from the intermediate value theorem, there exists a time $t_1 \in]0, t'[$ such that $e(t_1) = 0$.
Consider the equations of system (5) and let:

$$\xi_1(t) = \exp\left[\int_0^t \left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j + \kappa_1\right) d\tau\right], \quad \xi_3(t) = \exp\left(\int_0^t (\kappa_2 - \beta_1 s_1)\, d\tau\right),$$

$$\xi_2(t) = \exp(\kappa_3 t)\,, \qquad \xi_4(t) = \exp(\kappa_4 t)\,, \qquad \xi_5(t) = \exp(\kappa_6 t)\,, \qquad \xi_7(t) = \exp(bt)\,,$$

$$\xi_6(t) = \exp\left(\int_0^t (\kappa_5 - \beta_2 s_2)\, d\tau\right), \quad \xi_8(t) = \exp\left[\int_0^t \left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j + b\right) d\tau\right].$$

By differentiating each of the expressions $s_1\xi_1$, $i_1^1\xi_3$, $i_1^2\xi_2$, $i_1^3\xi_4$, $s_2\xi_8$, $i_2^1\xi_5$, $i_2^2\xi_6$ and $i_2^3\xi_7$ with respect to time t, we obtain:

$$\frac{ds_1\xi_1}{dt} = b\xi_1 \tag{8a}$$

$$\frac{di_1^1\xi_3}{dt} = s_1\xi_3\left(\beta_1\sum_{j=2}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right) \tag{8b}$$

$$\frac{di_1^2\xi_2}{dt} = a_1 i_1^1\xi_2 \tag{8c}$$

$$\frac{di_1^3\xi_4}{dt} = a_2 i_1^2\xi_4 \tag{8d}$$

$$\frac{ds_2\xi_8}{dt} = \gamma s_1\xi_8 \tag{8e}$$

$$\frac{di_2^1\xi_6}{dt} = \xi_6\left(\beta_1 s_2\sum_{j=1}^{3} i_1^j + \beta_2 s_2\sum_{j=2}^{3} i_2^j + c_1 i_1^1\right) \tag{8f}$$

$$\frac{di_2^2\xi_5}{dt} = \left(a_3 i_2^1 + c_2 i_1^2\right)\xi_5 \tag{8g}$$

$$\frac{di_2^3\xi_7}{dt} = \left(a_4 i_2^2 + c_3 i_1^3\right)\xi_7 \tag{8h}$$

By integrating the equations (8a)-(8h) between 0 and t_1, it holds:

$$s_1(t_1) = \frac{1}{\xi_1(t_1)}\left[s_1(0) + \int_0^{t_1} b\xi_1 dt\right] > 0 \tag{9a}$$

$$i_1^1(t_1) = \frac{1}{\xi_3(t_1)}\left[i_1^1(0) + \int_0^{t_1} s_1\xi_3\left(\beta_1\sum_{j=2}^{3} i_1^j + \beta_2\sum_{j=1}^{3} i_2^j\right)dt\right] > 0 \tag{9b}$$

$$i_1^2(t_1) = \frac{1}{\xi_2(t_1)}\left[i_1^2(0) + \int_0^{t_1} a_1\xi_2 i_1^1 dt\right] > 0 \tag{9c}$$

$$i_1^3(t_1) = \frac{1}{\xi_4(t_1)}\left[i_1^3(0) + \int_0^{t_1} a_2\xi_4 i_1^2 dt\right] > 0 \tag{9d}$$

$$s_2(t_1) = \frac{1}{\xi_8(t_1)}\left[s_2(0) + \int_0^{t_1} \gamma s_1\xi_8 dt\right] > 0 \tag{9e}$$

$$i_2^1(t_1) = \frac{1}{\xi_6(t_1)}\left[i_2^1(0) + \int_0^{t_1} \xi_6\left(\beta_1 s_2\sum_{j=1}^{3} i_1^j + \beta_2 s_2\sum_{j=2}^{3} i_2^j + c_1 i_1^1\right)dt\right] > 0 \tag{9f}$$

$$i_2^2(t_1) = \frac{1}{\xi_5(t_1)}\left[i_2^2(0) + \int_0^{t_1} \left(a_3 i_2^1 + c_2 i_1^2\right)\xi_5 dt\right] > 0 \tag{9g}$$

$$i_2^3(t_1) = \frac{1}{\xi_7(t_1)}\left[i_2^3(0) + \int_0^{t_1} \left(a_4 i_2^2 + c_3 i_1^3\right)\xi_7 dt\right] > 0 \tag{9h}$$

From (9a)-(9h), it follows that $e(t_1) > 0$. This is a contradiction according to the starting hypothesis. Then, $\forall t \geq 0$, $e(t) \geq 0$. Therefore, all solutions initiated in $\mathbb{R}^8_+$ are positive.

2. Invariant Region

Summing the equations of system (5), we obtain:

$$\frac{d}{dt}\left(s_1 + \sum_{j=1}^{3} i_1^j + s_2 + \sum_{j=1}^{3} i_2^j\right) = 0. \tag{10}$$

Integrating (10) using initial conditions, it holds:

$$\forall t \geq 0, \quad s_1(t) + \sum_{j=1}^{3} i_1^j(t) + s_2 t + \sum_{j=1}^{3} i_2^j(t) \leq 1.$$

This achieves the proof.

Consequently, in Γ, the model (5) is epidemiologically and mathematically well-posed. Therefore, it is sufficient to study the dynamics of the model in Γ.

4. Disease-Free Equilibrium (DFE) E_0 and Reproduction Number R_0

The Disease-Free Equilibrium (DFE) E_0 of the model (5) is determined by solving the following system:

$$\begin{cases} \dfrac{d\, s_1}{dt} = 0 \\ \dfrac{d\, i_1^1}{dt} = 0 \\ \dfrac{d\, i_1^2}{dt} = 0 \\ \dfrac{d\, i_1^3}{dt} = 0 \\ \dfrac{d\, s_2}{dt} = 0 \\ \dfrac{d\, i_2^1}{dt} = 0 \\ \dfrac{d\, i_2^2}{dt} = 0 \\ \dfrac{d\, i_2^3}{dt} = 0 \end{cases} \tag{11}$$

In the case of absence of disease, i.e. the population size is zero in all compartments except the susceptible compartment, the solution of (11) is given by:

$$E_0\left(\frac{b}{\kappa_1}, 0, 0, 0, \frac{\gamma}{\kappa_1}, 0, 0, 0\right).$$

We determine the basic reproduction number R_0 using the next generation matrix method at Disease-Free Equilibrium [12]. According to this method, R_0 is defined as the effective number of secondary infections caused by typical infected individual during his/her entire period of infectioussness [1, 4]. Let X be the vector of infected classes:

$$X = \begin{pmatrix} i_1^1, & i_1^2, & i_1^3, & i_2^1, & i_2^2, & i_2^3 \end{pmatrix}^T.$$

$$F = \begin{pmatrix} s_1\Big(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\Big), & 0, & 0, & s_2\Big(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\Big), & 0, & 0 \end{pmatrix}^T$$

denotes the vector of terms corresponding to new infections.

$$V = \left(\kappa_2 i_1^1,\ \kappa_3 i_1^2 - a_1 i_1^1,\ \kappa_4 i_1^3 - a_2 i_1^2,\ \kappa_5 i_2^1 - c_1 i_1^1,\ \kappa_6 i_2^2 - c_2 i_1^2 - a_3 i_2^1,\ b\ i_2^3 - c_3 i_1^3 - a_4 i_2^2\right)^T$$

refers to the vector of terms corresponding to individuals entering a given compartment and individuals leaving.

The partial derivatives of F and V with respect to i_1^1, i_1^2, i_1^3, i_2^1, i_2^2 and i_2^3 are given by the following matrices $\mathcal{F}$ and $\mathcal{V}$:

$$\mathcal{F} = \begin{pmatrix} \beta_1 \frac{b}{\kappa_1} & \beta_1 \frac{b}{\kappa_1} & \beta_1 \frac{b}{\kappa_1} & \beta_2 \frac{b}{\kappa_1} & \beta_2 \frac{b}{\kappa_1} & \beta_2 \frac{b}{\kappa_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 \frac{\gamma}{\kappa_1} & \beta_1 \frac{\gamma}{\kappa_1} & \beta_1 \frac{\gamma}{\kappa_1} & \beta_2 \frac{\gamma}{\kappa_1} & \beta_2 \frac{\gamma}{\kappa_1} & \beta_2 \frac{\gamma}{\kappa_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathcal{V} = \begin{pmatrix} \kappa_2 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & \kappa_3 & 0 & 0 & 0 & 0 \\ 0 & -a_2 & \kappa_4 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & \kappa_5 & 0 & 0 \\ 0 & -c_2 & 0 & -a_3 & \kappa_6 & 0 \\ 0 & 0 & -c_3 & 0 & -a_4 & b \end{pmatrix}$$

The next-generation matrix is defined by:

$$K = \mathcal{F}\mathcal{V}^{-1} = \frac{1}{M_7} \begin{pmatrix} M_1 b & M_2 b & M_3 b & M_4 b & M_5 b & M_6 b \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ M_1\gamma & M_2\gamma & M_3\gamma & M_4\gamma & M_5\gamma & M_6\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} M_1 &= \beta_1 b\kappa_3\kappa_4\kappa_5\kappa_6 + a_1\beta_1 b\kappa_4\kappa_5\kappa_6 + \beta_1 a_1 a_2 b\kappa_5\kappa_6 + \beta_2 c_1 b\kappa_3\kappa_4\kappa_6 + \beta_2 c_2 a_1 b\kappa_4\kappa_5 \\ &\quad + \beta_2 a_3 c_1 b\kappa_3\kappa_4 + \beta_2 c_3 a_2 a_1\kappa_5\kappa_6 + \beta_2 a_4 a_1 c_2\kappa_4\kappa_5 + \beta_2 a_4 a_3 c_1\kappa_3\kappa_4 \qquad (12) \\ M_2 &= \beta_1 b\kappa_2\kappa_4\kappa_5\kappa_6 + \beta_1 a_2 b\kappa_2\kappa_5\kappa_6 + \beta_2 c_2 b\kappa_2\kappa_4\kappa_5 + \beta_2 c_3 a_2\kappa_2\kappa_5\kappa_6 \\ &\quad + \beta_2 a_4 c_2\kappa_2\kappa_4\kappa_5 \qquad (13) \\ M_3 &= \beta_1 b\kappa_2\kappa_3\kappa_5\kappa_6 + \beta_2 c_3\kappa_2\kappa_3\kappa_5\kappa_6 \qquad (14) \\ M_4 &= \beta_2 b\kappa_2\kappa_3\kappa_4\kappa_6 + \beta_2 a_3 b\kappa_2\kappa_3\kappa_4 + \beta_2 a_3 a_4\kappa_2\kappa_3\kappa_4 \qquad (15) \\ M_5 &= \beta_2 b\kappa_2\kappa_3\kappa_4\kappa_5 + \beta_2 a_4\kappa_2\kappa_3\kappa_4\kappa_5 \qquad (16) \\ M_6 &= \beta_2\kappa_2\kappa_3\kappa_4\kappa_5\kappa_6 \qquad (17) \\ M_7 &= b\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\kappa_6 \qquad (18) \end{aligned}$$

The basic reproduction number R_0, computed from the spectral raduis of the next-generation matrix K, is given by:

$$R_0 = \frac{b\ M_1 + \gamma M_4}{M_7} \qquad (19)$$

where M_1, M_4 and M_7 are explicitly defined by (12), (15) and (18).

5. Local Stability of Disease-Free Equilibrium (DFE)

Theorem 2.
If $R_0 < 1$, then the disease-free equilibrium point

$$E_0\left(\frac{b}{\kappa_1}, 0, 0, 0, \frac{\gamma}{\kappa_1}, 0, 0, 0\right)$$

is locally asymptotically stable.

Proof. Consider the system (5). The jacobian of this system at E_0 is denoted by $J(E_0)$ and its eigenvalues are the solutions of the following equations:

$$r_8X^8 + r_7X^7 + r_6X^6 + r_5X^5 + r_4X^4 + r_3X^3 + r_2X^2 + r_1X + r_0 = 0.$$

Let us define:

$$\begin{aligned}
f_1 &= -a_1\kappa_5a_4c_2\kappa_4\beta_2b - a_1a_2\kappa_5\kappa_6b^2\beta_1 - \gamma\beta_2\kappa_6b\kappa_4\kappa_3\kappa_2 - a_3\gamma a_4\kappa_4\kappa_3\kappa_2\beta_2a_1\kappa_5\kappa_6b^2\kappa_4\beta_1 \\
&\quad -a_3a_4\kappa_4\kappa_3c_1\beta_2b \\
f_2 &= -\beta_2b^2\kappa_6\kappa_4\kappa_3c_1 - a_1a_2c_3\kappa_5\kappa_6\beta_2b - a_3\beta_2b^2\kappa_4\kappa_3c_1 - \kappa_5\kappa_6b^2\kappa_4\kappa_3\beta_1 - a_3\gamma\beta_2b\kappa_4\kappa_3\kappa_2 \\
&\quad -a_1\kappa_5b^2c_2\kappa_4\beta_2 \\
f_3 &= b\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\kappa_6 \\
f_4 &= -a_1\kappa_5a_4c_2\beta_2b - \beta_2b^2\kappa_6\kappa_3c_1 - a_3\beta_2b^2\kappa_3c_1 - \kappa_5\kappa_6b^2\kappa_3\beta_1 - a_3\gamma\beta_2b\kappa_3\kappa_2 - a_1\kappa_5b^2c_2\beta_2 \\
f_5 &= -\gamma\beta_2\kappa_6b\kappa_3\kappa_2 - a_3\gamma a_4\kappa_3\kappa_2\beta_2 - a_1\kappa_5\kappa_6b^2\beta_1 - a_3a_4\kappa_3c_1\beta_2b + \kappa_1\kappa_5\kappa_6b\kappa_3\kappa_2 \\
f_6 &= -a_3\gamma\beta_2b\kappa_4\kappa_3 - \gamma\beta_2\kappa_6b\kappa_4\kappa_3 - a_3\gamma a_4\kappa_4\kappa_3\beta_2 + \kappa_1\kappa_5\kappa_6b\kappa_4\kappa_3 \\
f_7 &= -\beta_2b^2\kappa_6\kappa_4c_1 - a_3\beta_2b^2\kappa_4c_1 - \kappa_5\kappa_6b^2\kappa_4\beta_1 - a_3\gamma\beta_2b\kappa_4\kappa_2 \\
f_8 &= -\gamma\beta_2\kappa_6b\kappa_4\kappa_2 - a_3\gamma a_4\kappa_4\kappa_2\beta_2 - a_3a_4\kappa_4c_1\beta_2b + \kappa_1\kappa_5\kappa_6b\kappa_4\kappa_2 \\
f_9 &= -a_1a_4c_2\kappa_4\beta_2b - a_1a_2c_3\kappa_6\beta_2b - \kappa_6b^2\kappa_4\kappa_3\beta_1 - a_1b^2c_2\kappa_4\beta_2 - a_1a_2\kappa_6b^2\beta_1 - a_1\kappa_6b^2\kappa_4\beta_1 \\
&\quad +\kappa_1\kappa_6b\kappa_4\kappa_3\kappa_2 \\
f_{10} &= -\beta_2b^2\kappa_4\kappa_3c_1 - a_1a_2c_3\kappa_5\beta_2b - \kappa_5b^2\kappa_4\kappa_3\beta_1 - a1a_2\kappa_5b^2\beta_1 - \gamma\beta_2b\kappa_4\kappa_3\kappa_2 - a_1\kappa_5b^2\kappa_4\beta_1 \\
&\quad +\kappa_1\kappa_5b\kappa_4\kappa_3\kappa_2 \\
f_{11} &= -a_1a_2\kappa_5\kappa_6\beta_1b - a_3\beta_2b\kappa_4\kappa_3c_1 - \kappa_5\kappa_6b\kappa_4\kappa_3\beta_1 - a_3\gamma\beta_2\kappa_4\kappa_3\kappa_2 - a_1\kappa_5c_2\kappa_4\beta_2b \\
f_{12} &= -\beta_2b\kappa_6\kappa_4\kappa_3c_1 - a_1\kappa_5\kappa_6\kappa_4\beta_1b - \gamma\beta_2\kappa_6\kappa_4\kappa_3\kappa_2 + \kappa_1\kappa_5\kappa_6\kappa_4\kappa_3\kappa_2 \\
f_{13} &= -\beta_2b^2\kappa_6c_1 - a_3\beta_2b^2c_1 - \kappa_5\kappa_6b^2\beta_1 - a_3\gamma\beta_2b\kappa_2 - \gamma\beta_2\kappa_6b\kappa_2 - a_3\gamma a_4\kappa_2\beta_2 - a_3a_4c_1\beta_2b \\
&\quad +\kappa_1\kappa_5\kappa_6b\kappa_2 \\
f_{14} &= -a_3\gamma\beta_2b\kappa_3 - \gamma\beta_2\kappa_6b\kappa_3 - a_3\gamma a_4\kappa_3\beta_2 + \kappa_1\kappa_5\kappa_6b\kappa_3 \\
f_{15} &= -\beta_2b\kappa_4\kappa_3c_1 - \kappa_5b\kappa_4\kappa_3\beta_1 - a_1a_2\kappa_5\beta_1b - \gamma\beta_2\kappa_4\kappa_3\kappa_2 - a_1\kappa_5\kappa_4\beta_1b + \kappa_1\kappa_5\kappa_4\kappa_3\kappa_2 \\
f_{16} &= -\beta_2b\kappa_6\kappa_4c_1 - a_3\beta_2b\kappa_4c_1 - \kappa_5\kappa_6b\kappa_4\beta_1 - a_3\gamma\beta_2\kappa_4\kappa_2 - \gamma\beta_2\kappa_6\kappa_4\kappa_2 + \kappa_1\kappa_5\kappa_6\kappa_4\kappa_2 \\
f_{17} &= -\beta_2b\kappa_6\kappa_3c_1 - a_3\beta_2b\kappa_3c_1 - \kappa_5\kappa_6b\kappa_3\beta_1 - a_3\gamma\beta_2\kappa_3\kappa_2 - a_1\kappa_5c_2\beta_2b - \gamma\beta_2\kappa_6\kappa_3\kappa_2 \\
&\quad -a_1\kappa_5\kappa_6\beta_1b + \kappa_1\kappa_5\kappa_6\kappa_3\kappa_2
\end{aligned}$$

$$
\begin{aligned}
f_{18} &= -a_1a_4c_2\beta_2 b - \kappa_6 b^2\kappa_3\beta_1 - a_1 b^2 c_2\beta_2 - a_1\kappa_6 b^2\beta_1 + \kappa_1\kappa_6 b\kappa_3\kappa_2\\
f_{19} &= -a_3\gamma\beta_2 b\kappa_4 - \gamma\beta_2\kappa_6 b\kappa_4 - a_3\gamma a_4\kappa_4\beta_2 + \kappa_1\kappa_5\kappa_6 b\kappa_4\\
f_{20} &= -\kappa_6 b^2\kappa_4\beta_1 + \kappa_1\kappa_6 b\kappa_4\kappa_2\\
f_{21} &= -a_3\gamma\beta_2\kappa_4\kappa_3 - \gamma\beta_2\kappa_6\kappa_4\kappa_3 + \kappa_1\kappa_5\kappa_6\kappa_4\kappa_3\\
f_{22} &= -\gamma\beta_2 b\kappa_4\kappa_3 + \kappa_1\kappa_5 b\kappa_4\kappa_3\\
f_{23} &= -\beta_2 b^2\kappa_3 c_1 - \kappa_5 b^2\kappa_3\beta_1 - \gamma\beta_2 b\kappa_3\kappa_2 - a_1\kappa_5 b^2\beta_1 + \kappa_1\kappa_6 b\kappa_4\kappa_3 + \kappa_1\kappa_5 b\kappa_3\kappa_2\\
f_{24} &= -\kappa_6 b\kappa_4\kappa_3\beta_1 - a_1c_2\kappa_4\beta_2 b - a_1a_2\kappa_6\beta_1 b - a_1\kappa_6\kappa_4\beta_1 b + \kappa_1\kappa_6\kappa_4\kappa_3\kappa_2\\
f_{25} &= -b^2\kappa_4\kappa_3\beta_1 - a_1a_2 b^2\beta_1 - a_1 b^2\kappa_4\beta_1 - a_1a_2c_3\beta_2 b + \kappa_1 b\kappa_4\kappa_3\kappa_2\\
f_{26} &= -\kappa_5 b^2\kappa_4\beta_1 - \gamma\beta_2 b\kappa_4\kappa_2 - \beta_2 b^2\kappa_4 c_1 + \kappa_1\kappa_5 b\kappa_4\kappa_2\\
f_{27} &= -\gamma\beta_2 b\kappa_4 + \kappa_1\kappa_5 b\kappa_4 - \kappa_6 b^2\beta_1 + \kappa_1\kappa_6 b\kappa_2 - \kappa_4\kappa_3\beta_1 b - a_1a_2\beta_1 b - a_1\kappa_4\beta_1 b + \kappa_1\kappa_4\kappa_3\kappa_2\\
f_{28} &= -b^2\kappa_3\beta_1 - a_1 b^2\beta_1 + \kappa_1 b\kappa_3\kappa_2 - \kappa_6 b\kappa_3\beta_1 - a_1c_2\beta_2 b - a_1\kappa_6\beta_1 b + \kappa_1\kappa_6\kappa_3\kappa_2\\
f_{29} &= -\kappa_6 b\kappa_4\beta_1 + \kappa_1\kappa_6\kappa_4\kappa_2 - \beta_2 b^2 c_1 - \kappa_5 b^2\beta_1 - \gamma\beta_2 b\kappa_2 + \kappa_1\kappa_5 b\kappa_2\\
f_{30} &= -\beta_2 b\kappa_6 c_1 - a_3\beta_2 bc_1 - \kappa_5\kappa_6 b\beta_1 - a_3\gamma\beta_2\kappa_2 - \gamma\beta_2\kappa_6\kappa_2 + \kappa_1\kappa_5\kappa_6\kappa_2\\
f_{31} &= -a_3\gamma\beta_2\kappa_3 - \gamma\beta_2\kappa_6\kappa_3 + \kappa_1\kappa_5\kappa_6\kappa_3 - a_3\gamma\beta_2 b - \gamma\beta_2\kappa_6 b - a_3\gamma a_4\beta_2 + \kappa_1\kappa_5\kappa_6 b\\
f_{32} &= -\gamma\beta_2\kappa_4\kappa_3 + \kappa_1\kappa_5\kappa_4\kappa_3 - \beta_2 b\kappa_4 c_1 - \kappa_5 b\kappa_4\beta_1 - \gamma\beta_2\kappa_4\kappa_2 + \kappa_1\kappa_5\kappa_4\kappa_2\\
f_{33} &= -\beta_2 b\kappa_3 c_1 - \kappa_5 b\kappa_3\beta_1 - \gamma\beta_2\kappa_3\kappa_2 - a_1\kappa_5\beta_1 b + \kappa_1\kappa_5\kappa_3\kappa_2 - a_3\gamma\beta_2\kappa_4 - \gamma\beta_2\kappa_6\kappa_4 + \kappa_1\kappa_5\kappa_6\kappa_4\\
f_{34} &= -\gamma\beta_2 b\kappa_3 + \kappa_1\kappa_5 b\kappa_3 - b^2\kappa_4\beta_1 + \kappa_1 b\kappa_4\kappa_2\\
f_{35} &= -b^2\beta_1 + \kappa_1 b\kappa_2 - \kappa_4\beta_1 b + \kappa_1\kappa_4\kappa_2 - \gamma\beta_2 b + \kappa_1\kappa_5 b - a_3\gamma\beta_2 - \gamma\beta_2\kappa_6 + \kappa_1\kappa_5\kappa_6 - \gamma\beta_2\kappa_4\\
&\quad +\kappa_1\kappa_5\kappa_4\\
f_{36} &= -\kappa_6 b\beta_1 + \kappa_1\kappa_6\kappa_2 - \kappa_3\beta_1 b - \beta_1 ba_1 + \kappa_1\kappa_3\kappa_2 - \beta_2 bc_1 - \kappa_5 b\beta_1 - \gamma\beta_2\kappa_2 + \kappa_1\kappa_5\kappa_2 - \gamma\beta_2\kappa_3\\
&\quad +\kappa_1\kappa_5\kappa_3\\
f_{37} &= \kappa_1\kappa_6 - \beta_1 b + \kappa_1\kappa_5 - \gamma\beta_2 + \kappa_1 b + \kappa_1\kappa_4 + \kappa_2\kappa_1 + \kappa_1\kappa_3
\end{aligned}
$$

and

$$
\begin{aligned}
F_1 &= f_1+f_2+f_3; \quad F_2 = f_{13}+f_{14}+f_{15}+f_{16}+f_{17}+f_{18}+f_{19}+f_{20}+f_{21}+f_{22}+f_{23}+f_{24}+f_{25}+f_{26}\\
F_3 &= f_4+f_5+f_6+f_7+f_8+f_9+f_{10}+f_{11}+f_{12}; \quad F_4 = f_{27}+f_{28}+f_{29}+f_{30}+f_{31}+f_{32}+f_{33}+f_{34}\\
F_5 &= f_{35}+f_{36}; \quad F_6 = f_{37}.
\end{aligned}
$$

Then

$$
\begin{aligned}
r_8 &= 1, \quad r_7 = \frac{1}{\kappa_1}\left(F_6 + \kappa_1 b + \kappa_1^2\right), \quad r_6 = \frac{1}{\kappa_1}\left[F_5 + F_6(\kappa_1+b) + \kappa_1^2 b\right],\\
r_5 &= \frac{1}{\kappa_1}\left[F_4 + F_5(\kappa_1+b) + F_6\kappa_1 b\right], \quad r_4 = \frac{1}{\kappa_1}[F_2 + F_4(\kappa_1+b) + F_5\kappa_1 b],\\
r_3 &= \frac{1}{\kappa_1}\left[F_3 + F_2(\kappa_1+b) + F_4\kappa_1 b\right], \quad r_2 = \frac{1}{\kappa_1}\left[F_1 + F_3(\kappa_1+b) + F_2\kappa_1 b\right],\\
r_1 &= \frac{1}{\kappa_1}\left[F_1(\kappa_1+b) + F_3\kappa_1 b\right], \quad r_0 = F_1 b.
\end{aligned}
$$

We now need to verify that all the coefficients r_0, r_1, r_2, r_3, r_4, r_5 and r_6 are positive. For this, it is sufficient to prove that F_1, F_2, F_3 , F_4, F_5 and F_6 are positive. We can clearly see that all the F_j, $j = 1$ to 6, are positive when $R_0 < 1$. Therefore, all the r_j, $j = 1$ to 6, are positive when $R_0 < 1$. Therefore, from Routh Hurwitz Criterion, E_0 is locally asymptotically stable if $R_0 < 1$.

6. Global Stability

In Theorem 2, we have proved that the disease-free equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$. We will now prove that, independently of the initial population size, if $R_0 < 1$, then the disease will die out. Let us define:

$$x_1 = \frac{b}{\kappa_1} - s_1 \quad \text{and} \quad x_2 = \frac{\gamma}{\kappa_2} - s_2.$$

The system (5) becomes:

$$\begin{cases} \dfrac{\mathrm{d}\,\mathrm{x}_1}{\mathrm{dt}} = \left(\dfrac{\mathrm{b}}{\kappa_1} - \mathrm{x}_1\right)\left(\beta_1 \displaystyle\sum_{j=1}^{3} \mathrm{i}_1^j + \beta_2 \sum_{j=1}^{3} \mathrm{i}_2^j\right) - \kappa_1 \mathrm{x}_1 \\ \dfrac{\mathrm{d}\,\mathrm{i}_1^1}{\mathrm{dt}} = \left(\dfrac{\mathrm{b}}{\kappa_1} - \mathrm{x}_1\right)\left(\beta_1 \displaystyle\sum_{j=1}^{3} \mathrm{i}_1^j + \beta_2 \sum_{j=1}^{3} \mathrm{i}_2^j\right) - \kappa_2 \mathrm{i}_1^1 \\ \dfrac{\mathrm{d}\,\mathrm{i}_1^2}{\mathrm{dt}} = \mathrm{a}_1 \mathrm{i}_1^1 - \kappa_3 \mathrm{i}_1^2 \\ \dfrac{\mathrm{d}\,\mathrm{i}_1^3}{\mathrm{dt}} = \mathrm{a}_2 \mathrm{i}_1^2 - \kappa_4 \mathrm{i}_1^3 \\ \dfrac{\mathrm{d}\,\mathrm{x}_2}{\mathrm{dt}} = -\gamma\left(\dfrac{\mathrm{b}}{\kappa_1} - \mathrm{x}_1\right) + \left(\dfrac{\gamma}{\kappa_1} - \mathrm{x}_2\right)\left(\beta_1 \displaystyle\sum_{j=1}^{3} \mathrm{i}_1^j + \beta_2 \sum_{j=1}^{3} \mathrm{i}_2^j\right) - \mathrm{b}\mathrm{x}_2 \\ \dfrac{\mathrm{d}\,\mathrm{i}_2^1}{\mathrm{dt}} = \left(\dfrac{\gamma}{\kappa_1} - \mathrm{x}_2\right)\left(\beta_1 \displaystyle\sum_{j=1}^{3} \mathrm{i}_1^j + \beta_2 \sum_{j=1}^{3} \mathrm{i}_2^j\right) - \kappa_5 \mathrm{i}_2^1 + \mathrm{c}_1 \mathrm{i}_1^1 \\ \dfrac{\mathrm{d}\,\mathrm{i}_2^2}{\mathrm{dt}} = \mathrm{a}_3 \mathrm{i}_2^1 - \kappa_6 \mathrm{i}_2^2 + \mathrm{c}_2 \mathrm{i}_1^2 \\ \dfrac{\mathrm{d}\,\mathrm{i}_2^3}{\mathrm{dt}} = \mathrm{a}_4 \mathrm{i}_2^2 - \mathrm{b}\ \mathrm{i}_2^3 + \mathrm{c}_3 \mathrm{i}_1^3 \end{cases} \tag{20}$$

$\mathrm{E}_0\left(\dfrac{\mathrm{b}}{\kappa_1}, 0, 0, 0, \dfrac{\gamma}{\kappa_1}, 0, 0, 0\right)$ is globally asymptotically stable for system (5) if and only if $\mathrm{E}_0\,(0,0,0,0,0,0,0,0)$ is globally asymptotically stable for system (20) .

Theorem 3.

If $\mathrm{R}_0 < 1$, then the disease-free equilibrium point $\mathrm{E}_0\,(0,0,0,0,0,0,0,0)$ is globally asymptotically stable for the system (20).

Proof. Consider the following function $\mathrm{V} : \Gamma \longrightarrow \mathbb{R}_+$ defined by:

$$V = \kappa_2^{-1}\left[(\mathrm{M}_7 - \gamma \mathrm{M}_4)\,\mathrm{i}_1^1 + \mathrm{b}\left(\mathrm{M}_2 \mathrm{i}_1^2 + \mathrm{M}_4 \mathrm{i}_2^1 + \mathrm{M}_5 \mathrm{i}_2^2 + (\mathrm{M}_3 + \mathrm{M}_6)\left(\sum_{j=1}^{2} \mathrm{x}_j - \sum_{j=1}^{3} \mathrm{i}_1^j - \sum_{j=1}^{2} \mathrm{i}_2^j\right)\right)\right]$$

If $R_0 < 1$, then $M_7 - \gamma M_4$, M_2, M_3, M_4, M_5 and M_6 are positive. Consequently, the function V is positive and vanishes at the disease-free equilibrium. The derivative of this Lyapunov function V along the trajectories of the ordinary differential system is

$$\begin{aligned}
\dot{V} &= \kappa_2^{-1}\left(M_7 - \kappa\gamma M_4\right)\left[\left(b\kappa_1^{-1} - x_1\right)\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) - \kappa_2 i_1^1\right] + b\kappa_2^{-1} M_2\left(a_1 i_1^1 - \kappa_3 i_1^2\right) \\
&+ b\kappa_2^{-1} M_3\left(a_2 i_1^2 - \kappa_4 i_1^3\right) + b\kappa_2^{-1} M_4\left[\left(\gamma\kappa_1^{-1} - x_2\right)\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) - \kappa_5 i_2^1 + c_1 i_1^1\right] \\
&+ b\kappa_2^{-1} M_5\left(a_3 i i_2^1 - \kappa_6 i_2^2 + c_2 i_1^2\right) + b\kappa_2^{-1} M_6\left(a_4 i_2^2 - b i_2^3 + c_3 i_1^3\right).
\end{aligned}$$

We can also write

$$\begin{aligned}
\dot{V} &= \kappa_2^{-1}\left(M_7 - \gamma M_4\right)\frac{b}{\kappa_1}\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) - \kappa_2^{-1}\left(M_7 - \gamma M_4\right)\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) x_1 \\
&\quad - \left(M_7 - \gamma M_4\right) i_1^1 + b\kappa_2^{-1} M_2 a_1 i_1^1 - b\kappa_2^{-1} M_2 \kappa_3 i_1^2 + b\kappa_2^{-1} M_3 a_2 i_1^2 - b\kappa_2^{-1} M_3 \kappa_4 i_1^3 - b\kappa_2^{-1} M_4 \kappa_5 i_2^1 \\
&\quad + b\kappa_2^{-1}\gamma M_4 \kappa_1{}^{-1}\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) - b\kappa_2^{-1} M_4\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) x_2 + b\kappa_2^{-1} M_4 c_1 i_1^1 \\
&\quad + b\kappa_2^{-1} M_5 a_3 i_2^1 - b\kappa_2^{-1} M_5 \kappa_6 i_2^2 + b\kappa_2^{-1} M_5 c_2 i_1^2 + b\kappa_2^{-1} M_6 a_4 i_2^2 - b\kappa_2^{-1} M_6 b i_2^3 + b\kappa_2^{-1} M_6 \gamma_3 i_2^3.
\end{aligned}$$

Following algebraic manipulations, it holds:

$$\begin{aligned}
\dot{V} &= -\kappa_2^{-1}\left(M_7 - \gamma M_4\right)\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) x_1 - b\kappa_2^{-1} M_4\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) x_2 \\
&\quad + \left(-M_7 + \gamma M_4 + b\kappa_2^{-1} M_2 a_1 + b\kappa_2^{-1} M_4 c_1 + \beta_1 b\kappa_1^{-1}\kappa_2^{-1} M_7\right) i_1^1
\end{aligned}$$

or

$$\dot{V} = -\kappa_2^{-1}\left(M_7 - \gamma M_4\right)\left(\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j\right) x_1 - b\kappa_2^{-1} M_4\left(k_h + k_f\right) x_2 - \left(M_7 - \gamma M_4 - b M_1\right) i_1^1.$$

If $R_0 < 1$, then $M_7 - \kappa\theta M_4$ and $M_7 - \kappa\theta M_4 - bM_1$ are positive, consequently, $\dot{V}$ is negative definite along the trajectories of the system (5). Therefore, the DFE E_0 is globally asymptotically stable for the system (5) if $R_0 < 1$.

This ends the proof of Theorem 3.

7. Existence of Endemic Equilibrium

In this section, we analyze the existence of non-trivial endemic equilibrium $E^*\left(s_1^*, i_1^{1*}, i_1^{2*}, i_1^{3*}, s_2^*, i_2^{1*}, i_2^{2*}, i_2^{3*}\right)$ of the system (5).

Theorem 4.

If $R_0 > 1$, then there exists an endemic equilibrium point E^* for the system (5).

Proof. Solving the equations of system (5) at equilibrium state, we obtain:

$$s_1 = \frac{\kappa_2 i_1^1}{\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j} \tag{21a}$$

$$s_2 = \frac{\kappa_5 i_2^1 - c_1 i_1^1}{\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j} \tag{21b}$$

$$i_1^2 = \frac{a_1}{\kappa_3} i_1^1 \tag{21c}$$

$$i_1^3 = \frac{a_1 a_2}{\kappa_3 \kappa_4} i_1^1 \tag{21d}$$

$$i_2^2 = \frac{a_3}{\kappa_6} i_2^1 + \frac{a_1 c_2}{\kappa_6 \kappa_3} i_1^1 \tag{21e}$$

$$i_2^3 = \frac{a_3 a_4}{b\kappa_6} i_2^1 + \left(\frac{a_4 a_1 c_2}{b\kappa_6 \kappa_3} + \frac{a_1 a_2 c_3}{b\kappa_4 \kappa_3} \right) i_1^1 \tag{21f}$$

$$i_2^1 = -i_1^1 \frac{f_{38} + f_{39} i_1^1}{\beta_2 \kappa_3 \kappa_4 \left(-f_{40} + f_{41} i_1^1 \right)} \tag{21g}$$

$$\beta_1 \sum_{j=1}^{3} i_1^j + \beta_2 \sum_{j=1}^{3} i_2^j = \frac{b - \kappa_1 s_1}{s_1} \tag{21h}$$

where

$$\begin{aligned}
f_{38} &= -\beta_1 \kappa_6 b^2 \kappa_4 \kappa_3 - \beta_1 \kappa_6 b^2 a_1 \kappa_4 - \beta_1 \kappa_6 b^2 a_1 a_2 - \beta_2 a_1 c_2 b^2 \kappa_4 - b\beta_2 a_1 a_4 c_2 \kappa_4 - b\beta_2 a_1 a_2 c_3 \kappa_6 \\
&\quad + \kappa_1 \kappa_2 \kappa_6 \kappa_3 b \kappa_4 \\
f_{39} &= \kappa_2 \beta_1 \kappa_6 b \kappa_4 \kappa_3 + \kappa_2 \beta_1 \kappa_6 b a_1 \kappa_4 + \kappa_2 \beta_1 \kappa_6 b a_1 a_2 + \kappa_2 \beta_2 a_1 c_2 b \kappa_4 + \kappa_2 \beta_2 a_1 a_4 c_2 \kappa_4 + \kappa_2 \beta_2 a_1 a_2 c_3 \kappa_6 \\
f_{40} &= b^2 \kappa_6 + a_3 b^2 + b a_3 a_4 \\
f_{41} &= \kappa_2 b \kappa_6 + \kappa_2 a_3 b + \kappa_2 a_3 a_4.
\end{aligned}$$

From (21b)-(21g), let

$$f_{44} \left(i_1^1 \right)^2 + f_{43} i_1^1 + f_{42} = 0 \tag{22}$$

where

$$\begin{aligned}
f_{42} &= b^2 \left(1 - R_0 \right) b \kappa_1 \kappa_2 \kappa_3 \kappa_4 \kappa_5 \kappa_6 \\
f_{43} &= \kappa_2 b \Big(2b^2 \kappa_6 \kappa_5 \beta_1 a_1 \kappa_4 + 2b^2 \kappa_6 \kappa_5 \beta_1 a_1 a_2 + 2b^2 \kappa_6 \kappa_5 \beta_1 \kappa_4 \kappa_3 + 2b^2 \kappa_6 \beta_2 \kappa_3 \kappa_4 c_1 + 2b^2 \kappa_5 \beta_2 a_1 c_2 \kappa_4 \\
&\quad + 2b^2 \beta_2 \kappa_3 \kappa_4 c_1 a_3 + 2b \beta_2 \kappa_3 \kappa_4 c_1 a_3 a_4 + 2b \kappa_6 \kappa_5 \beta_2 a_1 a_2 c_3 - b \kappa_6 c_1 \beta_2 \kappa_3 \kappa_4 \kappa_1 - b c_1 \beta_2 \kappa_3 \kappa_4 \kappa_1 a_3
\end{aligned}$$

$$-\mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\kappa_4\kappa_3 + 2\mathrm{b}\kappa_2\beta_2\kappa_3\kappa_4\gamma\mathrm{a}_3 - \mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\mathrm{a}_1\kappa_4 - \mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\mathrm{a}_1\mathrm{a}_2 - \mathrm{b}\kappa_2\kappa_6\kappa_5\kappa_1\kappa_3\kappa_4$$
$$+2\mathrm{b}\kappa_5\beta_2\mathrm{a}_1\mathrm{a}_4\mathrm{c}_2\kappa_4 - \mathrm{b}\kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{c}_2\kappa_4 + 2\mathrm{b}\kappa_2\kappa_6\beta_2\kappa_3\kappa_4\gamma - \kappa_6\kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{a}_2\mathrm{c}_3 + 2\kappa_2\beta_2\kappa_3\kappa_4\gamma\mathrm{a}_3\mathrm{a}_4$$
$$+\kappa_2\kappa_6\kappa_5\kappa_1^2\kappa_3\kappa_4 - \mathrm{c}_1\beta_2\kappa_3\kappa_4\kappa_1\mathrm{a}_3\mathrm{a}_4 - \kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{a}_4\mathrm{c}_2\kappa_4\Big)$$

$$\begin{aligned}
\mathrm{f}_{44} = \; & \kappa_2^2\Big(- \kappa_2\beta_2\kappa_3\kappa_4\gamma\mathrm{a}_3\mathrm{a}_4 - \mathrm{b}\kappa_2\kappa_6\beta_2\kappa_3\kappa_4\gamma - b\kappa_2\beta_2\kappa_3\kappa_4\gamma\mathrm{a}_3 + \mathrm{b}\kappa_6\mathrm{c}_1\beta_2\kappa_3\kappa_4\kappa_1 + \mathrm{b}\mathrm{c}_1\beta_2\kappa_3\kappa_4\kappa_1\mathrm{a}_3 \\
& +\mathrm{c}_1\beta_2\kappa_3\kappa_4\kappa_1 a_3\mathrm{a}_4 - \mathrm{b}^2\kappa_6\kappa_5\beta_1\kappa_4\kappa_3 - \mathrm{b}^2\kappa_6\kappa_5\beta_1\mathrm{a}_1\mathrm{a}_2 + \mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\mathrm{a}_1\kappa_4 + \kappa_6\kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{a}_2\mathrm{c}_3 \\
& -\mathrm{b}^2\kappa_6\kappa_5\beta_1\mathrm{a}_1\kappa_4 + \mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\mathrm{a}_1\mathrm{a}_2 + \kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{a}_4\mathrm{c}_2\kappa_4 - \mathrm{b}^2\beta_2\kappa_3\kappa_4\mathrm{c}_1\mathrm{a}_3 + \mathrm{b}\kappa_6\kappa_5\kappa_1\beta_1\kappa_4\kappa_3 \\
& -b\beta_2\kappa_3\kappa_4\mathrm{c}_1\mathrm{a}_3\mathrm{a}_4 - \mathrm{b}^2\kappa_6\beta_2\kappa_3\kappa_4\mathrm{c}_1 - \mathrm{b}^2\kappa_5\beta_2\mathrm{a}_1\mathrm{c}_2\kappa_4 - \mathrm{b}\kappa_6\kappa_5\beta_2\mathrm{a}_1\mathrm{a}_2\mathrm{c}_3 - \mathrm{b}\kappa_5\beta_2\mathrm{a}_1\mathrm{a}_4\mathrm{c}_2\kappa_4 \\
& +\mathrm{b}\kappa_5\kappa_1\beta_2\mathrm{a}_1\mathrm{c}_2\kappa_4\Big).
\end{aligned}$$

By replacing κ_1, κ_2, κ_3, κ_4, κ_5 and κ_6 by their expressions into f_{44}, we get:

$$\mathrm{f}_{44} = (\mathrm{a}_1 + \mathrm{c}_1 + \mathrm{b})^2\,\mathrm{b}\gamma\,(-\beta_2 + \beta_1)\,(\mathrm{b} + \mathrm{a}_4)\,(\mathrm{b} + \mathrm{a}_3)\Big(\mathrm{b}^2 + \mathrm{b}\mathrm{a}_2 + \mathrm{b}\mathrm{c}_2 + \mathrm{b}\mathrm{c}_3 + \mathrm{a}_1\mathrm{b} + \mathrm{a}_1\mathrm{c}_3 + \mathrm{a}_1\mathrm{a}_2 + \mathrm{c}_2\mathrm{c}_3 + \mathrm{a}_2\mathrm{c}_3 \Big).$$

If $\mathrm{R}_0 > 1$, we clearly see that $\mathrm{f}_{42} < 0$. We also note that:

(i) If $\beta_1 > \beta_2$, then $\mathrm{f}_{44} > 0$. Consequently, the discriminant of (22) is positive and the product of the solutions is negative. So, there exists a positive solution i_1^1 for (22).

(ii) If $\beta_1 < \beta_2$, then $\mathrm{f}_{44} < 0$ and $\mathrm{f}_{43} > 0$. Consequently, the discriminant of (22) is positive, the sum and the product of the solutions of (22) are positive. Therefore, there exist two distinct solutions.

(iii) If $\beta_1 = \beta_2$, then $\mathrm{f}_{44} = 0$ and $\mathrm{f}_{43} > 0$. So, (22) becomes:

$$\mathrm{f}_{43}\mathrm{i}_1^1 + \mathrm{f}_{42} = 0. \tag{23}$$

Therefore, there exists a unique positive solution i_1^1.

In all the cases discussed above, there exists at least a positive solution i_1^1 of (22). Let us denote by $\mathrm{i}_1^{1^*}$ this positive solution. We will now define and prove the positivity of $\mathrm{i}_2^{1^*}$, $\mathrm{i}_1^{3^*}$, $\mathrm{i}_2^{2^*}$, $\mathrm{i}_2^{3^*}$, $\mathrm{s}_1{}^*$ and $\mathrm{s}_2{}^*$ if $\mathrm{R}_0 > 1$.

From equation (21g), we get:

$$\mathrm{i}_2^{1^*} = -\mathrm{i}_1^{1^*}\frac{\mathrm{f}_{38} + \mathrm{f}_{39}\mathrm{i}_1^{1^*}}{\beta_2\kappa_3\kappa_4\left(-\mathrm{f}_{40} + \mathrm{f}_{41}\mathrm{i}_1^{1^*}\right)}$$

If $\mathrm{R}_0 > 1$, then $\mathrm{f}_{38} > 0$. From (21a) and (21h), it follows that $-\mathrm{f}_{40} + \mathrm{f}_{41}\mathrm{i}_1^{1^*} < 0$, hence $\mathrm{i}_2^{1^*} > 0$.

From equation (21c), it follows that $\mathrm{i}_1^{2^*} = \dfrac{a_1}{\kappa_3}\mathrm{i}_1^{1^*}$, hence $\mathrm{i}_1^{2^*} > 0$.

From equation (21d), it follows that, $i_1^{3*} = \dfrac{a_1 a_2}{\kappa_3 \kappa_4} i_1^{1*}$, hence $i_1^{3*} > 0$.

From equation (21e), it follows that, $i_2^{2*} = \dfrac{a_3}{\kappa_6} i_2^{1*} + \dfrac{a_1 c_2}{\kappa_6 \kappa_3} i_1^{1*}$, hence $i_2^{2*} > 0$.

From equation (21f), it follows that, $i_2^{3*} = \dfrac{a_3 a_4}{b\kappa_6} i_2^{1*} + \left(\dfrac{a_4 a_1 c_2}{b\kappa_6 \kappa_3} + \dfrac{a_1 a_2 c_3}{b\kappa_4 \kappa_3} \right) i_1^{1*}$, hence $i_2^{3*} > 0$.

From equation (21a), it follows that, $s_1^* = \dfrac{\kappa_2 i_1^{1*}}{\beta_1 \sum_{j=1}^{3} i_1^{j*} + \beta_2 \sum_{j=1}^{3} i_2^{j*}}$, hence $s_1^* > 0$.

From equation (21b), it follows that, $s_2^* = \dfrac{\kappa_5 i_2^{1*} - c_1 i_1^{1*}}{\beta_1 \sum_{j=1}^{3} i_1^{j*} + \beta_2 \sum_{j=1}^{3} i_2^{j*}}$, hence $s_2^* > 0$.

Therefore, if $R_0 > 1$, there exists a positive solution $E^* \left(s_1^*, i_1^{1*}, i_1^{2*}, i_1^{3*}, s_2^*, i_2^{1*}, i_2^{2*}, i_2^{3*} \right)$ for the system (5).

8. Stability Analysis of Endemic Equilibrium

Theorem 5.
If $R_0 > 1$, then the endemic equilibrium point

$$E^* \left(s_1^*, i_1^{1*}, i_1^{2*}, i_1^{3*}, s_2^*, i_2^{1*}, i_2^{2*}, i_2^{3*} \right)$$

is locally asymptotically stable.

Proof. Let us compute the jacobian of the system at the point $E^* \left(s_1^*, i_1^{1*}, i_1^{2*}, i_1^{3*}, s_2^*, i_2^{1*}, i_2^{2*}, i_2^{3*} \right)$.

$$J(E^*) = \begin{pmatrix} -H^* - \kappa_1 & -\beta_1 s_1^* & -\beta_1 s_1^* & -\beta_1 s_1^* & 0 & -\beta_2 s_1^* & -\beta_2 s_1^* & -\beta_2 s_1^* \\ H^* & \beta_1 s_1^* - \kappa_2 & \beta_1 s_1^* & \beta_1 s_1^* & 0 & \beta_2 s_1^* & \beta_2 s_1^* & \beta_2 s_1^* \\ 0 & a_1 & -\kappa_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & -\kappa_4 & 0 & 0 & 0 & 0 \\ \gamma & -\beta_1 s_2^* & -\beta_1 s_2^* & -\beta_1 s_2^* & -H^* - b & -\beta_2 s_2^* & -\beta_2 s_2^* & -\beta_2 s_2^* \\ 0 & \beta_1 s_2^* + c_1 & \beta_1 s_2^* & H^* & \beta_2 s_2^* - \kappa_5 & \beta_2 s_2^* & \beta_2 s_2^* & \\ 0 & 0 & c_2 & 0 & 0 & a_3 & -\kappa_6 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & a_4 & -b \end{pmatrix}$$

where $H^* = \beta_1 \sum_{j=1}^{3} {i_1^j}^* + \beta_2 \sum_{j=1}^{3} {i_2^j}^*$.

The characteristic equation of $J(E^*)$ is given by:

$$A_8X^8 + A_7X^7 + A_6X^6 + A_5X^5 + A_4X^4 + A_3X^2 + A_2X^2 + A_1X + A_0 = 0 \qquad (24)$$

where $A_0,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8$ are obtained as a result of a boring calculation. They are all positive. Applying the Routh-Hurwitz criterion [7], it follows that all eigenvalues of the characteristic equation (24) have negative real part if $R_0 > 1$. Therefore, the endemic solution E^* is locally asymptotically stable if $R_0 > 1$.

9. Numerical Simulations and Discussion

In this section, we perform numerical simulations to support the theoretical results from the mathematical analysis of model (5). In addition to the verification of the theoretical results, these numerical solutions are very important from a practical point of view.

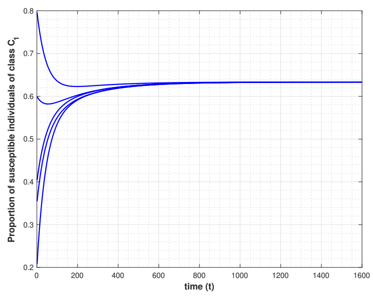
We first consider the case where $R_0 = 0.67893 < 1$ using the parameter values reported in Table 4.

Using different initial conditions, the dynamics of the susceptible and infected populations of the model are plotted in Figures 2, 3, 4 and 5.

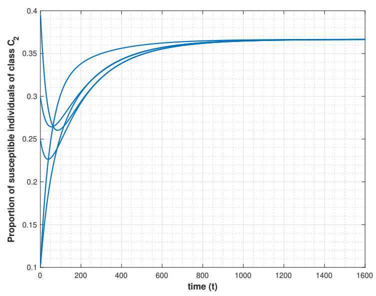
In Figure 2, we can observe that the proportions of susceptible individuals in classes C_1 and C_2 are consistent, ($s_1 = 0.63343,\ s_2 = 0.36657$). In contrast, as shown in Figures 3, 4 and 5, the proportions of infected individuals in classes C_1 and C_2 decline to zero $\left(i_1^1 = 0,\ i_1^2 = 0,\ i_1^3 = 0,\ i_2^1 = 0,\ i_2^2 = 0,\ i_2^3 = 0\right)$, i.e. approach the disease-free equilibrium

Table 4: Parameter values

Parameter	Value	Source	Parameter	Value	Source
b	0.0432	Estimated	a_2	0.01	Estimated
m	0.0096	Estimated	a_3	0.01	Estimated
γ	0.025	Estimated	a_4	0.01	Estimated
β_1	0.035	Estimated	c_1	0.02	Estimated
β_2	0.025	Estimated	c_2	0.02	Estimated
a_1	0.01	Estimated	c_3	0.02	Estimated

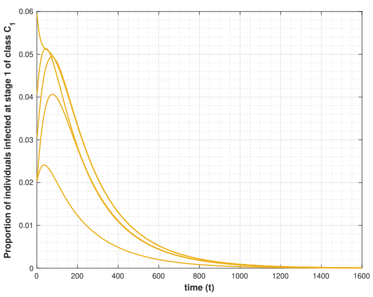


(a) Proportion of susceptible individuals in C_1

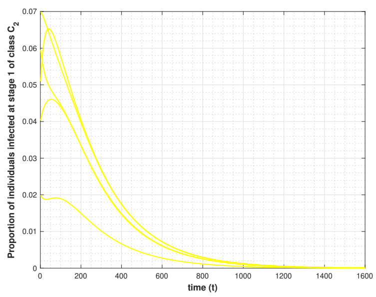


(b) Proportion of susceptible individuals in C_2

Figure 2: Time series plots of the proportions of susceptible individuals in classes C_1 and C_2 for $R_0 = 0.67893 < 1$ using various initial conditions and parameter values reported in Table 4.



(a) Proportion of individuals infected at stage 1 in C_1

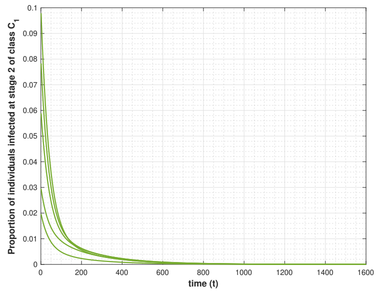


(b) Proportion of individuals infected at stage 1 in C_2

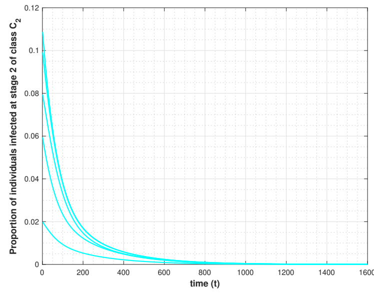
Figure 3: Time series plots of the proportions of infected individuals at stage 1 in classes C_1 and C_2 for $R_0 = 0.67893 < 1$ using various initial conditions and parameter values reported in Table 4.

(DFE). They show that DFE is locally asymptotically stable when $R_0 < 1$. These numerical simulations support the result stated in Theorem 2 on the stability of DFE.

Further using the parameter values given in Table 5, we consider the case when $R_0 =$

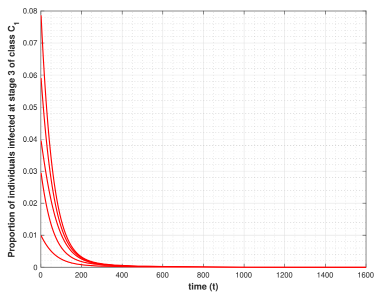


(a) Proportion of individuals infected at stage 2 in C_1

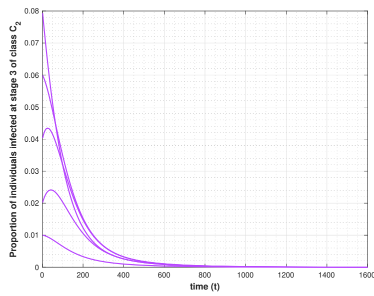


(b) Proportion of individuals infected at stage 2 in C_2

Figure 4: Time series plots of the proportions of infected individuals at stage 2 in classes C_1 and C_2 for $R_0 = 0.67893 < 1$ using various initial conditions and parameter values reported in Table 4.



(a) Proportion of individuals infected at stage 3 in C_1



(b) Proportion of individuals infected at stage 3 in C_2

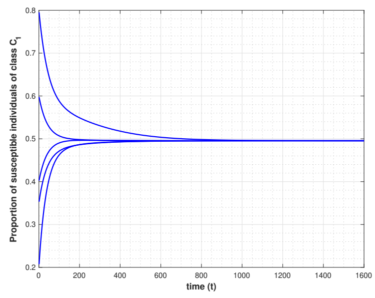
Figure 5: Time series plots of the proportions of infected individuals at stage 3 in classes C_1 and C_2 for $R_0 = 0.67893 < 1$ using various initial conditions and parameter values reported in Table 4.

$1.408 > 1$.

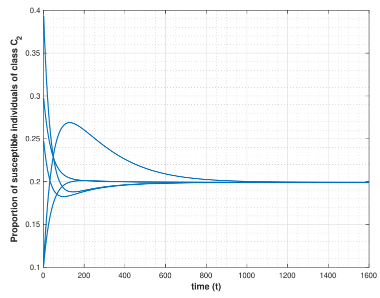
Table 5: Parameter values

Parameter	Value	Source	Parameter	Value	Source
b	0.0432	Estimated	a_2	0.01	Estimated
m	0.0096	Estimated	a_3	0.01	Estimated
γ	0.025	Estimated	a_4	0.01	Estimated
β_1	0.075	Estimated	c_1	0.02	Estimated
β_2	0.05	Estimated	c_2	0.02	Estimated
a_1	0.01	Estimated	c_3	0.02	Estimated

Using different initial conditions, the dynamics of the susceptible and infected populations of the model are plotted in Figures 6, 7, 8 and 9.

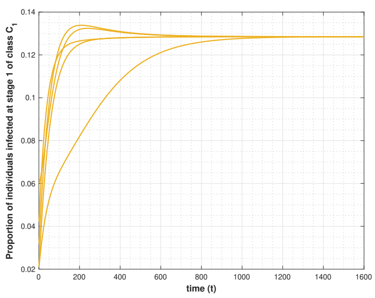


(a) Proportion of susceptible individuals in C_1

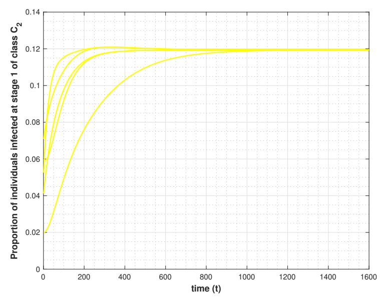


(b) Proportion of susceptible individuals in C_2

Figure 6: Time series plot of the proportions of susceptible individuals in classes C_1 and C_2 for $R_0 = 1.408 > 1$ using various initial conditions and parameter values reported in Table 5.



(a) Proportion of individuals infected at stage 1 in C_1



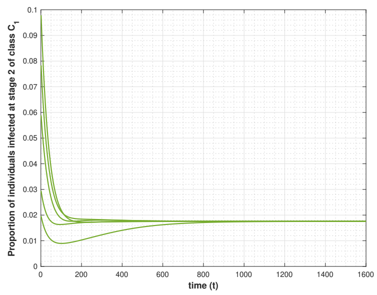
(b) Proportion of individuals infected at stage 1 in C_2

Figure 7: Time series plot of the proportions of individuals infected at stage 1 in classes C_1 and C_2 for $R_0 = 1.408 > 1$ using various initial conditions and parameter values reported in Table 5.

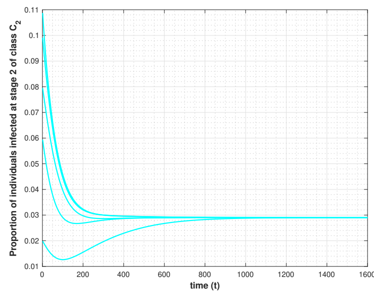
As shown in Figures 6, 7, 8 and 9, the proportions of susceptible and infected individuals in classes C_1 and C_2 are consistent, $\Big[\big(s_1{}^*,\ i_1^{1*},\ i_1^{2*},\ i_1^{3*},\ s_2{}^*,\ i_2^{1*},\ i_2^{2},\ i_2^{3*}\ \big) = \big(0.49548,\ 0.12853,\ 0.017559,\ 0.0027783,\ 0.19918,\ 0.11941,\ 0.029047,\ 0.0080157\ \big)\Big]$, i.e., the population tends to endemic equilibrium E^* when $R_0 > 1$. This indicates that, regardless of initial conditions, the infected population eventually reaches endemic equilibrium over time and the disease-free equilibrium point becomes unstable when $R_0 > 1$. These numerical simulations support our theoretical results.

10. Conclusions

In this paper, we have developed a mathematical model for the spread of HIV disease within two different age classes. We proposed a basic reproduction number that depends

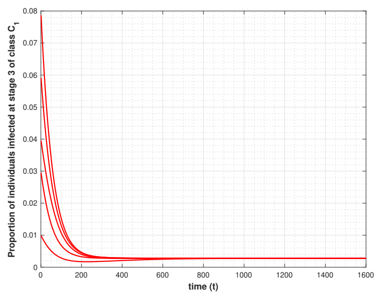


(a) Proportion of individuals infected at stage 2 in C_1

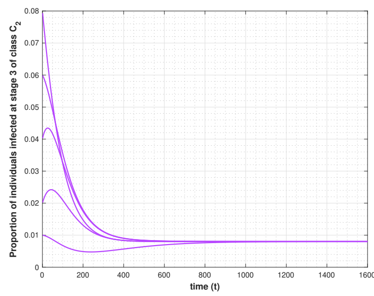


(b) Proportion of individuals infected at stage 2 in C_2

Figure 8: Time series plot of the proportions of individuals infected at stage 2 in classes C_1 and C_2 for $R_0 = 1.408 > 1$ using various initial conditions and parameter values reported in Table 5.



(a) Proportion of individuals infected at stage 3 in C_1



(b) Proportion of individuals infected at stage 3 in C_2

Figure 9: Time series plot of the proportions of individuals infected at stage 3 in classes C_1 and C_2 for $R_0 = 1.408 > 1$ using various initial conditions and parameter values reported in Table 5.

on the characteristics of the two age classes. We have proved that if the Routh-Hurwitz criterion are satisfied, then the disease-free equilibrium (DFE) E_0 is locally asymptotically stable. We constructed a Lyapunov function to prove that the disease-free equilibrium (DFE) E_0 is globally stable when $R_0 < 1$. For $R_0 > 1$, we obtain from mathematical analysis a quadratic equation in i_1^1. It has been proven that the existence of an endemic equilibrium depends on the existence of at least one real positive value for i_1^1. The stability analysis of endemic equilibrium produces that if the Routh-Hurwitz criterion are satisfied, then the endemic equilibrium E^* is locally asymptotically stable. The important mathematical results in this paper were all corroborated by numerical simulations performed using MATLAB. Indeed, we verified through numerical experiments that the disease-free equilibrium E_0 is stable when $R_0 < 1$. On the other hand, we numerically verified that, if $R_0 > 1$, then the endemic equilibrium E^* becomes stable.

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