EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1716-1737
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Boundedness of non regular pseudo-differential operators and their adjoints on variable exponent Besov-Morrey spaces

Mohamed Congo[1,*], Marie Françoise Ouedraogo[1]

[1] *Département de Mathematiques. UFR Sciences Exactes et Appliquées/ Université Joseph KI-ZERBO, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso*

Abstract. This paper deal with the boundedness property of non regular pseudo-differential operators $a(x, D)$ and their adjoints $a(x, D)^*$ on variable exponent BM spaces. For this purpose, given such an operator, we use the technique of decomposition of its symbol into elementary symbols already used in other spaces.

2020 Mathematics Subject Classifications: 42B37, 46E30, 35S05

Key Words and Phrases: Pseudo-differential operators, adjoints, Non regular symbols, Elementary symbol, Variable exponent Besov-Morrey spaces.

1. Introduction

Besov-Morrey spaces denoted $\mathcal{N}^s_{p,u,q}$ were initially investigated by Kozono and Yamazaki in [8] to study the solutions of the Navier-Stokes equations with critical regularity. The theory of Besov-Morrey spaces and their applications to non-linear PDEs were further studied by Mazzucato [13]. They are modified Besov spaces where the base norm is of Morrey-type. A first generalisation of the Besov-Morrey spaces $\mathcal{N}^s_{p,u,q}$ into $\mathcal{N}^s_{p(\cdot),u(\cdot),q}$ where only the exponents p and u varied was introduced by Fu and Xu in [6] and a full generalisation to variable exponent Besov-Morrey spaces denoted $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ with all exponents variable is due to Almeida and Caetano [1].
Now the boundedness of an operator is a fundamental property for it's use. One can find in several works the study of boundedness of pseudo-differential operators: on Lebesgue spaces, Besov spaces, Triebel-Lizorkin spaces and Sobolev spaces (see [2], [3], [11] and [12]). In particular, the boundedness of pseudo-differential operators on Besov-Morrey (BM) spaces with **constant exponents** denoted $\mathcal{N}^s_{p,u,q}$ was studied by Mazzucato in [13]. We are concerned in this paper with the boundedness of pseudo-differential operators on

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4553

Email addresses: mohamed.congo@yahoo.fr (M. Congo), omfrancoise@yahoo.fr (M. F. Ouedraogo)

Besov-Morrey spaces with **variable exponents** denoted $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$(see [1]). Since the symbol class $S^m_{1,\delta}$ is too restrictive for applications to non-linear equations, we use symbols in the class $C^\ell_* S^m_{1,\delta}$ where the x regularity is measured in Hölder-Zygmund spaces. The results of this paper generalize those of [13] and complement the studies done on Triebel-Lizorkin-Morrey spaces (see [4]). We further extended the study of the boundedness of such pseudo-differential operators to their adjoints.

Our approach is as follows: we consider pseudo-differential operators in $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ whose symbols belong to the class $C^\ell_* S^m_{1,\delta}$. We use the decomposition of these symbols into elementary symbols following the method of [2], [11] and [13]. We then set up intermediary results useful to prove the main results in theorem [2] and theorem [3].

This paper is structured in **4** sections: the section **2** concerns preliminaries and set up notations as well as definitions and properties of Morrey spaces and Besov-Morrey spaces with variable smoothness and integrability. In section **3**, we recall tools that are necessary to establish lemmas and the main theorems of the next section. The section **4** contains the results and the proof of the main theorem of the boundedness of non regular pseudo-differential operators in the space $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ as well as the adjoint estimate.

2. Preliminaries

2.1. General Notation

We denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write $B(x, r)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We use c as a generic positive constant, i.e. a constant whose value may change with each appearance. If ξ belongs to $\mathbb{R}^n$ and r to $\mathbb{R}$, the expression $|\xi| \sim r$ means that there exists two constants $c_1, c_2 > 0$ such that $c_1 r \leq |\xi| \leq c_2 r$. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c, and $f \approx g$ means $f \lesssim g \lesssim f$. Throughout the paper we denote by $\mathcal{M}(\mathbb{R}^n)$ the family of all complex or extended real-valued measurable functions on $\mathbb{R}^n$.

By supp$f$ we denote the support of the function f , i.e. the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then χ_E denotes its characteristic function. We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform denoted $\mathcal{F}f(\xi)$ or $\hat{f}$ is defined on $\mathcal{S}$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x) \mathrm{d}x$$

and extended to $\mathcal{S}'$ by duality. The inverse Fourier transform denoted $\mathcal{F}^{-1}f(x)$ or $\check{f}$ is defined by

$$\check{f}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(\xi) \mathrm{d}\xi.$$

For two complex or extended real-valued measurable functions f, g on $\mathbb{R}^n$ the convolution $f * g$ is given, in the usual way, by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y, \quad x \in \mathbb{R}^n \quad \text{and } \operatorname{supp}(f*g) \subset \operatorname{supp}f + \operatorname{supp}g.$$

2.2. Variable exponents

In this sub-section we recall the definition and some properties of variable exponents. For more details see [10] and [5].

• We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p : \mathbb{R}^n \to (0, \infty]$ (called variable exponents) which are essentially bounded away from zero. We denote $p^+_{\mathbb{R}^n} := \text{ess sup}_{\mathbb{R}^n} p(x)$ and $p^-_{\mathbb{R}^n} := \text{ess inf}_{\mathbb{R}^n} p(x)$; we abbreviate $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$.

• The function ϕ_p is defined as follows:

$$\phi_{p(x)}(t) = \begin{cases} t^{p(x)} & \text{if } p(x) \in (0,\infty), \\ 0 & \text{if } p(x) = \infty \text{ and } t \in [0,1], \\ \infty & \text{if } p(x) = \infty \text{ and } t \in (1,\infty]. \end{cases}$$

The variable exponent modular associated to $p(\cdot)$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \phi_{p(x)}(|f(x)|)\mathrm{d}x.$$

The variable exponent Lebesgue space $L_{p(\cdot)} := L_{p(\cdot)}(\mathbb{R}^n)$ is the family of (equivalence classes of) functions $f \in \mathcal{M}(\mathbb{R}^n)$ such that $\varrho_{p(\cdot)}(f/\lambda)$ is finite for some $\lambda > 0$.
$L_{p(\cdot)}$ is a quasi-Banach space equipped with the quasinorm

$$\|f\|_{p(\cdot)} := \inf\left\{\mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \le 1\right\}.$$

• We say that a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{log}_{loc}(\mathbb{R}^n)$, if there exists $c_{log}(g) \ge 0$ such that

$$|g(x) - g(y)| \le \frac{c_{log}(g)}{\log(\mathrm{e} + 1/|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{1}$$

The function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-*Hölder continuous* and there exists $g_\infty \in \mathbb{R}$ and $c_\infty(g) \ge 0$ such that

$$|g(x) - g_\infty| \le \frac{c_\infty(g)}{\log(\mathrm{e} + |x|)} \quad \text{for all } x \in \mathbb{R}^n.$$

We define the following class of variable exponents

$$\mathcal{P}^{log}(\mathbb{R}^n) := \left\{p \in \mathcal{P} : \frac{1}{p} \in C^{log}(\mathbb{R}^n)\right\}.$$

We define $\dfrac{1}{p_\infty} := \lim\limits_{|x|\to\infty} \dfrac{1}{p(x)}$ and we use the convention $\dfrac{1}{\infty} = 0$.

2.3. Variable exponent Besov-Morrey spaces

We recall the definition of variable exponent Besov-Morrey spaces. We refer to the papers [1], [18], [17] and [8], for further results on these spaces.

Definition 1. *For $p, u \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \le p(x) \le u(x) \le \infty$, the variable exponent Morrey space $M_{p(\cdot),u(\cdot)} := M_{p(\cdot),u(\cdot)}(\mathbb{R}^n)$ consists of all functions $f \in \mathcal{M}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p(\cdot),u(\cdot)}} := \sup_{x\in\mathbb{R}^n,\ r>0} r^{\frac{n}{u(x)}-\frac{n}{p(x)}} \left\|f\chi_{B(x.r)}\right\|_{L_{p(\cdot)}}. \tag{2}$$

Definition 2. *Let $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \le u(x)$. Given a sequence $(f_\nu)_\nu \subset \mathcal{M}(\mathbb{R}^n)$, we set*

$$\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}((f_\nu)_\nu) := \sum_{\nu\ge 0} \sup_{x\in\mathbb{R}^n,\ r>0} \inf\left\{\lambda > 0 :\ \varrho_{p(\cdot)}\left(r^{\frac{n}{u(x)}-\frac{n}{p(x)}} f_\nu \chi_{B(x.r)}/\lambda^{\frac{1}{q(\cdot)}}\right) \le 1\right\}. \tag{3}$$

Remark 1. *When $q^+ < \infty$ or $q^+ = \infty$ and $p(x) \ge q(x)$ we can simplify (3) to obtain*

$$\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}((f_\nu)_\nu) := \sum_{\nu\ge 0} \sup_{x\in\mathbb{R}^n,\ r>0} \left\|\phi_{q(\cdot)}\left(r^{\frac{n}{u(x)}-\frac{n}{p(x)}} |f_\nu|\chi_{B(x.r)}\right)\right\|_{L_{\frac{p(\cdot)}{q(\cdot)}}}$$

Definition 3. *Let $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \le u(x)$.The mixed Morrey-sequence space $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ consists of all sequences $(f_\nu)_\nu \subset \mathcal{M}(\mathbb{R}^n)$ such that, $\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}(\mu(f_\nu)) < \infty$ for some $\mu > 0$. For $(f_\nu)_\nu \in \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ we define*

$$\|(f_\nu)_\nu\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} := \inf\left\{\mu > 0 :\ \varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}\left(\frac{1}{\mu}(f_\nu)\right) \le 1\right\}. \tag{4}$$

Proposition 1. *Let $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \le u(x)$. Let $(f_\nu)_\nu \in \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$*

(i) The functional $\|\cdot\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$ is a quasinorm in $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ and

$$\|(f_\nu)_\nu\|^t_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} = \left\|(|f_\nu|^t)_\nu\right\|_{\ell_{q(\cdot)/t}(M_{p(\cdot)/t,u(\cdot)/t})}, \quad \forall t > 0.$$

(ii) If $f_{\nu_0} = f$ for some $f \in M_{p(\cdot),u(\cdot)}(\mathbb{R}^n)$ and $\nu_0 \in \mathbb{N}_0$, and $f_\nu = 0$ for all $\nu \ne \nu_0$, then

$$\|(f_\nu)_\nu\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} = \|f\|_{M_{p(\cdot),u(\cdot)}}.$$

Theorem 1. *The functional (4) defines a quasinorm in the vector space $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ for any $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \le u(x)$. Moreover, it induces a norm in the following cases (each one understood for almost every $x \in \mathbb{R}^n$):*

(i) $p(x) \ge 1$ and $q \in [1, \infty]$ is constant;

(ii) $1 \le q(x) \le p(x) \le u(x) \le \infty$;

(iii) $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1.$

Besov-Morrey spaces: To define Besov spaces based on $M_{p(\cdot),u(\cdot)}$, let us first recall the définition of a *Littlewood-Paley partition of unity* $\{\varphi_\nu\}$, $\nu \geq 0$.
The functions φ_ν are defined as follows. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ a real function such that $\varphi_0 \equiv 1$ on $B(0;1)$ and supp$\varphi_0 \subset B(0;2)$.
Set

$$\varphi_\nu(\xi) = \varphi_0(2^{-\nu}\xi) - \varphi_0(2^{-\nu+1}\xi) \text{ for all } \nu \in \mathbb{N}.$$

Then φ_ν is supported on the dyadic shell

$$D_\nu = \left\{\xi \in \mathbb{R}^n : \ 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\right\}$$

with $D_\nu \cap D_\mu = \emptyset$ if $|\nu - \mu| > 1$. One has

$$\sum_{\nu \geq 0} \varphi_\nu = 1.$$

Then for all $f \in \mathcal{S}'$,

$$f = \sum_{\nu \geq 0} \varphi_\nu f.$$

The Littlewood-Paley partition of unity is used to define the Fourier multiplier $\varphi_j(D)$ as followed

$$\varphi_\nu(D)f(x) = \mathcal{F}^{-1}(\varphi_\nu \cdot \hat{f})(x) = \int_{\mathbb{R}^n} \varphi_\nu(\xi)\hat{f}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

Definition 4. *Let $\{\varphi_\nu\}$ be the Littlewood-Paley partition of unity. Let $s \in C_{loc}^{log}$ and $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ such that $0 < p^- \leq p(x) \leq u(x) \leq \infty$. The Besov-Morrey spaces $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} := \|\varphi_0(D)f\|_{M_{p(\cdot),u(\cdot)}} + \left\|\left(2^{\nu s(\cdot)}\varphi_\nu(D)f\right)_{\nu\geq 1}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} < \infty. \quad (5)$$

Remark 2.

(i) Let us notice that Besov-Morrey spaces $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ are defined by the composite $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ while Triebel-Lizorkin-Morrey spaces $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ are defined by $M_{p(\cdot),u(\cdot)}\left(\ell_{q(\cdot)}\right)$.

(ii) The case of the boundedness of non-regular PDOs on variable exponent Triebel-Lizorkin-Morrey spaces has been studied in [4].

Proposition 2. *Let $s \in C_{loc}^{log}$, $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \leq u(x)$ and $1/q$ locally log-Hölder continuous. Then it holds*

$$\mathcal{S} \hookrightarrow \mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'.$$

3. Basic tools

In the following, we present some results which will be useful in the last section. First of all, we recall the η-functions defined on $\mathbb{R}^n$ by

$$\eta_{\nu,m}(x) = 2^{n\nu}\left(1+2^{\nu}|x|\right)^{-m}, \quad \nu \in \mathbb{N}_0,\ m>0.$$

Note that $\eta_{\nu,m} \in L^1$ for $m>n$ and the corresponding L_1-norm does not depend on ν.

The next lemmas can be found in [7](Lemma 19) and [9](Lemma 6.1.).

Lemma 1. *Let $\alpha \in C^{log}_{loc}(\mathbb{R}^n)$ and let $m \geq 0,\ l \geq c_{log}(\alpha)$, where c_{log} is the constant from (1) for α. Then*

$$2^{\nu\alpha(x)}\eta_{\nu,m+l}(x-y) \leq c2^{\nu\alpha(y)}\eta_{\nu,m}(x-y)$$

with $c>0$ independent of $x,y \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.

Lemma 2. *Let $t>0, \nu \in \mathbb{N}_0$ and $m>n$. Then there exist $c=c(t,m,n)$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $supp\mathcal{F}g \subset \left\{\xi \in \mathbb{R}^n :\ |\xi| \leq 2^{\nu+1}\right\}$,*

$$|g(x)| \leq c\left(\eta_{\nu,m} * |g|^t(x)\right)^{1/t}, \quad x \in \mathbb{R}^n.$$

Lemma 3. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u,q \in \mathcal{P}$ such that $\frac{1}{q} \in C^{log}_{loc}$ with $1 \leq p^- \leq p(x) \leq u(x) \leq \infty$. If $m > n+c_{log}(1/q)+n\max\left\{0,\ \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)}-\frac{1}{u(x)}\right)-\frac{1}{p_\infty}\right\}$, then there exist $c>0$ such that*

$$\left\|(\eta_{\nu,m} * f_\nu)_\nu\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \leq c\left\|(f_\nu)_\nu\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

for all sequences $(f_\nu)_\nu \subset \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$.

Lemma 4. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u \in \mathcal{P}$ with $1 \leq p^- \leq p(x) \leq u(x) \leq \infty$. If $m > n+n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)}-\frac{1}{u(x)}\right)-\frac{1}{p_\infty}\right\}$. Then there exists $c>0$ such that*

$$\|\eta_{\nu,m} * f\|_{M_{p(\cdot),u(\cdot)}} \leq \|f\|_{M_{p(\cdot),u(\cdot)}}.$$

Lemma 5. *Let $p,u,q \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \leq u(x)$. Let $\delta>0$. For any sequence $(g_j)_{j\in\mathbb{N}_0}$ of non-negative measurable functions on $\mathbb{R}^n$, let*

$$G_\nu(x) := \sum_{j=0}^{\infty} 2^{-|\nu-j|\delta} g_j(x), \quad x \in \mathbb{R}^n,\ \nu \in \mathbb{N}_0.$$

Then it holds

$$\|(G_\nu)_\nu\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \lesssim \left\|(g_j)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

4. Boundedness of pseudo-differential operators

We will use symbols for which x-regularity is measured in Hölder-Zygmund spaces.

Definition 5. *The function $a(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ belongs to the symbol class $C^\ell_* S^m_{1,\delta}$, $\delta \in [0,1]$, $\ell > 0$ if it is smooth in ξ and satisfies the following estimates:*

$$\begin{cases} \left\| \partial^\alpha_\xi a(\cdot,\xi) \right\|_{C^\ell_*} \leq c_\alpha \langle \xi \rangle^{m-|\alpha|+\ell\delta} \\ \left| \partial^\alpha_\xi a(x,\xi) \right| \leq c'_\alpha \langle \xi \rangle^{m-|\alpha|} \end{cases} \tag{6}$$

where $\langle \xi \rangle$ stands for $\left(1+|\xi|^2\right)^{1/2}$. A pseudo-differential operator on $\mathcal{S}$ with symbol $a \in C^\ell_* S^m_{1,\delta}$ is defined by

$$a(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} a(x,\xi) \mathcal{F}f(\xi) \mathrm{d}\xi \ , \quad f \in \mathcal{S}.$$

We write $a(x,D) \in C^\ell_* OPS^m_{1,\delta}$ if $a(x,\xi)$ belongs to the class $C^\ell_* S^m_{1,\delta}$.

To study the boundedness of $a(x,D)$, we will resolve its symbol a into elementary symbols. Therefore, the operator $a(x,D)$ with symbol a can be resolved into "elementary operators" $a_k(x,D)$ with symbols a_k. This idea has been exploited to establish boundedness of pseudo-differential operators with non-regular symbols in Sobolev spaces $H^{s,p}$ and Hölder-Zygmund spaces C^ℓ_* (see [11], [2]). We will proceed in the same way with the adjoint operator.

Definition 6. *[13] We call elementary symbol in the class $C^\ell_* S^m_{1,\delta}$, $\delta \in [0,1]$, $\ell > 0$ an expression of the form*

$$a(x,\xi) = \sum_{j \geq 0} \sigma_j(x) \varphi_j(\xi)$$

where φ_0 is smooth supported on the ball $B(0,2)$, $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\varphi \in C^\infty_0$ is supported on the dyadic shell $D_0 = \{\xi \in \mathbb{R}^n |\ 1/2 \leq |\xi| \leq 2\}$, while σ_j is a uniformly bounded sequence such that

$$\|\sigma_j\|_{C^\ell_* S^m_{1,\delta}} \leq c 2^{j(m+\ell\delta)}.$$

Example 1. *Let $\{\varphi_j\}$ be a Littlewood-Paley partition of unity and $\{\sigma_j\}$ a sequence uniformly bounded in C^ℓ_*.*
Set

$$a(x,D)f(x) = \sum_{j \geq 0} \sigma_j(2^{j\delta}x) \varphi_j(D) f(x) \ , \quad f \in \mathcal{S}$$

then $\quad a(x,D) \in C^\ell_* OPS^0_{1,\delta}$.

Lemma 6. *[13] Let $f = \sum_{j \geq 0} f_j$ in $\mathcal{S}'$, with supp$\hat{f}_j \subset B(0, A2^j)$ for some $A > 0$. Then, for $\ell > 0$,*

$$\|f\|_{C^\ell_*} \leq c(A) \sup_{j \geq 0} \left\{ 2^{j\ell} \|f_j\|_{L^\infty} \right\}. \tag{7}$$

Let us establish the two following lemmas which play a fundamental role in the proof of the boundedness of pseudo-differential operators on $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$.

Lemma 7. *Let $c_1, c_2 > 0$, $p \in \mathcal{P}^{log}(\mathbb{R}^n)$, $u, q \in \mathcal{P}$ such that , $\frac{1}{q} \in C^{log}_{loc}$ with $1 \leq p^- \leq p(x) \leq u(x) \leq \infty$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of tempered distributions such that*

$$supp \mathcal{F} f_0 \subset B(0, 2c_2)$$

and

$$supp \mathcal{F} f_k \subset \left\{\xi \in \mathbb{R}^n : \ c_1 2^{k-1} \leq |\xi| \leq c_2 2^{k+1}\right\} \quad for\ k > 0.$$

Then

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\|\left(2^{ks(\cdot)} f_k\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

Proof. Using (5) we have

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} + \left\|\left\{2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=1}^{\infty} f_k\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

Since φ_j is supported on the dyadic shell D_j, while φ_0 is supported on the ball $B(0;2)$, there are $N_1, N_2 \in \mathbb{N}_0$ such that

$$\begin{aligned}\left\|\sum_{k=0}^{\infty} f_k\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\|\varphi_0(D)\left(\sum_{k=0}^{N_1} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} + \left\|\left\{2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=j-N_1}^{j+N_2} f_k\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &= \left\|\sum_{k=0}^{N_1} \check{\varphi}_0 * f_k\right\|_{M_{p(\cdot),u(\cdot)}} + \left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2} \check{\varphi}_j * f_k\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.\end{aligned}$$

Let us now estimate these two terms.

• Estimation of the term $\left\|\sum_{k=0}^{N_1} \check{\varphi}_0 * f_k\right\|_{M_{p(\cdot),u(\cdot)}}$

One has

$$\operatorname{supp}\mathcal{F}\left(\check{\varphi}_0 * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\right\}.$$

Then by lemma 2, $|\check{\varphi}_0 * f_k| \lesssim |f_k|$. It follows that

$$\left\|\sum_{k=0}^{N_1} \check{\varphi}_0 * f_k\right\|_{M_{p(\cdot),u(\cdot)}} \lesssim \sum_{k=0}^{N_1} \|f_k\|_{M_{p(\cdot),u(\cdot)}}$$

$$= \sum_{k=0}^{N_1} \|(0,\ldots,f_k,0,\ldots)\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \quad \text{by proposition 1}$$
$$\lesssim \left\|\left(2^{ks(\cdot)}f_k\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

- Estimation of the term $\left\|\left\{2^{js(\cdot)}\sum_{k=j-N}^{j+N}\check{\varphi}_j * f_k\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$.

Since $\check{\varphi}_j * f_k \in \mathcal{S}'$ and $\text{supp}\mathcal{F}(\check{\varphi}_j * f_k) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \le 2^{j+1}\right\}$, then by lemma 2,

$$|\check{\varphi}_j * f_k| \lesssim \left(\eta_{j,m} * |f_k|^t\right)^{1/t}$$

for any $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$ and any $t > 0$.

Thus with $t = 1$,

$$\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\varphi}_j * f_k\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \lesssim \left\|\left\{\sum_{k=j-N_1}^{j+N_2} 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

By lemma 1, we can move $2^{js(\cdot)}$ inside the convolution and get

$$2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right) \lesssim \eta_{j,m-c_{log}(s)} * 2^{js(\cdot)}|f_k|.$$

Let us notice that if $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$ then $m - c_{log}(s)$ verifies the hypothesis of the lemma 3. Therefore

$$\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\varphi}_j * f_k\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \lesssim \left\|\left\{\sum_{k=j-N_1}^{j+N_2}\left(\eta_{j,m-c_{log}(s)} * 2^{js(\cdot)}|f_k|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$
$$\le \sum_{k=j-N_1}^{j+N_2}\left\|\left\{\left(\eta_{j,m-c_{log}(s)} * 2^{js(\cdot)}|f_k|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

By lemma 3 ,

$$\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\varphi}_j * f_k\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \lesssim \sum_{k=0}^{N_1+N_2}\left\|\left(2^{js(\cdot)}f_{j+k-N_1}\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$
$$\lesssim \left\|\left(2^{ks(\cdot)}f_k\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

The two estimations yield the desired estimate. □

Lemma 8. *Let $c > 0, p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u, q \in \mathcal{P}$ such that $\frac{1}{q} \in C_{loc}^{log}$ with $1 \leq p^- \leq p(x) \leq u(x) \leq \infty$. Let $s \in C_{loc}^{log}$ such that $s^- > 0$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of tempered distributions such that*

$$supp\mathcal{F}f_k \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq c2^{k+1}\right\}.$$

Then

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\|\left(2^{ks(\cdot)}f_k\right)_k\right\|_{\ell_q(M_{p(\cdot),u(\cdot)})}.$$

Proof. Using the hypothesis on Suppφ_j, there is $N \in \mathbb{N}_0$ such that

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} + \left\|\left\{2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=j-N}^{\infty} f_k\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}. \tag{8}$$

(i) Let us first estimate the term $\left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} = \left\|\sum_{k=0}^{N} \check{\varphi}_0 * f_k\right\|_{M_{p(\cdot),u(\cdot)}}$.

Since

$$\text{supp}\mathcal{F}\left(\check{\psi}_0 * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2^{k+1}\right\},$$

By lemma 2

$$\left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} \lesssim \left\|\sum_{k=0}^{\infty} \eta_{k,m} * |f_k|\right\|_{M_{p(\cdot),u(\cdot)}},$$

for any $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$.

Then

$$\left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} \lesssim \left\|\sum_{k=0}^{\infty} 2^{-ks^-}\left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)}|f_k|\right)\right\|_{M_{p(\cdot),u(\cdot)}} \quad \text{by lemma 1.}$$

Using proposition 1, we have

$$\begin{aligned}\left\|\sum_{k=0}^{\infty} 2^{-ks^-}\left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)}|f_k|\right)\right\|_{M_{p(\cdot),u(\cdot)}} &= \left\|\left\{\sum_{k=0}^{\infty} 2^{-ks^-}\left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)}f_k\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\lesssim \left\|\left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)}f_k\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \quad \text{by lemma 5}\end{aligned}$$

Thus, by lemma 2, it follows that

$$\left\|\varphi_0(D)\left(\sum_{k=0}^{\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} \lesssim \left\|\left(2^{ks(\cdot)}f_k\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

(ii) Now let us estimate the term $\left\|\left\{2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=j-N}^{\infty} f_k\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$.

$$\left\|\left\{2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=j-N}^{\infty} f_k\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} = \left\|\left\{\sum_{k=j-N}^{\infty} 2^{js(\cdot)}\left(\check{\varphi}_j * f_k\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

Since

$$\begin{cases}\operatorname{supp}\mathcal{F}\left(\check{\varphi}_j * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2^{j+1}\right\}\\ \operatorname{supp}\mathcal{F}\left(\check{\varphi}_j * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2^{k+1}\right\},\end{cases}$$

by lemma 2

$$\begin{cases}2^{js(\cdot)}\left(\check{\varphi}_j * f_k\right) \lesssim 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right)\\ 2^{js(\cdot)}\left(\check{\varphi}_j * f_k\right) \lesssim 2^{js(\cdot)}\left(\eta_{k,m} * |f_k|\right)\end{cases}$$

for $m > n + c_{log}(1/q) + c_{log}(s) + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$.

Therefore

$$\begin{aligned}\left\|2^{js(\cdot)}\varphi_j(D)\left(\sum_{k=j-N}^{\infty} f_k\right)\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} &\lesssim \left\|\left\{\sum_{k=j-N}^{j} 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}\\ &+ \left\|\left\{\sum_{k=j+1}^{\infty} 2^{-(k-j)s(\cdot)}2^{ks(\cdot)}\left(\eta_{k,m} * |f_k|\right)\right\}_{j\geq N}\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.\end{aligned}$$

We are left now to estimate each terms on the right-hand side.
Using lemma 1 we can move $2^{\nu s(\cdot)}$ inside the convolution $2^{\nu s(\cdot)}\left(\eta_{\nu,m} * |f_k|\right)$ and get

$$2^{\nu s(\cdot)}\left(\eta_{\nu,m} * |f_k|\right) \lesssim \left(\eta_{\nu,m_0} * 2^{\nu s(\cdot)}|f_k|\right)\ ,\nu = j \text{ or } k \ \text{ where } m_0 = m - c_{log}(s).$$

Thus

- $$\begin{aligned}\left\|\left\{\sum_{k=j-N}^{j} 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} &\lesssim \sum_{\ell=-N}^{0}\left\|\left\{\left(\eta_{j,m_0} * 2^{js(\cdot)}|f_{j+\ell}|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}\\ &\lesssim \left\|\left(2^{js(\cdot)}f_j\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \quad \text{by lemma 3.}\end{aligned}$$

Also

- $$\left\|\left\{\sum_{k=j+1}^{\infty} 2^{-(k-j)s(\cdot)}2^{ks(\cdot)}\left(\eta_{k,m} * |f_k|\right)\right\}_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

$$\lesssim \left\| \left\{ \sum_{k=j+1}^{\infty} 2^{-|j-k|s(\cdot)} \left(\eta_{k,m_0} * 2^{ks(\cdot)} |f_k| \right) \right\}_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

$$\lesssim \left\| \left\{ \sum_{k=0}^{\infty} 2^{-|j-k|s^-} \left(\eta_{k,m_0} * 2^{ks(\cdot)} |f_k| \right) \right\}_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

$$\lesssim \left\| \left(\eta_{k,m_0} * 2^{ks(\cdot)} |f_k| \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

$$\lesssim \left\| \left(2^{ks(\cdot)} f_k \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \quad \text{by lemma 5 and 3.}$$

. □

4.1. The main estimate

In this subsection, we will establish the boundedness of pseudo-differential operators on $\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$.

Theorem 2. *Let $a(x,\xi) \in C^{\ell}_{*} S^{m}_{1,\delta}$ where $m \in \mathbb{R}$, $\delta \in [0,1]$ and $\ell > 0$. Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u, q \in \mathcal{P}$ such that $\frac{1}{q} \in C^{log}_{loc}$ with $1 \leq p^- \leq p(x) \leq u(x) \leq \infty$. Let $s \in C^{log}_{loc}$ such that $0 < s^- \leq s(x) < \ell$. Then*

$$a(x,D) : \mathcal{N}^{s(\cdot)+m}_{p(\cdot),u(\cdot),q(\cdot)} \longrightarrow \mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$$

is bounded.

Remark 3. *The operator $(1-\Delta)^{\frac{m}{2}}$, $m \in \mathbb{R}$ is an isomorphism that composes well with pseudo-differential operators (see[13] and [15]). Thus, it is enough to treat the case $m = 0$. Therefore let us set $a(x,\xi) \in C^{\ell}_{*} S^{0}_{1,\delta}$.*
The symbol reduction method due to Coifman and Meyer[3], makes it possible to be limited to symbols of the form(see [13], [11], [2] and [16])

$$a(x,\xi) = \sum_{j \geq 0} \sigma_j(x) \varphi_j(\xi)$$

where σ_j satisfies

$$\|\sigma_j\|_{C^{\ell}_{*}} \leq c 2^{j\ell\delta} \tag{9}$$

$$\text{and} \quad \|\sigma_j\|_{L^\infty} \leq c \tag{10}$$

with c depending on δ and ℓ but not on j and φ_j is exactly a Littlewood-Paley function.

Proof. Let

$$a(x,\xi) = \sum_{j\geq 0} \sigma_j(x)\varphi_j(\xi)$$

with the conditions given above. Let us decompose this symbol into three parts.

First of all we have $\sigma_j(x) = \sum_{k=0}^{\infty} \varphi_k(D)\sigma_j(x)$.

By multiplying each member by $\varphi_j(\xi)$, we obtain $\sigma_j(x)\varphi_j(\xi) = \left(\sum_{k=0}^{\infty} \varphi_k(D)\sigma_j(x)\right)\varphi_j(\xi)$.

$$\text{and then} \quad a(x,\xi) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} \varphi_k(D)\sigma_j(x)\right)\varphi_j(\xi).$$

By setting $a_{kj} = \varphi_k(D)\sigma_j$ we have

$$a(x,\xi) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{kj}\right)\varphi_j(\xi). \tag{11}$$

Now we rewrite (11) as a sum of three parts

$$\begin{aligned} a(x,\xi) &= \sum_{j\geq 0}\left(\sum_{k=0}^{j-4} a_{kj}(x) + \sum_{k=j-3}^{j+3} a_{kj}(x) + \sum_{k=j+4}^{\infty} a_{kj}(x)\right)\varphi_j(\xi) \\ &= a_1(x,\xi) + a_2(x,\xi) + a_3(x,\xi) \end{aligned}$$

(i) Let $\varphi_j(D)f = f_j$. Then we define three " elementary" pseudo-differential operators:

$$\begin{aligned} a_1(x,D)f &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j-4} a_{kj}f_j\right), \\ a_2(x,D)f &= \sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+3} a_{kj}f_j\right), \\ a_3(x,D)f &= \sum_{j=0}^{\infty}\left(\sum_{k=j+4}^{\infty} a_{kj}f_j\right). \end{aligned}$$

(ii) It remains to estimate each of these three pseudo-differential operators.
For this purpose, it's necessary to estimate $\|a_{kj}\|_{L_\infty}$.
Let us recall the quasinorm of C_*^ℓ: $\|\varphi_k(D)\sigma_j\|_{C_*^\ell} = \sup_k 2^{k\ell}\|\varphi_k(D)\sigma_j\|_{L^\infty}$.
Since

$$\|\varphi_k(D)\sigma_j\|_{C_*^\ell} \leq c\|\sigma_j\|_{C_*^\ell}.$$

Then

$$\sup_k 2^{k\ell}\|\varphi_k(D)\sigma_j\|_{L_\infty} \leq c\|\sigma_j\|_{C_*^\ell}.$$

Using (9), we obtain

$$\|a_{kj}\|_{L^\infty} \leq c2^{j\ell\delta}2^{-k\ell}. \tag{12}$$

We are ready to estimate the pseudo-differential operators $a_1(x,D), a_2(x,D)$ and $a_3(x,D)$.

• The estimation of $a_1(x,D)$.

We have

$$\begin{aligned}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj} f_j\right) &= \sum_{k=0}^{j-4} \mathcal{F}\left(\varphi_k(D)\sigma_j\right) * \mathcal{F}\left(\varphi_j(D)f\right) \\ &= \sum_{k=0}^{j-4} \left(\psi_k \mathcal{F}\sigma_j\right) * \left(\psi_j \mathcal{F} f\right).\end{aligned}$$

Using the fact that $\text{supp}(f*g) \subset \text{supp} f + \text{supp} g$ for all compactly supported distributions $f, g \in \mathcal{S}'$, we have

$$\text{supp}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj} f_j\right) \subset \left\{\xi \in \mathbb{R}^n :\ |\xi| \sim 2^{j+1}\right\}.$$

Then lemma 7 yields

$$\begin{aligned}\|a_1(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\|\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j-4} a_{kj} f_j\right)\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\ &\lesssim \left\|\left(2^{js(\cdot)}\sum_{k=0}^{j-4} a_{kj} f_j\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\lesssim \left(\sup_{j\in\mathbb{N}_0}\sum_{k=0}^{\max\{j-4,0\}} \|a_{kj}\|_{L^\infty}\right)\left\|\left(2^{js(\cdot)}\varphi_j(D)f\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\lesssim \left\|\left(2^{js(\cdot)}\varphi_j(D)f\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.\end{aligned}$$

It follows that

$$\|a_1(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}.$$

• Estimation of $a_2(x,D)$

For the second part $\|a_2(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\|\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+3} a_{kj} f_j\right)\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}$,

Let us first observe that

$$\mathcal{F}\left(\sum_{k=j-3}^{j+3} a_{kj} f_j\right) = \sum_{k=j-3}^{j+3} \mathcal{F}\left(\varphi_k(D)\sigma_j\right) * \mathcal{F}\left(\varphi_j(D)f\right)$$

$$= \sum_{k=j-3}^{j+3} (\varphi_k \mathcal{F}\sigma_j) * (\varphi_j \mathcal{F} f)\,.$$

Therefore $\mathcal{F}\left(\sum_{k=j-3}^{j+3} a_{kj} f_j\right)$ is supported on the ball $B(0, 2^{j+4})$. By lemma 8,

$$\begin{aligned}
\|a_2(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\| \left(2^{js(\cdot)} \sum_{k=j-3}^{j+3} a_{kj} f_j \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\
&\leq 2^{-m} \left\| \left(\sum_{k=j-3}^{j+3} \|a_{kj}\|_{L_\infty} 2^{js(\cdot)} \varphi_j(D) f \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} .
\end{aligned}$$

Now using (12) one has $\sum_{k=j-3}^{j+3} \|a_{kj}\|_{L^\infty} \lesssim \sum_{k=-3}^{3} 2^{-k\ell} < \infty$ (with $\delta = 1$)

and then

$$\begin{aligned}
\|a_2(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\| \left(2^{js(\cdot)} \varphi_j(D) f \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\
&\lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}
\end{aligned}$$

• Estimation of $a_3(x,D)$

Since $\mathcal{F}\left(\sum_{k=j+4}^{\infty} a_{kj} f_j\right)$ is not supported on any ball or shell, we cannot directly use neither lemma 7 nor lemma 8. However, in $\mathcal{S}'$ we can write

$$\sum_{j=0}^{\infty} \sum_{k=j+4}^{\infty} a_{kj} f_j = \sum_{k=4}^{\infty} \sum_{j=0}^{k-4} a_{kj} f_j .$$

We have

$$\mathcal{F}\left(\sum_{j=0}^{k-4} a_{kj} f_j\right) = \sum_{j=0}^{k-4} (\psi_k \mathcal{F} a_j) * (\psi_j \mathcal{F} f)$$

then

$$\operatorname{supp}\mathcal{F}\left(\sum_{j=0}^{k-4} a_{kj} f_j\right) \subset \left\{ \xi \in \mathbb{R}^n : \ |\xi| \sim 2^{k+1} \right\} .$$

Then lemma 7 yields

$$\begin{aligned}\|a_3(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\|\sum_{k=4}^{\infty}\left(\sum_{j=0}^{k-4}a_{kj}f_j\right)\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\ &\lesssim \left\|\left(2^{ks(\cdot)}\sum_{j=0}^{k-4}a_{kj}f_j\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\lesssim \left\|\left(\sum_{j=0}^{k-4}\|a_{kj}\|_{L^\infty}2^{ks(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.\end{aligned}$$

If we use (12) with $\delta = 1$, we have

$$\begin{aligned}\|a_3(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\|\left(\sum_{j=0}^{k-4}2^{j\ell}2^{-k\ell}2^{ks(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &= \left\|\left(\sum_{j=0}^{k-4}2^{(k-j)(s(\cdot)-\ell)}2^{js(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\leq \left\|\left(\sum_{j=0}^{k-4}2^{-|k-j||s^--\ell|}2^{js(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\ &\leq \left\|\left(\sum_{j=0}^{\infty}2^{-|k-j||s^--\ell|}2^{js(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.\end{aligned}$$

By hypothesis we have $|s^- - \ell| > 0$. Therefore, by lemma 5

$$\left\|\left(\sum_{j=0}^{k-4}2^{-|k-j||s^--\ell|}2^{js(\cdot)}\varphi_j(D)f\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \lesssim \left\|\left(2^{js(\cdot)}\varphi_j(D)f\right)_k\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

Then

$$\|a_3(x,D)f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}.$$

The proof is completed. □

4.2. Thet adjoint operator estimate

In this subsection, let us go further by studying the boundedness of the adjoint operator. Let the adjoint operator A^* of the operator A defined by

$$\int (Af)\overline{g}\mathrm{d}x = \int f\overline{A^*g}\mathrm{d}x \quad f,g\in\mathcal{S}. \tag{13}$$

Since an operator $a(x,D)$ with symbol $a(x,\xi)\in C^{\ell}_{*}S^{0}_{1,\delta}$ can be decomposed in the form

$$a(x,D)=a_1(x,D)+a_2(x,D)+a_3(x,D),$$

then its adjoint $a(x,D)^*$ can be written as follow

$$a(x,D)^*=a_1(x,D)^*+a_2(x,D)^*+a_3(x,D)^*.$$

This method has already been used by Marschall in [11] and [12].
Let's calculate $a_\lambda(x,D)^*$ for $\lambda=1,2,3$. For that, let $a_\lambda(x,D)f=\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}a_{kj}f_j$ where $I_\lambda\subset\mathbb{N}_0,\ I'_\lambda\subset\mathbb{N}_0$.
Using (13),

$$\begin{aligned}\int\left(a_\lambda(x,D)f(x)\right)\bar{g}(x)\mathrm{d}x &= \int\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}a_{kj}\varphi_j(D)f(x)\overline{g}(x)\mathrm{d}x\\ &= \int\left(\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)\overline{\overline{a_{kj}}g}\right)(x)\mathrm{d}x.\end{aligned}$$

Plancherel's theorem yields

$$\int\left(a_\lambda(x,D)f(x)\right)\bar{g}(x)\mathrm{d}x=\int f(x)\overline{\left(\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\mathcal{F}^{-1}\left(\varphi_j\mathcal{F}(\overline{a_{kj}}g)\right)\right)}(x)\mathrm{d}x.$$

Then, the adjoint of $a_\lambda(x,D)$ is

$$a_\lambda(x,D)^*f = \sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\varphi_j(D)\left(\overline{a_{kj}}f\right)=\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=0}^{\infty}\varphi_{k'}(D)f\right).$$

Thus

$$a_\lambda(x,D)^*f=\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=0}^{\infty}f_{k'}\right)=\sum_{j\in I_\lambda}\sum_{k\in I'_\lambda}\sum_{k'=0}^{\infty}\varphi_j(D)\left(\overline{a_{kj}}f_{k'}\right)\quad\text{for }\lambda=1,2,3 \tag{14}$$

One has

$$\mathcal{F}\left\{\varphi_j(D)\left(\overline{a_{kj}}f_{k'}\right)\right\}=\varphi_j\mathcal{F}\left\{\mathcal{F}^{-1}(\varphi_k\mathcal{F}\sigma_j)\cdot\mathcal{F}^{-1}(\varphi_{k'}\mathcal{F}f)\right\}.$$

The intersection of the supports of φ_j and $\mathcal{F}\left\{\mathcal{F}^{-1}(\varphi_k\mathcal{F}\sigma_j)\cdot\mathcal{F}^{-1}(\varphi_{k'}\mathcal{F}f)\right\}$ is empty if the non-negative integer k' dœs not verify the following cases (See [14] and [11]):

$$\begin{cases}j-3\le k'\le j+3\ \text{ and }\ k=0,\dots,j+3\\ j-3\le k\le j+3\ \text{ and }\ k'=0,\dots,j+3\\ k\ge j+4,\ k'\ge j+4\ \text{ and }\ |k'-k|\le 3.\end{cases}.$$

It follows that

$$\begin{aligned} a_1(x,D)^*f &= \sum_{j=0}^{\infty}\sum_{k=0}^{j-4}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=j-3}^{j+3}f_{k'}\right), \\ a_2(x,D)^*f &= \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+3}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=0}^{j+6}f_{k'}\right), \\ a_3(x,D)^*f &= \sum_{j=0}^{\infty}\sum_{k=j+4}^{\infty}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=k-3}^{k+3}f_{k'}\right). \end{aligned}$$

Theorem 3. *Let $a(x,\xi)\in C^{\ell}_{*}S^{m}_{1,\delta}$ where $m\in\mathbb{R}$, $\delta\in[0,1]$ and $\ell>0$. Let $p\in\mathcal{P}^{log}(\mathbb{R}^n)$ and $u,q\in\mathcal{P}$ such that $\frac{1}{q}\in C^{log}_{loc}$ with $1\leq p^-\leq p(x)\leq u(x)\leq\infty$. Let $s\in C^{log}_{loc}$ such that $0<s^-\leq s(x)<\ell$. Then*

$$a(x,D)^*:\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}\longrightarrow\mathcal{N}^{s(\cdot)-m}_{p(\cdot),u()\cdot,q(\cdot)}$$

is bounded.

Proof. As for the proof of theorem 2, we will proceed by estimating the three above operators. Let $m=0$.
• The estimation of $a_1(x,D)^*$

$$\begin{aligned} \|a_1(x,D)^*f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\|\sum_{j=0}^{\infty}\sum_{k=0}^{j-4}\varphi_j(D)\left(\overline{a_{kj}}\sum_{k'=j-3}^{j+3}f_{k'}\right)\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\ &= \left\|\sum_{j=0}^{\infty}\sum_{k=0}^{j-4}\varphi_j(D)\left(\overline{a_{kj}}f_j\right)\right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}. \end{aligned}$$

Here $f_j:=\varphi'_jf$ where φ'_j is a suitably chosen smooth function supported in the annulus $|\xi|\sim 2^j$.
Moreover we have

$$\mathrm{supp}\mathcal{F}\left\{\sum_{k=0}^{j-4}\varphi_j(D)\left(\overline{a_{kj}}f_j\right)\right\}\subset\left\{\xi\in\mathbb{R}^n:\ |\xi|\sim 2^j\right\}.$$

By applying lemma 7 and lemma 1 we obtain,

$$\|a_1(x,D)^*f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\|\left(2^{js(\cdot)}\check{\varphi}_j*\sum_{k=0}^{j-4}\overline{a_{kj}}f_j\right)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

$$\lesssim \left\| \left(2^{js(\cdot)} \eta_{j,m} * \left| \sum_{k=0}^{j-4} \overline{a_{kj}} f_j \right| \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

for any $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n \max\left\{0, \sup_{x\in\mathbb{R}^n} \left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$
Thus, by lemmas 2,

$$\|a_1(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(\eta_{j,m-C_{log}(s)} * \sum_{k=0}^{j-4} \left| \overline{a_{kj}} 2^{js(\cdot)} f_j \right| \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

Then by lemma 3 ,

$$\|a_1(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(\sum_{k=0}^{j-4} \left| \|\overline{a_{kj}}\|_{L^\infty} 2^{js(\cdot)} \varphi'_j(D) f \right| \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

Therefore

$$\|a_1(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}. \tag{15}$$

- The estimation of $a_2(x,D)^*$

$$\begin{aligned}
\|a_2(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\| \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} \varphi_j(D) \left(\overline{a_{kj}} \sum_{k'=0}^{j+6} f_{k'} \right) \right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\
&= \left\| \sum_{k=-3}^{3} \sum_{j=0}^{\infty} \varphi_j(D) \left(\overline{a_{(k+j)j}} \sum_{k'=0}^{j+6} f_{k'} \right) \right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\
&\lesssim \left\| \sum_{j=0}^{\infty} \varphi_j(D) \left(\overline{a_{(k+j)j}} \sum_{k'=0}^{j+6} f_{k'} \right) \right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}
\end{aligned}$$

One has

$$\text{supp } \mathcal{F} \left\{ \varphi_j(D) \left(\overline{a_{(k+j)j}} \sum_{k'=0}^{j+6} f_{k'} \right) \right\} \subset B(0, c2^{j+1}).$$

By lemma 8 ,

$$\begin{aligned}
\|a_2(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\| \left(2^{js(\cdot)} \varphi_j(D) \left(\overline{a_{(k+j)j}} \sum_{k'=0}^{j+6} f_{k'} \right) \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\
&\lesssim \left\| \left(\|\overline{a_{(k+j)j}}\|_{L^\infty} 2^{js(\cdot)} \eta_{j,m-C_{log}(s)} * \left| \sum_{k'=0}^{j+6} f_{k'} \right| \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}
\end{aligned}$$

for any $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n \max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$.
Then by lemma 3

$$\|a_2(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(2^{js(\cdot)} \sum_{k'=0}^{j+6} \varphi_{k'}(D) f \right)_j \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$$

Then

$$\|a_2(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}. \tag{16}$$

- The estimation of $a_3(x,D)^*$

$$\begin{aligned}
\|a_3(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\| \sum_{j=0}^{\infty} \sum_{k=j+4}^{\infty} \varphi_j(D) \left(\overline{a_{kj}} \sum_{k'=k-3}^{k+3} f_{k'} \right) \right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\
&= \left\| \sum_{k=4}^{\infty} \sum_{j=0}^{k-4} \varphi_j(D) \left(\overline{a_{kj}} f_j \right) \right\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}.
\end{aligned}$$

Here $f_j := \varphi'_j f$ where φ'_j is a suitably chosen smooth function supported in the annulus $|\xi| \sim 2^j$.
Moreover we have

$$\operatorname{supp}\mathcal{F}\left\{ \sum_{j=0}^{k-4} \varphi_j(D) \left(\overline{a_{kj}} f_j \right) \right\} \subset \left\{ \xi \in \mathbb{R}^n : \ |\xi| \sim 2^k \right\}.$$

Then by lemma 7 ,

$$\|a_3(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\| \left(2^{ks(\cdot)} \sum_{j=0}^{k-4} \varphi_j(D) \left(\overline{a_{kj}} f_j \right) \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

For any $m > n + c_{log}(s) + c_{log}(\frac{1}{q}) + n \max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$ we have

$$\begin{aligned}
\|a_3(x,D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\| \left(\sum_{j=0}^{k-4} \eta_{j,m-C_{log}(s)} * \left| \overline{a_{kj}} 2^{ks(\cdot)} \varphi'_j(D) f \right| \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \\
&\lesssim \left\| \left(\sum_{j=0}^{k-4} \left| \overline{a_{kj}} 2^{ks(\cdot)} \varphi'_j(D) f \right| \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}
\end{aligned}$$

$$\lesssim \left\| \left(\sum_{j=0}^{k-4} \|\overline{a_{kj}}\|_{L^\infty} \left| 2^{ks(\cdot)} \varphi_j'(D) f \right| \right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

The rest is the same as that of $a_3(x, D)$ in the proof of theorem 2.
We obtain

$$\|a_3(x, D)^* f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{N}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \tag{17}$$

The three estimates (15) , (16) and (17) yield the desired estimate. □

References

[1] A. Almeida and A. Caetano. Variable exponent besov-morrey spaces. *Journal of Fourier Analysis and Applications*, 2020.

[2] G. Bourdaud. Une algèbre maximale d'opérateurs pseudo-différentiels. *Comm. Partial Differential Equations*, 13(9):1059–1083, 1988.

[3] R. Coifman and Y. Meyer. *Au delà des opérateurs pseudo-différentiels.* Société Mathématique de France, Paris ., 1978.

[4] M. Congo and M. F. Ouedraogo. Boundedness of nonregular pseudo-differential operators on variable exponent triebel-lizorkin-morrey spaces. *ejpam*, 15(1):47–63, 2022.

[5] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue Spaces.* Birkhäuser, Basel., 2013.

[6] J. Xu J. Fu. Characterizations of morrey type besov and triebel-lizorkin spaces with variable exponents . *J. Math. Anal. Appl.*, 381:280–298, 2011.

[7] H. Kempka and J. Vybíral. Spaces of variable smoothness and integrability: Characterizations by local means and ball means of differences. *Fourier Anal. Appl.*, 18(4):852–891, 2012.

[8] H. Kozono and M. Yamazaki. Semilinear heat equations and the navier-stokes equation with distributions in new function spaces as initial data. *Comm. Partial Differential Equations*, 19:959–1014, 1994.

[9] P. Hästö L. Diening and S. Roudenko. Function spaces of variable smoothness and integrability. *Funct. Anal.*, 256(6):1731–1768, 2009.

[10] P. Hästö L. Diening, P. Harjulehto and M. Ruzicka. *Lebesgue and Sobolev Spaces with Variable Exponents.*, volume 2017. Springer-Verlag, Berlin, 2011.

[11] J. Marschall. Pseudodifferential operators with coefficients in sobolev spaces. *Trans. Amer. Math. Soc.*, 307(1):335–361, 1988.

[12] J. Marschall. Nonregular pseudo-differential operators. *Z. Anal. Anwend*, 15(1):109–148, 1996.

[13] A. Mazzucato. Besov-morrey spaces: Function space theory and applications to nonlinear pde. *Trans. Amer. Math. Soc.*, 355:1297–1364, 2003.

[14] T. Runst. Para-differential operators in spaces of triebel-lizorkin and besov typee. *Z. Anal. Anwendungen.*, 4:557–573, 1985.

[15] Y. Sawano. A note on besov-morrey spaces and triebel-lizorkin-morrey spaces. *Acta Math. Sin.*, 25:1223–1242, 2009.

[16] M. E. Taylor. *Partial Differential Equations III.* Springer-Verlag New York, Inc, 1996.

[17] H. Triebel. *Besov spaces with variable smoothness and integrability.* Birkhauser Verlag, Basel and al., 1983.

[18] W. Sickel W. Yuan and D. Yang. Morrey and campanato meet besov, lizorkin and triebel. lecture notes in mathematics. *Springer, Berlin.*, 2005, 2010.