EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1750-1759
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Analytical solution of the Ginzburg-Landau equation

Yanick Alain Servais Wellot[1,*], Gires Dimitri Nkaya[2]

[1] *Mathematics, Ecole Normale Supérieure, Université Marien Ngouabi, Brazzaville, Congo*
[2] *Mathematics, Faculté des Sciences et Technique, Université Marien Ngouabi, Brazzaville, Congo*

Abstract. In this paper, we will construct the solution of the Landau-Ginzburg equation by the Adomian decomposition method. This method avoids linearization of space and discretization of time, it often gives a good approximation of the exact solution.

2020 Mathematics Subject Classifications: 47H14, 34G20, 47J25, 65J15

Key Words and Phrases: Adomian method, Ginzburg-Landau equation.

1. Introduction

The Landau equation, also called the Fokker-Planck-Landau equation, is a nonlinear partial differential equation that describes the motion of particles in a plasma. This equation was obtained by Lev Davidovitch Landau in 1936 from the Bolzmann equation [3, 4].

In 1950, Ginzburg and Landau proposed an extension of the Landau energy to describe the superconductor in the presence of a magnetic field. In this article, we are interested in the difference in these two models try to describe the field applied to systems of partial differential equations with initial conditions. Then we apply the Adomian Decomposition Method [1, 2], an analytical method that allows us to obtain an exact solution when it exists without space linearization and time discretization. However, the existence and uniqueness of solution of the Ginzburg-Landau equation has been proved in [5–7].

In the mathematical and physical context, explanations are being made available to the research community.

During the last two decades, the mathematics of superconductivity has been the subject of intense activity [8, 10, 13, 14].

The Ginzburg-Landau function is a commonly used model to describe the behavior of a superconductor involving, a wave function (called order parameter) and a vector field (called magnetic potential), both defined on an open set [9].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4551

Email addresses: yanick.wellot@umng.cg (Y.A.S. Wellot), mail:giresnkaya@gmail.com (G. D. Nkaya)

2. Proposition

On the basis of a suitable choice of physical data the model can be written in a mathematical way in the form [11]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (1+i\alpha)\dfrac{\partial^2 u(x,t)}{\partial x^2} + \gamma u(x,t) - (1+ib)|u(x,t)|^{2n}u(x,t) - (1-4i)|u(x,t)|^{4n}u(x,t) \\ n \geqslant 1, \quad i^2 = -1, \quad \alpha, \gamma, b \quad \text{are real} \\ u(x,0) = f(x). \end{cases} \tag{1}$$

3. About Adomian Decomposition Method

The Adomian Decomposition Method allows to solve functional problems of different types: algebraic equations, differential, integral, integro-differential and partial differential equations. It was introduced in the 1980s by Professor , **GEORGES ADOMIAN** (1922 - 1996).
It is a decomposition method which consists in seeking the solution in the form of a convergent series .
It has the advantage of not linearizing and discretizing the equations. This method allows to determine the exact solution of the problem if it exists or to give an approximation of the solution of the problem by preserving the physical properties of the studied phenomenon.

Let us solve the functional equation :

$$Au = h, \tag{2}$$

where A is an operator of a real Hilbert space H, $u \in H$ an unknown function and $h \in H$ a known function.

The principle of the Adomian Decomposition Method consists in decomposing the operator A into two parts : one linear and the other non-linear:

$$A = \underbrace{L+R}_{\text{linear}} + \underbrace{N}_{\text{nonlinear}}, \tag{3}$$

where L is assumed to be invertible in the Adomian sense. A in (2) leads to

$$Lu + Ru + Nu = h ; \tag{4}$$

By applying L^{-1} to (4), we obtain the **canonical form of Adomian**:

$$u = \theta + L^{-1}h - L^{-1}Ru - L^{-1}Nu, \tag{5}$$

where θ verifies the relationship $L\theta = 0$ (θ is the integration constant if L is a differential operator).

As for the **Adomian algorithm**, it is obtained by assuming that the solution of (2) has the form of a series

$$u = \sum_{n=0}^{\infty} u_n. \tag{6}$$

The nonlinear operator also has the form of a series

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{7}$$

A_n are Adomian polynomials. That are defined by the formula [5]:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{+\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{8}$$

λ is a real parameter introduced by convenience then we obtain

$$\sum_{n=0}^{\infty} u_n = \theta + L^{-1}h - L^{-1}\left[R\left(\sum_{n=0}^{\infty} u_n\right)\right] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{9}$$

Assuming that the $\sum\limits_{n=0}^{\infty} u_n$ series and $\sum\limits_{n=0}^{\infty} A_n$

$$\begin{cases} u_0 = \theta + L^{-1}(h\ ) \\ u_{n+1} = -L^{-1}\left[R\left(u_n\right)\right] - L^{-1}\left(A_n\right), \forall n \geq 0 \end{cases} \tag{10}$$

Equation (10) allows to compute all u_n recursively, then the analytical solution of (2) is defined by the sum of u_n:

$$u = \sum_{n=0}^{\infty} u_n \tag{11}$$

If

$$\varphi_n = \sum_{n=0}^{n-1} u_n$$

is the truncated series, and if

$$\sum_{n=0}^{\infty} u_n \quad \text{is} \quad \text{convergent, we} \quad \text{have} \ :$$

$$u = \lim_{n \to +\infty} \varphi_n \tag{12}$$

4. Application

4.1. Problem 1

We consider the following initial value problem [11]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - (1+i)\dfrac{\partial^2 u(x,t)}{\partial x^2} + (1+2i)|u(x,t)|^2 u(x,t) - 3u(x,t) + (1-4i)|u(x,t)|^4 u(x,t) = 0 \\ \\ u(x,0) = e^{ix}, \end{cases} \tag{13}$$

with $u(x,t)$ a complex function.
Notation:
$|u|^2 = u\overline{u}; \Rightarrow |u|^2 u = u^2\overline{u}$

$|u|^4 = u^2\overline{u}^2 \Rightarrow |u|^4 u = u^3\overline{u}^2$
The Adomian decomposition method consists to look the solution of $u(x,t)$ in the form:

$$u(x,t) = u(x,0) + (1+i)\int_0^t \frac{\partial^2 u(x,s)}{\partial x^2}ds - (1+2i)\int_0^t |u(x,s)|^2 u(x,s)ds +$$

$$3\int_0^t u(x,s)ds - (1-4i)\int_0^t |u(x,s)|^4 u(x,s)ds \tag{14}$$

Let's look for the solution in the following form:

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) \tag{15}$$

with $N = N_1 + N_2$, and let us note

$$\left.\begin{array}{l} N_1 u = |u(x,t)|^2 u(x,t) \\ N_2 u = |u(x,t)|^4 u(x,t) \end{array}\right\} \tag{16}$$

Suppose that

$$\left.\begin{array}{c} N_1 u = \displaystyle\sum_{n=0}^{\infty} A_n(x,t) \\ \text{and} \\ N_2 u = \displaystyle\sum_{n=0}^{\infty} B_n(x,t) \end{array}\right\} \tag{17}$$

with

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_1\left(\sum_{j=0}^{+\infty}\lambda^j u_j\right)\right]_{\lambda=0}, n = 0,1,2,... \tag{18}$$

and

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_2\left(\sum_{j=0}^{+\infty}\lambda^j u_j\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{19}$$

λ is a real parameter introduced by convenience, we obtain the following Adomian polynomials:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{+\infty}\lambda^i u_i\right)^2 \overline{\left(\sum_{j=0}^{+\infty}\lambda^j u_j\right)}\right]_{\lambda=0} \tag{20}$$

and

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{+\infty}\lambda^i u_i\right)^3 \overline{\left(\sum_{j=0}^{+\infty}\lambda^j u_j\right)^2}\right]_{\lambda=0} \tag{21}$$

This gives

$$A_0 = u_0^2\overline{u_0}. \tag{22}$$

$$A_1 = u_0^2\overline{u_1} + 2u_0u_1\overline{u_0}. \tag{23}$$

$$A_2 = u_0^2\overline{u_2} + 2u_0\overline{u_0}u_2 + 2u_1\overline{u_1}u_0 + u_1^2\overline{u_0}. \tag{24}$$

$$A_3 = u_0^2\overline{u_3} + 2u_0\overline{u_0}u_3 + 2\overline{u_0}u_1u_2 + 2u_0\overline{u_1}u_2 + 2u_0u_1\overline{u_2} + u_1^2\overline{u_1}. \tag{25}$$

and

$$B_0 = u_0^3\overline{u_0}^2. \tag{26}$$

$$B_1 = 3u_1u_0^2\overline{u_0^2} + 2u_0^3\overline{u_1u_0}. \tag{27}$$

$$B_2 = 3u_2u_0^2\overline{u_0^2} + 3u_1^2u_0\overline{u_0^2} + 6u_1u_0^2\overline{u_1u_0} + 2u_0^3\overline{u_2u_0} + u_0^3\overline{u_1^2}. \tag{28}$$

The equation of problem (1), becomes:

$$\sum_{n=0}^{+\infty} u_n(x,t) = u(x,0) + (1+i)\sum_{n=0}^{+\infty}\int_0^t \frac{\partial^2 u_n(x,s)}{\partial x^2}ds - (1+2i)\sum_{n=0}^{+\infty}\int_0^t A_n(x,s)ds +$$

$$3\sum_{n=0}^{+\infty}\int_0^t u_n(x,s)ds - (1-4i)\sum_{n=0}^{+\infty}\int_0^t B_n(x,s)ds. \tag{29}$$

From the above equation (4.1) we get the Adomian canonical form. This leads us to obtain the following Adomian algorithm (30) below:

$$\begin{cases} u_0(x,t) = u(x,0) \\ u_{n+1}(x,t) = (1+i)\int_0^t \frac{\partial^2 u_n(x,s)}{\partial x^2}ds - (1+2i)\int_0^t A_n(x,s)\,ds + \\ \qquad 3\int_0^t u_n(x,s)ds - (1-4i)\int_0^t B_n(x,s)\,ds \end{cases} \tag{30}$$

Either:

$$\begin{cases} u_0(x,t) = e^{ix} \\ u_{n+1}(x,t) = (1+i)\int_0^t \frac{\partial^2 u_n(x,s)}{\partial x^2}ds - (1+2i)\int_0^t A_n(x,s)\,ds + \\ \qquad 3\int_0^t u_n(x,s)ds - (1-4i)\int_0^t B_n(x,s)\,ds \end{cases} \tag{31}$$

$$u_1(x,t) = (1+i)\int_0^t \frac{\partial^2 u_0(x,s)}{\partial x^2}ds - (1+2i)\int_0^t A_0 ds + 3\int_0^t u_0(x,s)ds - (1-4i)\int_0^t B_0 ds \tag{32}$$

With $A_0 = B_0 = e^{ix}$, we can easily calculate $u_1(x,t)$

$$u_1(x,t) = -(1+i)te^{ix} + 3te^{ix} - (1+2i)te^{ix} - (1-4i)te^{ix} = ite^{ix} \tag{33}$$

The same calculation procedure leads us to obtain A_1 as follows

$$A_1 = u_0^2\overline{u_1} + 2u_0\overline{u_0}u_0 = ite^{ix}. \tag{34}$$

Then

$$B_1 = 3u_1u_0^2\overline{u_0^2} + 2u_0^3\overline{u_1u_0} = ite^{ix}. \tag{35}$$

Thus, the simple formula to calculate the expression $u_2(x,t)$ follows from this. Let it be:

$$u_2(x,t) = (1+i)\int_0^t \frac{\partial^2 u_1(x,s)}{\partial x^2}ds - (1+2i)\int_0^t A_1 ds + 3\int_0^t u_1(x,s)ds - (1-4i)\int_0^t B_1 ds \tag{36}$$

$$u_2(x,t) = [-i(1+i) + 3i - (1+2i)(i) - (1-4i)(i)]\frac{t^2}{2}e^{ix} \tag{37}$$

Thus:

$$u_2(x,t) = -\frac{t^2}{2}e^{ix} \tag{38}$$

We proceed in the same way for the expression of $u_3(x,t)$.

$$u_3(x,t) = (1+i)\int_0^t \frac{\partial^2 u_2(x,s)}{\partial x^2}ds - (1+2i)\int_0^t A_2 ds + 3\int_0^t u_2(x,s)ds - (1-4i)\int_0^t B_2 ds \tag{39}$$

With

$$A_2 = u_0^2\overline{u_2} + 2u_0\overline{u_0}u_2 + 2u_1\overline{u_1}u_0 + u_1^2\overline{u_0} = -\frac{t^2}{2}e^{ix}, \tag{40}$$

and

$$B_2 = 3u_2u_0^2\overline{u_0^2} + 3u_1^2u_0\overline{u_0^2} + 6u_1u_0^2\overline{u_1u_0} + 2u_0^3\overline{u_2u_0} + u_0^3\overline{u_1^2} = -\frac{t^2}{2}e^{ix}. \tag{41}$$

Therefore:

$$u_3(x,t) = [1+i+1+2i-3+1-4i]\frac{t^3}{3!}e^{ix} = -i\frac{t^3}{3!}e^{ix}. \tag{42}$$

Thus we have:
$$\begin{cases} u_0(x,t) = e^{ix} \\ u_1(x,t) = ite^{ix} \\ u_2(x,t) = \dfrac{(it)^2}{2!}e^{ix} \\ u_3(x,t) = \dfrac{(it)^3}{3!}e^{ix} \\ \vdots \\ u_n(x,t) = \dfrac{(it)^n}{n!}e^{ix} \end{cases} \tag{43}$$

Hence, the exact solution of the Ginzburg-Landau equation is:

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = \sum_{n=0}^{+\infty} \frac{(it)^n}{n!}e^{ix} = e^{i(x+t)}. \tag{44}$$

4.2. Problem 2

We consider the following initial value problem [12, 15]:

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} - (1+2i)\left[\dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2}\right] + (1+2i)|u(x,y,t)|^2 u(x,y,t) - \gamma u(x,y,t) = 0 \\ u(x,y,0) = e^{i\frac{\pi}{3}(x+y)}, \end{cases} \tag{45}$$

With $u(x,y,t)$ a complex function, $\gamma \in \mathbb{R}$.
According to the Adomian decomposition method, the equation (45)

$$\frac{\partial u(x,y,t)}{\partial t} - (1+2i)\Delta u(x,y,t) + (1+2i)|u(x,y,t)|^2 u(x,y,t) - \gamma u(x,y,t) = 0 \tag{46}$$

$$u(x,y,t) = u(x,y,0) + \int_0^t (1+2i)\Delta u(x,y,s)ds - (1+2i)\int_0^t |u(x,y,s)|^2 u(x,y,s)ds + \gamma \int_0^t u(x,y,s)ds) = 0 \tag{47}$$

Let's put $Nu = |u(x,y,t)|^2 u(x,y,t)$, we have

$$u(x,y,t) = u(x,y,0) + \int_0^t (1+2i)\Delta u(x,y,s)ds - (1+2i)\int_0^t Nu(x,y,s)ds + \gamma \int_0^t u(x,y,s)ds) = 0 \tag{48}$$

Let us look for the solution of (45) in the form of a series and the nonlinear part

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t) \quad \text{and} \qquad Nu(x,y,t)=\sum_{n=0}^{\infty}A_n(x,y,t) \tag{49}$$

with

$$A_0=u_0^2\overline{u_0}. \tag{50}$$

$$A_1=u_0^2\overline{u_1}+2u_0u_1\overline{u_0}. \tag{51}$$

$$A_2=u_0^2\overline{u_2}+2u_0\overline{u_0}u_2+2u_1\overline{u_1}u_0+u_1^2\overline{u_0}. \tag{52}$$

We obtain the following canonical form:

$$\sum_{n=0}^{\infty}u_n(x,y,t)=u(x,y,0)+(1+2i)\int_0^t\Delta u_n(x,y,s)ds-(1+2i)\int_0^t A_n ds+\gamma\int_0^t u_n(x,y,s)ds \tag{53}$$

From (53), for $\gamma=1+\frac{2\pi^2}{9}$ (because γ is a constant), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x,y,t)=e^{i\dfrac{\pi}{3}(x+y)} \\ u_{n+1}(x,y,t)=\displaystyle\int_0^t(1+2i)\Delta u_n(x,y,s)ds-(1+2i)\int_0^t A_n ds+(1+\frac{2\pi^2}{9})\int_0^t u_n(x,y,s)ds \end{cases} \tag{54}$$

For $n=0$,

$$u_1(x,y,t)=\int_0^t(1+2i)\Delta u_0(x,y,s)ds-(1+2i)\int_0^t A_0 ds+(1+\frac{2\pi^2}{9})\int_0^t u_0(x,y,s)ds \tag{55}$$

With $\Delta u_0(x,y,t)=-\frac{2\pi^2}{9}e^{i\dfrac{\pi}{3}(x+y)}$ and $A_0=e^{i\dfrac{\pi}{3}(x+y)}$ (55), gives us

$$u_1(x,y,t)=-2i\left(1+\frac{2\pi^2}{9}\right)te^{i\dfrac{\pi}{3}(x+y)}$$

For $n=2$,

$$u_2(x,y,t)=\int_0^t(1+2i)\Delta u_1(x,y,s)ds-(1+2i)\int_0^t A_1 ds+(1+\frac{2\pi^2}{9})\int_0^t u_1(x,y,s)ds \tag{56}$$

With $\Delta u_1(x,y,t)=4i\frac{\pi^2}{9}(1+\frac{2\pi^2}{9})te^{i\dfrac{\pi}{3}(x+y)}$ and $A_1=-2i(1+\frac{2\pi^2}{9})te^{i\dfrac{\pi}{3}(x+y)}$
So

$$u_2(x,y,t)=-4\left(1+\frac{2\pi^2}{9}\right)^2\frac{t^2}{2}e^{i\dfrac{\pi}{3}(x+y)} \tag{57}$$

For $n = 3$,

$$u_3(x,y,t) = \int_0^t (1+2i)\Delta u_2(x,y,s)ds - (1+2i)\int_0^t A_2 ds + (1+\frac{2\pi^2}{9})\int_0^t u_2(x,y,s)ds \quad (58)$$

With $\Delta u_2(x,y,t) = 8\frac{\pi^2}{9}\left(1+\frac{2\pi^2}{9}\right)^2 \frac{t^2}{2} e^{i\frac{\pi}{3}(x+y)}$ and $A_2 = -4\left(1+\frac{2\pi^2}{9}\right)^2 \frac{t^2}{2} e^{i\frac{\pi}{3}(x+y)}$
So

$$u_3(x,y,t) = 8i\left(1+\frac{2\pi^2}{9}\right)^3 \frac{t^3}{3!} e^{i\frac{\pi}{3}(x+y)} \quad (59)$$

Then

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + \ldots \quad (60)$$

$$u(x,y,t) = \left[1 - 2i\left(1+\frac{2\pi^2}{9}\right)t - 4\left(1+\frac{2\pi^2}{9}\right)^2 \frac{t^2}{2} + 8i\left(1+\frac{2\pi^2}{9}\right)^3 \frac{t^3}{3} + \ldots\right] e^{i\frac{\pi}{3}(x+y)} \quad (61)$$

This implies

$$u(x,y,t) = \sum_{n=0}^{\infty} \frac{\left[-2i\left(1+\frac{2\pi^2}{9}\right)\right]^n}{n!} e^{i\frac{\pi}{3}(x+y)}. \quad (62)$$

Then the exact solution of the Landau-Ginzburg equation in dimension two is:

$$u(x,y,t) = e^{i\left[\frac{\pi}{3}(x+y) - 2\left(1+\frac{2\pi^2}{9}\right)t\right]}. \quad (63)$$

5. Conclusion

In this paper, the Adomian decomposition method has been used to solve the complex model of the Landau Ginzburg equation.
In order to show the importance and applicability of the proposed method, the Adomian polynomial manipulated in this article allowed us to find the exact solution of the studied problem.

References

[1] George Adomian. A review of the decomposition method in applied mathematics. *Journal of mathematical analysis and applications.*

[2] George Adomian. *Solving frontier problems of physics: the decomposition method.* Springer Science & Business Media.

[3] Kamel Attar. *The Ginzburg-Landau model with a variable magnetic field.* PhD thesis, Université Paris Sud-Paris XI; Université Libanaise. Faculté des Sciences ....

[4] Kamel Attar. The ground state energy of the two dimensional ginzburg–landau functional with variable magnetic field. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 32, pages 325–345. Elsevier, 2015.

[5] Charles Bu. An initial-boundary value problem for the ginzburg-landau equation. *Applied mathematics letters*, 5(5):31–34, 1992.

[6] Qiang Du. Global existence and uniqueness of solutions of the time-dependent ginzburg-landau model for superconductivity. *Applicable Analysis*, 53(1-2):1–17, 1994.

[7] Qiang Du. Numerical approximations of the ginzburg–landau models for superconductivity. *Journal of mathematical physics*, 46(9):095109, 2005.

[8] Francis Filbet and Lorenzo Pareschi. A numerical method for the accurate solution of the fokker–planck–landau equation in the nonhomogeneous case. *Journal of Computational Physics*, 179(1):1–26, 2002.

[9] Ayman Kachmar and Marwa Nasrallah. On the ginzburg–landau energy with a magnetic field vanishing along a curve.

[10] Mohammed Lemou and Luc Mieussens. Fast implicit schemes for the fokker–planck–landau equation. *Comptes Rendus Mathematique*, 338(10):809–814, 2004.

[11] Soheila Naghshband and Mohammad Ali Fariborzi Araghi. Solving generalized quintic complex ginzburg–landau equation by homotopy analysis method. *Ain Shams Engineering Journal*, 9(4):607–613, 2018.

[12] Eduardo Salete, Antonio M Vargas, Ángel García, Mihaela Negreanu, Juan J Benito, and Francisco Ureña. Complex ginzburg–landau equation with generalized finite differences. *Mathematics*.

[13] Etienne Sandier and Sylvia Serfaty. The decrease of bulk-superconductivity close to the second critical field in the ginzburg–landau model. *SIAM journal on mathematical analysis*.

[14] Etienne Sandier and Sylvia Serfaty. *Vortices in the magnetic Ginzburg-Landau model.* Springer Science & Business Media.

[15] Shao-Wen Yao, Esin Ilhan, P Veeresha, and Haci Mehmet Baskonus. A powerful iterative approach for quintic complex ginzburg–landau equation within the frame of fractional operator. *Fractals*.