



## Mathematical Analysis and Simulation of an Age-Structured Model with Two-Patch and an Uncontrolled Migration: Application to Tuberculosis

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**Abstract.** In this paper, we studied a two-patch age-structured model of tuberculosis in a context where the migration is not controlled. Motivated by the fact that no author has highlighted the impact of migration on the dynamics of transmission of tuberculosis. Each subpopulation is subdivided into five compartments: susceptible; latent, vaccinated, infective and treated. After the determination of the reproductive numbers  $\mathcal{R}(\psi, \rho)$  and  $\mathcal{R}_0(\rho)$ , we established the conditions of the global and local stability of the equilibrium point without disease. It has been shown that there is only one point of endemic equilibrium. Numerical simulations show that uncontrolled migration negatively influences the dynamics of tuberculosis.

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**Key Words and Phrases:** Age-structured, reproductive number, two-patch, tuberculosis, stability, migration, simulation, uncontrolled, endemic

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### 1. Introduction

Tuberculosis (TB) is one of the oldest diseases humanity has known, which is still relevant today; archaeological excavations in Pharaonic Egypt, ancient India and the Far East show that the date of its first “official” recognition was 2400 BC [8, 10, 16]. Despite enormous scientific advances and the availability of effective treatment, TB remains one of the top five killers worldwide. Understanding its propagation dynamics has become a concern for the entire scientific community. It is an infection caused by a bacterium called *Mycobacterium tuberculosis*. Tuberculosis most commonly affects the lungs, but can also affect other parts of the body, such as the skin, bones, lymph nodes, liver, digestive tract, and central nervous system [11]. Tuberculosis is transmitted from one person with active

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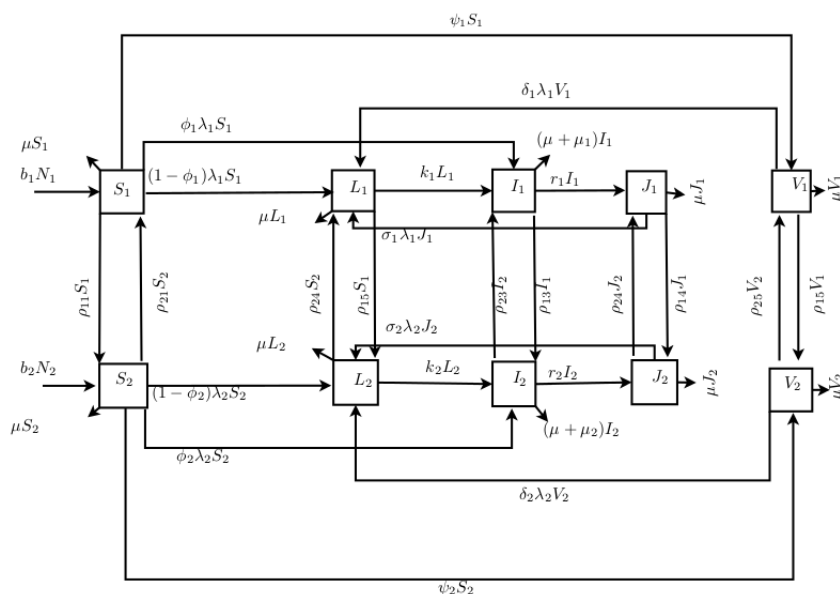
pulmonary tuberculosis to another when the latter is exposed to *M. tuberculosis*. Living conditions that contribute to a high risk of exposure to TB include overcrowded housing, homeless shelters and correctional facilities. According to WHO reports, the People at increased risk of exposure to tuberculosis include those with a history of alcohol and drug abuse and those from areas where TB is prevalent, including many countries in the Caribbean, Africa and Asia. Until the emergence of the COVID-19 pandemic (widely studied, among recent works we can cite [1, 17]), tuberculosis was the leading cause of death attributable to a single infectious agent, ahead of HIV/AIDS. According to the 2021 WHO report, most people (nearly 90%) who develop the disease are adults and more often men than women. Nearly a quarter of the world's population is infected with *M.tuberculosis*. Despite the fact tuberculosis is a preventable disease that can be cured. Several mathematical models have been developed in order to make contributions in the decision making of the strategies to carry out the control of this disease. We must, the first mathematical model dealing with the case of tuberculosis to Frost, He predicted that with the low and falling rate of transmission of tuberculous infection in the United States, it was most likely that the tubercle bacilli were fighting a losing battle, since they would be unable to transmit sufficient infection to maintain the balance in their own favor. (See [21, 22]. Nowadays, we have a fairly broad literature on the mathematical epidemiology of tuberculosis (We refer readers to the documents [2, 4, 18] and articles cited in these papers for a good overview of the work done in this field as the list is by no means exhaustive), but this work neglects a number of factors (either all or some of its parameters) such as the vaccination program, the compartment of the treaties, individuals with rapid progression to the state of the disease, the mortality rate related to the disease, the birth rate of the population in addition, very few of these models are structured in age and the migration is considered in very few works. On this subject Tewa J.J has dedicated two articles modeled by an EDO [13, 14]. These factors are far from negligible in African countries. Recently Marwin Ramli and al, studied the stability of the endemic equilibrium of a SIR model of tuberculosis where the compartment of the susceptible is subdivided into two subgroups, that of susceptible vaccinated and those of susceptible unvaccinated, unstructured by age [12],they made an extinction of this model by adding the latent compartment[20]; Rui Xu and al Motivated by the work of Yang and al [24] and Mc Cluskey [5], in their paper, they investigated the effects of incomplete treatment and age structure in latently infected and infectious individuals the dynamics of tuberculosis [23].Yu Zhao and al. analyzed a model type SEIR where the compartment of susceptible is subdivided into three age groups: childhood, middle-age and senior. Our model is an extension of the model proposed in [19], by the fact that migration is allowed without distinction of the epidemiological status of individuals.

This type of migration is undoubtedly one of the factors, delaying the process of eradicating this disease. To evaluate the impact of migration on the propagation dynamics of this disease, we expressed the reproduction numbers as a function of the migration rate  $\mathfrak{R}(\psi, \rho)$  and  $\mathfrak{R}_0(\rho)$ . After establishing the conditions of existence and uniqueness of the positive solutions of our model by the semi-group method. We studied the global and

local stability of the equilibrium point without disease, showed the existence of an endemic equilibrium point and finally the results of the numerical simulations confirm the negative effects of the uncontrolled migration that would cause tuberculosis in our community. This paper is organized as follows: Section 1 introduces the two-patch model structured in age to study the dynamics of TB transmission case of a non controlled migration. The mathematical model formulation is given in the section 2. The existence of positive and unique solutions is demonstrated in Section 3. The point of equilibrium without disease, reproductive numbers  $\mathfrak{R}(\psi, \rho)$  and  $\mathfrak{R}_0(\rho)$  are defined in the section 4 and the local stability of the disease-free equilibrium point in the presence of vaccine. The global stability of the disease-free equilibrium point to the absence of the vaccine is established in Section 5. The existence of an endemic equilibrium is proven in Section 6. Some numerical simulation results are given in Section 7. In Section 8, we have a discussion, conclusion and further work.

## 2. Mathematical model formulation

Two-patch age structured model of tuberculosis was considered. The model is to split the population into two subpopulations. The recruitment is only possible in the class of susceptible and the vaccinated individuals were able to migrate between the two subpopulations. Each subpopulation is divided into five classes based on their epidemiological status: susceptible, vaccinated, latent, infectious or treated. We denote these subgroups  $S_i(t, a)$ ,  $v_i(t, a)$ ,  $L_i(t, a)$ ,  $I_i(t, a)$  and  $J_i(t, a)$  respectively. The birth rate of the patch  $i$  is  $b_i(a)$ ;  $\mu_i(a)$  and  $\mu(a)$  denote the mortality rate related to the disease relative to the patch  $i$  and the rate of natural mortality. The time and age depended of the force of infection of the subpopulation  $i$  is  $\lambda_i(t, a)$  and vaccination rate is  $\psi_i(a)$ ;  $p_i(a, a')$  is the probability that an infective individual of age  $a'$  will have contact with and successfully infect a susceptible individual of age  $a$ ,  $c_i(a)$  is the age-specific per-capita contact/activity rate (all of these functions are assumed to be continuous and to be zero beyond some maximum age). A fraction  $\phi_i$  of newly infected individuals of the sub-population  $i$  is assumed to undergo a fast progression directly to the infectious class  $I_i$ . Rate of susceptible passage to latent infectious state and treatment are respectively  $k_i$  and  $r_i$ .  $\sigma_i$  and  $\delta_i$  are the reductions in risk due to prior exposure to TB and vaccination, respectively,  $0 \leq \sigma_i \leq (1 - \phi_i)$ ,  $0 \leq \delta_i \leq (1 - \phi_i)$ . The migration rates of the susceptible, latently infected, infectious, treated, and vaccinated between the two populations are respectively  $\rho_{i1}$ ,  $\rho_{i2}$ ,  $\rho_{i3}$ ,  $\rho_{i4}$  and  $\rho_{i5}$ , in this paper  $i = 1, 2$ .



**fig.1. Flow chart of the two-patch model for tuberculosis disease transmission.**

The age-structured model for the transmission of TB is described by the following system of partial differential equations:

$$\left\{ \begin{aligned}
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) S_1(t, a) &= b_1(a)N_1(t, a) - [\lambda_1(t, a) + \psi_1(a) + \mu(a) + \rho_{11}]S_1(t, a) + \rho_{21}S_2(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) L_1(t, a) &= \lambda_1(t, a)[(1 - \phi_1)S_1(t, a) + \sigma_1 J_1(t, a) + \delta_1 V_1(t, a)] - (k_1 + \mu(a) + \rho_{12})L_1(t, a) + \rho_{22}L_2(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) I_1(t, a) &= k_1 L_1(t, a) - (r_1 + \mu(a) + \mu_1(a) + \rho_{13})I_1(t, a) + \phi_1 \lambda_1(t, a)S_1(t, a) + \rho_{23}I_2(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) J_1(t, a) &= r_1 I_1(t, a) - (\sigma_1 \lambda_1(t, a) + \mu(a) + \rho_{14})J_1(t, a) + \rho_{24}J_2(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) V_1(t, a) &= \psi_1(a)S_1(t, a) + \rho_{25}V_2(t, a) - (\rho_{15} + \mu(a) + \delta_1 \lambda_1(t, a))V_1(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) S_2(t, a) &= b_2(a)N_2(t, a) - [\lambda_2(t, a) + \psi_2(a) + \mu(a) + \rho_{21}]S_2(t, a) + \rho_{11}S_1(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) L_2(t, a) &= \lambda_2(t, a)[(1 - \phi_2)S_2(t, a) + \sigma_2 J_2(t, a) + \delta_2 V_2(t, a)] - (k_2 + \mu(a) + \rho_{22})L_2(t, a) + \rho_{12}L_1(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) I_2(t, a) &= k_2 L_2(t, a) - (r_2 + \mu(a) + \mu_2(a) + \rho_{23})I_2(t, a) + \phi_2 \lambda_2(t, a)S_2(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) J_2(t, a) &= r_2 I_2(t, a) - (\sigma_2 \lambda_2(t, a) + \mu(a) + \rho_{24})J_2(t, a) + \rho_{14}J_1(t, a) \\
 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) V_2(t, a) &= \psi_2(a)S_2(t, a) + \rho_{15}V_1(t, a) - (\rho_{25} + \mu(a) + \delta_2 \lambda_2(t, a))V_2(t, a)
 \end{aligned} \right. \tag{1}$$

with initial and boundary conditions :

$$\left\{ \begin{aligned}
 S_i(t, 0) &= \int_{a_1}^{a_2} b_i(a)N_i(t, a)da \\
 L_i(t, 0) &= V_i(t, 0) = I_i(t, 0) = J_i(t, 0) = 0 \\
 S_i(0, a) &= S_{0i}(a); L_i(0, a) = L_{0i}(a); V_i(0, a) = V_{0i}(a) \\
 I_i(0, a) &= I_{0i}(a); J_i(0, a) = J_{0i}(a).
 \end{aligned} \right.$$

And  $\lambda_i(t, a) = \beta_i(a)c_i(a) \int_0^{a+} \frac{I_i(t, a')}{N_i(t, a')} p_i(a, a') da'$ , assume that

$$p_i(a, a') = g_i(a) \widehat{\beta}_i(a') \tag{2}$$

(see Dietz and Schenzle 1985 [9] ; Greenhalgh 1988 [7]).

Let  $N(t, a) = N_1(t, a) + N_2(t, a)$  and  $N_i(t, a) = \alpha_i N(t, a)$ , with  $\alpha_1 + \alpha_2 = 1$   
 $N(t, a) = S_1(t, a) + L_1(t, a) + I_1(t, a) + J_1(t, a) + V_1(t, a) + S_2(t, a) + L_2(t, a) + I_2(t, a) + J_2(t, a) + V_2(t, a)$ .

By summing equations of system (1) and (2), we obtain the following equations for the total population  $N(t, a)$ :

$$\begin{cases} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) N(t, a) = (b(a) - \mu(a))N(t, a) - \mu_1(a)I_1(t, a) - \mu_2(a)I_2(t, a) \\ N(t, 0) = \int_{a_1}^{a_2} b(a)N(t, a)da. \end{cases} \tag{3}$$

Where  $b(a) = \alpha_1 b_1(a) + \alpha_2 b_2(a)$ ;  $a_1$  and  $a_2$  are respectively the minimum and maximum age of procreation and  $a_+$  is the maximum age of an individual, with  $a_+ < +\infty$ .

Let

$$\begin{cases} s_i(t, a) = \frac{S_i(t, a)}{N(t, a)}; l_i(t, a) = \frac{L_i(t, a)}{N(t, a)}; i_i(t, a) = \frac{I_i(t, a)}{N(t, a)} \\ j_i(t, a) = \frac{J_i(t, a)}{N(t, a)}; v_i(t, a) = \frac{V_i(t, a)}{N(t, a)}. \end{cases} \tag{4}$$

The system (1) can be normalized as the following system:

$$\begin{cases} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) s_1(t, a) = \alpha_1 b_1(a) - [\lambda_1(t, a) + \psi_1(a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{11}s_1(t, a) + \rho_{21}s_2(t, a)] \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) l_1(t, a) = \lambda_1(t, a)[(1 - \phi_1)s_1(t, a) + \sigma_1 j_1(t, a) + \delta_1 v_1(t, a)] - (k_1 + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{12})l_1(t, a) + \rho_{22}l_2(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i_1(t, a) = -(r_1 + \mu_1(a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{13})i_1(t, a) + \rho_{23}i_2(t, a) + \phi_1 \lambda_1(t, a) s_1(t, a) + k_1 l_1(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) j_1(t, a) = r_1 i_1(t, a) - (\sigma_1 \lambda_1(t, a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{14})j_1(t, a) + \rho_{24}j_2(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) v_1(t, a) = \psi_1(a)s_1(t, a) + \rho_{25}v_2(t, a) - (b(a) + \rho_{15} + \delta_1 \lambda_1(t, a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a))v_1(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) s_2(t, a) = \alpha_2 b_2(a) - [\lambda_2(t, a) + \psi_2(a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{21})s_2(t, a) + \rho_{11}s_1(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) l_2(t, a) = \lambda_2(t, a)[(1 - \phi_2)s_2(t, a) + \sigma_2 j_2(t, a) + \delta_2 v_2(t, a)] - (k_2 + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{22})l_2(t, a) + \rho_{12}l_1(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i_2(t, a) = -(r_2 + \mu_2(a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{23})i_2(t, a) + \rho_{13}i_1(t, a) + \phi_2 \lambda_2(t, a) s_2(t, a) + k_2 l_2(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) j_2(t, a) = r_2 i_2(t, a) - (\sigma_2 \lambda_2(t, a) + b(a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a) + \rho_{24})j_2(t, a) + \rho_{14}j_1(t, a) \\ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) v_2(t, a) = \psi_2(a)s_2(t, a) + \rho_{15}v_1(t, a) - (b(a) + \rho_{25} + \delta_2 \lambda_2(t, a) - \mu_1(a)i_1(t, a) - \mu_2(a)i_2(t, a))v_2(t, a) \end{cases} \tag{5}$$

with boundary conditions

$$s_i(t, 0) = \Lambda_i; v_i(t, 0) = l_i(t, 0) = i_i(t, 0) = j_i(t, 0) = 0$$

$$\text{with } \Lambda_1 + \Lambda_2 = 1.$$

The problem is well posed, the method of proof is the same used in [3].

### 3. Existence of positive solutions

In this section we will prove that the system (5) has a unique positive solution, and to achieve this we will write the system (5) in compact form (abstract Cauchy problem).

Consider the Banach space  $X$  defined by  $X = (L^1(0, a_+))^{10}$ , endowed with the norm

$$\|\varphi\| = \sum_{i=1}^2 \sum_{j=1}^5 \|\varphi_{ij}\| \tag{6}$$

Where  $\varphi(a) = (\varphi_{11}(a), \varphi_{12}(a), \varphi_{13}(a), \varphi_{14}(a), \varphi_{15}(a), \varphi_{21}(a), \varphi_{22}(a), \varphi_{23}(a), \varphi_{24}(a), \varphi_{25}(a))^T \in X$  and  $\|\cdot\|$  is the norm of  $L^1(0, a_+)$ . Let denote by

$$\Omega = \{(s_1, l_1, i_1, j_1, v_1, s_2, l_2, i_2, j_2, v_2) \in X_+ \setminus 0 \leq s_1 + l_1 + i_1 + j_1, v_1 + s_2 + l_2 + i_2 + j_2 + v_2 \leq 1\} \tag{7}$$

The state space of system (5). Where  $X_+ = (L^1_+(0, a_+))^{10}$ , et  $L^1_+(0, a_+)$  denotes the positive cone of  $L^1(0, a_+)$ . Let  $A$  be a linear operator defined by

$$(A\varphi)(a) = (A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{21}, A_{22}, A_{23}, A_{24}, A_{25})^T \tag{8}$$

$$\left\{ \begin{array}{l} A_{11} = (-\frac{d}{da}\varphi_{11} - (\psi_1(a) + b(a) + \rho_{11})\varphi_{11}, 0, 0, 0, 0, \rho_{21}\varphi_{21}, 0, 0, 0, 0) \\ A_{12} = (0, -\frac{d}{da}\varphi_{12} - (b(a) + k_1 + \rho_{12})\varphi_{12}, 0, 0, 0, 0, \rho_{22}\varphi_{22}, 0, 0, 0) \\ A_{13} = (0, k_1\varphi_{12}, -\frac{d}{da}\varphi_{13} - (r_1 + \mu_1(a) + b(a) + \rho_{13})\varphi_{13}, 0, 0, 0, 0, \rho_{23}\varphi_{23}, 0, 0) \\ A_{14} = (0, 0, r_1\varphi_{13}, -\frac{d}{da}\varphi_{14} - (b(a) + \rho_{14})\varphi_{14}, 0, 0, 0, 0, \rho_{24}\varphi_{24}, 0) \\ A_{15} = (\psi_1(a)\varphi_{11}, 0, 0, 0, -\frac{d}{da}\varphi_{15} - (\rho_{15} + b(a))\varphi_{15}, 0, 0, 0, 0, \rho_{25}\varphi_{25}) \\ A_{21} = (\rho_{11}\varphi_{11}, 0, 0, 0, 0, -\frac{d}{da}\varphi_{21} - (\psi_2(a) + b(a) + \rho_{21})\varphi_{21}, 0, 0, 0, 0) \\ A_{22} = (0, \rho_{12}\varphi_{12}, 0, 0, 0, 0, -\frac{d}{da}\varphi_{22} - (b(a) + k_2 + \rho_{22})\varphi_{22}, 0, 0, 0) \\ A_{23} = (0, 0, \rho_{13}\varphi_{13}, 0, 0, 0, k_2\varphi_{22}, -\frac{d}{da}\varphi_{23} - (r_2 + \mu_2(a) + b(a) + \rho_{23})\varphi_{23}, 0, 0) \\ A_{24} = (0, 0, 0, \rho_{14}\varphi_{14}, 0, 0, 0, r_2\varphi_{23}, -\frac{d}{da}\varphi_{24} - (b(a) + \rho_{24})\varphi_{24}, 0) \\ A_{25} = (0, 0, 0, 0, \rho_{15}\varphi_{15}, \psi_2(a)\varphi_{21}, 0, 0, 0, -\frac{d}{da}\varphi_{25} - (\rho_{25} + b(a))\varphi_{25}) \end{array} \right. \tag{9}$$

With

$$\varphi(a) = (\varphi_{11}(a), \varphi_{12}(a), \varphi_{13}(a), \varphi_{14}(a), \varphi_{15}(a), \varphi_{21}(a), \varphi_{22}(a), \varphi_{23}(a), \varphi_{24}(a), \varphi_{25}(a))^T \in D(A)$$

where  $D(A)$  is the domain given by:

$$D(A) = \{\varphi \in X \setminus \varphi_{ij} \in AC[0, a_+), \varphi(0) = (\Lambda_1, 0, 0, 0, 0, 0, \Lambda_2, 0, 0, 0, 0)^T\}$$

And  $AC[0, a_+)$  denotes the set of absolutely continuous functions on  $[0, a_+)$ . We also define a nonlinear operator  $F : X \rightarrow X$  by :

$$(F\varphi)(a) = \begin{pmatrix} \alpha_1 b_1(a) - ((Q_1\varphi_{13})(a))\varphi_{11} + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{11} \\ ((Q_1\varphi_{13})(a))((1 - \phi_1)\varphi_{11} + \sigma_1\varphi_{14} + \delta_1\varphi_{15}) + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{12} \\ \phi_1((Q_1\varphi_{13})(a))\varphi_{11} + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{13} \\ (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23}) - \delta_1((Q_1\phi_{13})(a))\varphi_{14} \\ (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23}) - \sigma_1((Q_1\varphi_{13})(a))\varphi_{15} \\ \alpha_2 b_2(a) - ((Q_2\varphi_{23})(a))\varphi_{21} + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{21} \\ ((Q_2\varphi_{23})(a))((1 - \phi_2)\varphi_{21} + \sigma_2\varphi_{24} + \delta_2\varphi_{25}) + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{22} \\ \phi_2((Q_2\varphi_{23})(a))\varphi_{21} + (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23})\varphi_{23} \\ (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23}) - \delta_2((Q_2\phi_{23})(a))\varphi_{24} \\ (\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23}) - \sigma_2((Q_2\varphi_{23})(a))\varphi_{25} \end{pmatrix} \quad (10)$$

where  $Q_i$  is a bounded linear operator on  $L^1(0, a_+)$  given by

$$(Q_i f)(a) = c_i(a)\beta_i(a)g_i(a) \int_0^{a_+} \widehat{\beta}_i(a')f(a')da' \quad (11)$$

Let

$$u(t) = (s_1(\cdot, t), l_1(\cdot, t), i_1(\cdot, t), j_1(\cdot, t), v_1(\cdot, t), s_2(\cdot, t), l_2(\cdot, t), i_2(\cdot, t), j_2(\cdot, t), v_2(\cdot, t))$$

Thus, we can rewrite the system as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + F(u(t)) \\ u(0) = u_0 \end{cases} \quad (12)$$

where

$$u_0(a) = (s_{01}(a), l_{01}(a), i_{01}(a), j_{01}(a), v_{01}(a), s_{02}(a), l_{02}(a), i_{02}(a), j_{02}(a), v_{02}(a))^T.$$

According to these results we have the following results, ( see [6, 10]).

**Lemma 1.** *The operator  $F$  is continuously Fréchet differentiable on  $X$ .*

**Lemma 2.** *The operator  $A$  generates a  $C_0$ -semigroup of the bounded linear operators  $\{T(t)\}_{t \geq 0}$  and the space  $\Omega$  is positively invariant by  $\{T(t)\}_{t \geq 0}$ .*

**Theorem 1.** *For each  $u_0 \in X_+$  there are a maximal interval of existence  $[0, t_{max})$  and a unique continuous mild solution for (12) such that*

$$u(t) = u_0 e^{tA} + \int_0^t e^{A(t-\xi)} F(u(\xi)) d\xi$$

#### 4. The disease-free steady state

##### 4.1. Determination of the point of disease-free equilibrium

A steady state  $(s_1(a), l_1(a), i_1(a), j_1(a), v_1(a), s_2(a), l_2(a), i_2(a), j_2(a), v_2(a))$  of system (5) must satisfy the following time-independent system of ordinary differential equations:

$$\left\{ \begin{array}{l} \frac{d}{da} s_1(a) = \alpha_1 b_1(a) - [\beta_1(a)c_1(a)g_1(a)\Gamma_1 + \psi_1(a) + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{11}]s_1(a) + \rho_{21}s_2(a) \\ \frac{d}{da} l_1(a) = \beta_1(a)c_1(a)g_1(a)\Gamma_1[(1 - \phi_1)s_1(a) + \delta_1 v_1(a) + \sigma_1 j_1(a)] \\ \quad - (b(a) + k_1 - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{12})l_1(a) + \rho_{22}l_2(a) \\ \frac{d}{da} i_1(a) = k_1 l_1(a) + \phi_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1 s_1(a) - [r_1 + b(a) + \mu_1(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{13}]i_1(a) + \rho_{23}i_2(a) \\ \frac{d}{da} j_1(a) = r_1 i_1(a) - (\sigma_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{14})j_1(a) + \rho_{24}j_2(a) \\ \frac{d}{da} v_1(a) = \psi_1(a)s_1(a) - (\delta_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{15})v_1(a) + \rho_{25}v_2(a) \\ \frac{d}{da} s_2(a) = b_2(a) - [\beta_2(a)c_2(a)g_2(a)\Gamma_2 + \psi_2(a) + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{21}]s_2(a) + \rho_{11}s_1(a) \\ \frac{d}{da} l_2(a) = \beta_2(a)c_2(a)g_2(a)\Gamma_2[(1 - \phi_2)s_2(a) + \delta_2 v_2(a) + \sigma_2 j_2(a)] - (b(a) + k_2 - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{22})l_2(a) + \rho_{12}l_1(a) \\ \frac{d}{da} i_2(a) = k_2 l_2(a) + \phi_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2 s_2(a) - [r_2 + b(a) + \mu_2(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{23}]i_2(a) + \rho_{13}i_1(a) \\ \frac{d}{da} j_2(a) = r_2 i_2(a) - (\sigma_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{24})j_2(a) + \rho_{14}j_1(a) \\ \frac{d}{da} v_2(a) = \psi_2(a)s_2(a) + \rho_{15}v_1(a) - (\delta_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_{25})v_2(a) \\ \Gamma_i = \frac{1}{\alpha_i} \int_0^{a+} \widehat{\beta}_i(a)i_i(a) da \end{array} \right. \tag{13}$$

with initial value conditions

$$s_i(0) = \Lambda_i; l_i(0) = i_i(0) = j_i(0) = v_i(0) = 0$$

Therefore, we obtain the disease-free steady state

$$\left\{ \begin{array}{l} s_i^0(a) = \Lambda_i e^{-\int_0^a (b(\tau) + \psi_i(\tau) + \rho_{i1}) d\tau} + \int_0^a e^{-\int_\eta^a (b(\tau) + \psi_i(\tau) + \rho_{i1}) d\tau} (\alpha_i b_i(\eta) + \rho_{j1} s_j^0(a)) d\eta \\ v_i^0(a) = \frac{\Lambda_i - s_i^0(a) + \rho_{j5} v_j^0(a)}{1 - \rho_{i5}}; l_i^0(a) = i_i^0(a) = j_i^0(a) = 0 \end{array} \right. \tag{14}$$



### 4.2. Calculation of $\mathfrak{R}(\psi, \rho)$ - $\mathfrak{R}_0(\rho)$ and stability of the infection-free state

To study the stability of the disease-free steady state, we denote the perturbations of system by

$$\left\{ \begin{array}{l} s_i(t, a) = \bar{s}_i(t, a) + s_i^0(a) \\ l_i(t, a) = \bar{l}_i(t, a) + l_i^0(a) \\ i_i(t, a) = \bar{i}_i(t, a) + i_i^0(a) \\ j_i(t, a) = \bar{j}_i(t, a) + j_i^0(a) \\ v_i(t, a) = \bar{v}_i(t, a) + v_i^0(a) \end{array} \right. \quad (15)$$

The perturbations satisfy the following equations:

$$\left\{ \begin{array}{l} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{s}_1(t, a) = \rho_{21}\bar{s}_2(a) - (b(a) + \psi_1(a) + \rho_{11})\bar{s}_1(t, a) - [\bar{\gamma}_1(t)\beta_1(a)c_1(a)g_1(a) - \mu_1(a)\bar{i}_1(t, a) - \mu_2\bar{i}_2(t, a)]s_1^0(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{l}_1(t, a) = \rho_{22}\bar{l}_2(a) - (b(a) + k_1 + \rho_{12})\bar{l}_1(t, a) + \bar{\gamma}_1(t)\beta_1(a)c_1(a)g_1(a)[(1 - \phi_1)s_1^0(a) + \delta_1v_1^0(a)] \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{i}_1(t, a) = \rho_{23}\bar{i}_2(a) + k_1\bar{l}_1(t, a) + \phi_1\beta_1(a)c_1(a)g_1(a)\bar{\gamma}_1(t)s_1^0(a) - (r_1 + \mu_1(a) + b(a) + \rho_{13})\bar{i}_1(t, a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{j}_1(t, a) = r_1\bar{i}_1(t, a) + \rho_{24}\bar{j}_2(a) - (\rho_{14} + b(a))\bar{j}_1(t, a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{v}_1(t, a) = \psi_1(a)\bar{s}_1(t, a) + \rho_{25}\bar{v}_2(t, a) - (\rho_{15} + b(a))\bar{v}_1(t, a) \\ \quad - (\delta_1\beta_1(a)c_1(a)g_1(a)\bar{\gamma}_1(t) - \mu_1(a)\bar{i}_1(t, a) - \mu_2(a)\bar{i}_2(t, a))v_1^0(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{s}_2(t, a) = \rho_{11}\bar{s}_1(a) - (b(a) + \psi_2(a) + \rho_{21})\bar{s}_2(t, a) - [\bar{\gamma}_2(t)\beta_2(a)c_2(a)g_2(a) - \mu_1(a)\bar{i}_1(t, a) - \mu_2\bar{i}_2(t, a)]s_2^0(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{l}_2(t, a) = \rho_{12}\bar{l}_1(a) - (b(a) + k_2 + \rho_{22})\bar{l}_2(t, a) + \bar{\gamma}_2(t)\beta_2(a)c_2(a)g_2(a)[(1 - \phi_2)s_2^0(a) + \delta_2v_2^0(a)] \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{i}_2(t, a) = \rho_{13}\bar{i}_1(a) + k_2\bar{l}_2(t, a) + \phi_2\beta_2(a)c_2(a)g_2(a)\bar{\gamma}_2(t)s_2^0(a) - (r_2 + \mu_2(a) + b(a) + \rho_{23})\bar{i}_2(t, a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{j}_2(t, a) = \rho_{14}\bar{j}_1(a) + r_2\bar{i}_2(t, a) - (\rho_{24} + b(a))\bar{j}_2(t, a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \bar{v}_2(t, a) = \psi_2(a)\bar{s}_2(t, a) + \rho_{15}\bar{v}_1(t, a) - (\rho_{25} + b(a))\bar{v}_2(t, a) \\ \quad - (\delta_2\beta_2(a)c_2(a)g_2(a)\bar{\gamma}_2(t) - \mu_1(a)\bar{i}_1(t, a) - \mu_2(a)\bar{i}_2(t, a))v_2^0(a) \\ \bar{\gamma}_i(t) = \frac{1}{\alpha_i} \int_0^{a+} \widehat{\beta}_i(a)\bar{i}_i(t, a)da \end{array} \right. \quad (16)$$

with boundary conditions :

$$\bar{s}_i(t, 0) = \bar{l}_i(t, 0) = \bar{i}_i(t, 0) = \bar{j}_i(t, 0) = \bar{v}_i(t, 0) = 0$$

we consider the exponential solutions of system (16) of the form:

$$\left\{ \begin{array}{l} \bar{s}_i(t, a) = \bar{s}_i(a)e^{\lambda t}; \bar{l}_i(t, a) = \bar{l}_i(a)e^{\lambda t}; \bar{v}_i(t, a) = \bar{v}_i(a)e^{\lambda t} \\ \bar{i}_i(t, a) = \bar{i}_i(a)e^{\lambda t}; \bar{j}_i(t, a) = \bar{j}_i(a)e^{\lambda t} \end{array} \right. \quad (17)$$

The system (16) becomes:

$$\left\{ \begin{aligned} \frac{d}{da} \bar{s}_1(a) &= \rho_{21} \bar{s}_2(a) - (b(a) + \psi_1(a) + \lambda + \rho_{11}) \bar{s}_1(a) - [\bar{\Gamma}_1 \beta_1(a) c_1(a) g_1(a) - \mu_1(a) \bar{i}_1(a) - \mu_2(a) \bar{i}_2(a)] s_1^0(a) \\ \frac{d}{da} \bar{l}_1(a) &= \rho_{22} \bar{l}_2(a) - (b(a) + k_1 + \lambda + \rho_{12}) \bar{l}_1(a) + \bar{\Gamma}_1 \beta_1(a) c_1(a) g_1(a) [(1 - \phi_1) s_1^0(a) + \delta_1 v_1^0(a)] \\ \frac{d}{da} \bar{i}_1(a) &= \rho_{23} \bar{i}_2(a) + \phi_1 \beta_1(a) c_1(a) g_1(a) \bar{\Gamma}_1 + k_1 \bar{l}_1(a) - (r_1 + \mu_1(a) + b(a) + \lambda + \rho_{13}) \bar{i}_1(a) \\ \frac{d}{da} \bar{j}_1(a) &= \rho_{24} \bar{j}_2(a) + r_1 \bar{i}_1(a) - (b(a) + \lambda + \rho_{14}) \bar{j}_1(a) \\ \frac{d}{da} \bar{v}_1(a) &= \psi_1(a) \bar{s}(a) + \rho_{25} \bar{v}_2(a) - (\rho_{15} + b(a) + \lambda) \bar{v}_1(a) - (\delta_1 \bar{\Gamma} \beta_1(a) c_1(a) g_1(a) - \mu_1(a) \bar{i}_1(a) - \mu_2(a) \bar{i}_2(a)) v_1^0(a) \\ \frac{d}{da} \bar{s}_2(a) &= \rho_{11} \bar{s}_1(a) - (b(a) + \psi_2(a) + \lambda + \rho_{21}) \bar{s}_2(a) - [\bar{\Gamma}_2 \beta_2(a) c_2(a) g_2(a) - \mu_1(a) \bar{i}_1(a) - \mu_2(a) \bar{i}_2(a)] s_2^0(a) \\ \frac{d}{da} \bar{l}_2(a) &= \rho_{12} \bar{l}_1(a) - (b(a) + k_2 + \lambda + \rho_{22}) \bar{l}_2(a) + \bar{\Gamma}_2 \beta_2(a) c_2(a) g_2(a) [(1 - \phi_2) s_2^0(a) + \delta_2 v_2^0(a)] \\ \frac{d}{da} \bar{i}_2(a) &= \rho_{13} \bar{i}_1(a) + \phi_2 \beta_2(a) c_2(a) g_2(a) \bar{\Gamma}_2 + k_2 \bar{l}_2(a) - (r_2 + \mu_2(a) + b(a) + \lambda + \rho_{13}) \bar{i}_2(a) \\ \frac{d}{da} \bar{j}_2(a) &= \rho_{14} \bar{j}_1(a) + r_2 \bar{i}_2(a) - (b(a) + \lambda + \rho_{14}) \bar{j}_2(a) \\ \frac{d}{da} \bar{v}_2(a) &= \psi_2(a) \bar{s}_2(a) + \rho_{15} \bar{v}_1(a) - (\rho_{25} + b(a) + \lambda) \bar{v}_2(a) - (\delta_2 \bar{\Gamma} \beta_2(a) c_2(a) g_2(a) - \mu_1(a) \bar{i}_1(a) - \mu_2(a) \bar{i}_2(a)) v_2^0(a) \\ \bar{\Gamma}_i &= \frac{1}{\alpha_i} \int_0^a \widehat{\beta}_i(a) \bar{i}_i(a) da \end{aligned} \right. \tag{18}$$

with boundary conditions :

$$\bar{s}_i(0) = \bar{l}_i(0) = \bar{i}_i(0) = \bar{j}_i(0) = \bar{v}_i(0) = 0$$

Let

$$N_{\psi_i}(a) = (1 - \phi_i) s_i^0(a) + \delta_i v_i^0(a) \tag{19}$$

$$\rho_{j2} \bar{l}_j(a) = \widehat{\rho}_{j2} \bar{l}_i(a) \text{ and } \rho_{j3} \bar{i}_j(a) = \widehat{\rho}_{j3} \bar{i}_i(a) \tag{20}$$

From equation (18) and (19), we obtain:

$$\bar{l}_i(a) = \bar{\Gamma}_i \int_0^a e^{-\int_\eta^a (b(\tau) + k_i + \rho_{i2} - \widehat{\rho}_{j2} + \lambda) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) N_{\psi_i}(\eta) d\eta \tag{21}$$

$$\bar{i}_i(a) = \int_0^a e^{-\int_\eta^a (b(\tau) + r_i + \lambda + \mu_i(a) + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} (k_i \bar{l}_i(\eta) + \phi_i \bar{\Gamma}_i \beta_i(\eta) c_i(\eta) g_i(\eta)) d\eta \tag{22}$$

Hence, by equations (21) and (22) after changing order of integration, we obtain:

$$\begin{aligned} \bar{i}_i(a) &= \bar{\Gamma}_i \int_0^a e^{-\int_\eta^a (b(\tau) + \mu_i(\tau) + r_i + \lambda + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) \\ &[\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta) \int_\eta^a e^{-\int_x^\eta (\mu_i(\tau) + r_i - k_i - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} dx] d\eta \end{aligned} \tag{23}$$

Injecting (23) in the expression of  $\bar{\Gamma}_i$ , and dividing both sides the expression by  $\bar{\Gamma}_i$  (since  $\bar{\Gamma}_i \neq 0$ ), we get the characteristic equation:

$$1 = \frac{1}{\alpha_i} \int_0^{a+} \hat{\beta}_i(a) \int_0^a e^{-\int_{\eta}^a (b(\tau)+\mu_i(\tau)+r_i+\lambda+\rho_{i3}-\hat{\rho}_{j3})d\tau} \beta_i(\eta)c_i(\eta)g_i(\eta) [\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta) \int_{\eta}^a e^{-\int_x^{\eta} (\mu_i(\tau)+r_i-k_i-\rho_{i2}+\hat{\rho}_{j2}+\rho_{i3}-\hat{\rho}_{j3})d\tau} dx] d\eta da \tag{24}$$

Denote the right-hand side of equation (24) by  $G_{ij}(\lambda)$  i.e :

$$G_{ij}(\lambda) = \frac{1}{\alpha_i} \int_0^{a+} \hat{\beta}_i(a) \int_0^a e^{-\int_{\eta}^a (b(\tau)+\mu_i(\tau)+r_i+\lambda+\rho_{i3}-\hat{\rho}_{j3})d\tau} \beta_i(\eta)c_i(\eta)g_i(\eta) [\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta) \int_{\eta}^a e^{-\int_x^{\eta} (\mu_i(\tau)+r_i-k_i-\rho_{i2}+\hat{\rho}_{j2}+\rho_{i3}-\hat{\rho}_{j3})d\tau} dx] d\eta da \tag{25}$$

We define the net reproductive number as  $\mathfrak{R}^i(\psi, \rho_{ij}) = G_i(0)$ , i.e

$$\mathfrak{R}^i(\psi_i, \rho_{ij}) = \frac{1}{\alpha_i} \int_0^{a+} \hat{\beta}_i(a) \int_0^a e^{-\int_{\eta}^a (b(\tau)+\mu_i(\tau)+r_i+\rho_{i3}-\hat{\rho}_{j3})d\tau} \beta_i(\eta)c_i(\eta)g_i(\eta) [\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta) \int_{\eta}^a e^{-\int_x^{\eta} (\mu_i(\tau)+r_i-k_i-\rho_{i2}+\hat{\rho}_{j2}+\rho_{i3}-\hat{\rho}_{j3})d\tau} dx] d\eta da \tag{26}$$

Where

$$\rho_{ij} = (\rho_{i2}, \hat{\rho}_{j2}, \rho_{i3}, \hat{\rho}_{j3})$$

We can obtain an expression for  $\mathfrak{R}_0^i(\psi_i, \rho_{ij})$  in a similar way as the derivation of  $\mathfrak{R}^i(\psi, \rho_{ij})$  by considering Equation (1) without vaccination; i.e., by assuming that  $\psi_i(a) \equiv 0$  and neglecting the equation of vaccinated. It can be shown that  $\mathfrak{R}_0^i(\rho_{ij}) = \mathfrak{R}^i(0, \rho_{ij})$  which is called the basic reproductive number (when a purely susceptible population is considered) (see [3]).

$$\mathfrak{R}_0^i(\rho_{ij}) = \frac{\Lambda_i}{\alpha_i} \int_0^{a+} \hat{\beta}_i(a) \int_0^a e^{-\int_{\eta}^a (b(\tau)+\mu_i(\tau)+r_i+\rho_{i3}-\hat{\rho}_{j3})d\tau} \beta_i(\eta)c_i(\eta)g_i(\eta) [\phi_i + k_i(1 - \phi_i) \int_{\eta}^a e^{-\int_x^{\eta} (\mu_i(\tau)+r_i-k_i-\rho_{i2}+\hat{\rho}_{j2}+\rho_{i3}-\hat{\rho}_{j3})d\tau} dx] d\eta da \tag{27}$$

Let

$$\mathfrak{R}(\psi, \rho) = \max_{i,j} \mathfrak{R}^i(\psi_i, \rho_{ij}) \quad \text{and} \quad \mathfrak{R}_0(\rho) = \max_{i,j} \mathfrak{R}_0^i(\rho_{ij})$$

The infectious and latent individuals' migration has a meaningful influence on the spreading of the disease because of the expression of the so many reproduction.

### 4.3. Local stability of the infection-free equilibrium

**Theorem 2.** *The infection-free steady-state (5) is locally asymptotically stable (l.a.s.) if  $\mathfrak{R}(\psi, \rho) < 1$  and unstable if  $\mathfrak{R}(\psi, \rho) > 1$ .*

*Proof.* Noticing that

$$G'_{ij}(\lambda) < 0; \quad \lim_{\lambda \rightarrow +\infty} G_{ij}(\lambda) = 0; \quad \lim_{\lambda \rightarrow -\infty} G_{ij}(\lambda) = +\infty.$$

We know that equation (23) has a unique negative real solution  $\lambda^*$  if, and only if,  $G_{ij}(0) < 1$ , hence,  $\Re^i(\psi_i, \rho_{ij}) < 1$  ( Also, equation (23) has a unique positive (zero) real solution if  $\Re^i(\psi_i, \rho_{ij}) > 1$  ( $\Re^i(\psi_i, \rho_{ij}) = 1$ ). To show that  $\lambda^*$  is the dominant real part of roots of  $G_{ij}(\lambda)$ , we let  $\lambda = x + iy$  be an arbitrary complex solution to equation (23). Note that

$$1 = G_{ij}(\lambda) = |G_{ij}(x + iy)| \leq G_{ij}(x),$$

indicating that  $R_e(\lambda) \leq \lambda^*$ . It follows that the infection-free steady state is l.a.s. if  $\Re(\psi, \rho) < 1$ , and unstable if  $\Re(\psi, \rho) > 1$ .

### 5. Global stability of the infection-free state

Since  $\mu_i(a)$  and  $i_i(t, a)$  are bounded, there exists a positif constant  $R_c$  that satisfies

$$0 \leq \int_{\eta}^a \sum_{i=1}^2 \mu_i(\tau) i_i(t - a + \tau, \tau) d\tau \leq R_c \tag{28}$$

**Theorem 3.** *The disease-free equilibrium of system (5) is globally asymptotically stable if  $\Re_0(\rho) < 1$  and  $R_c < \ln(\frac{1}{\Re_0(\rho)})$ .*

*Proof.*

The proof consist to show that

$$i_i(t, a) \longrightarrow 0; \quad j_i(t, a) \longrightarrow 0; \quad l_i(t, a) \longrightarrow 0;$$

$$s_i(t, a) \longrightarrow s_i^0(a) \quad \text{and} \quad v_i(t, a) \longrightarrow \Lambda_i - s_i^0(a), \quad \text{when } t \rightarrow +\infty$$

Integrating system (5) along characteristic lines we get

$$l_i(t, a) = \int_0^a e^{-\int_{\eta}^a (b(\tau) - \mu_i(\tau) i_i(t-a+\tau, \tau) - \mu_j(\tau) j_j(t-a+\tau, \tau) d\tau + k_i + \rho_{i2} - \widehat{\rho}_{j2}) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) \lambda_i(t-a+\eta) \times$$

$$[\sigma_i j_i(t-a+\eta, \eta) + \delta_i v_i(t-\eta+a, \eta) + (1 - \phi_i) s_i(t-a+\eta, \eta)] d\eta, \quad a < t \tag{29}$$

$$i_i(t, a) = \int_0^a e^{-\int_{\xi}^a (b(\tau) - \mu_i(\tau) i_i(t-a+\tau, \tau) - \mu_j(\tau) j_j(t-a+\tau, \tau) + r_i + \mu_i(\tau) + \rho_{i3} - \widehat{\rho}_{j3}) d\tau}$$

$$[\phi_i \beta_i(\xi) c_i(\xi) g_i(\xi) \lambda_i(t-a+\xi) + k_i l(t-a+\xi, \xi)] d\xi, \quad a < t \tag{30}$$

Injecting (29) in (30), and changing order of integration, we obtain:

$$\begin{aligned}
 i_i(t, a) = & \int_0^a e^{-\int_\eta^a (\mu_i(\tau) - \mu_i(\tau) i_i(t-a+\tau, \tau) - \mu_j(\tau) i_j(t-a+\tau, \tau) r_i + \rho_{i2} - \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) \lambda_i(t-a+\eta) \times \\
 & [\phi_i s_i(t-a+\eta, \eta) + k_i(\sigma_i j_i(t-a+\eta, \eta) + \delta_i v_i(t-\eta+a, \eta) \\
 & + (1-\phi_i) s_i(t-a+\eta, \eta) \int_\eta^a e^{-\int_\xi^\eta (r_i + \mu_i(\tau) - k_i - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} d\xi] d\eta \tag{31}
 \end{aligned}$$

Injecting (31) in  $\lambda_i(t)$ , and changing order of integration, we obtain:

$$\begin{aligned}
 \lambda_i(t) = & \int_0^a e^{-\int_\eta^a (\mu_i(\tau) - \mu_i(\tau) i_i(t-a+\tau, \tau) - \mu_j(\tau) i_j(t-a+\tau, \tau) r_i + \rho_{i2} - \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) \lambda_i(t-a+\eta) \times \\
 & [\phi_i s_i(t-a+\eta, \eta) + k_i(\sigma_i j_i(t-a+\eta, \eta) + \delta_i v_i(t-\eta+a, \eta) \\
 & + (1-\phi_i) s_i(t-a+\eta, \eta) \int_\eta^a e^{-\int_\xi^\eta (r_i + \mu_i(\tau) - k_i - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} d\xi] d\eta \tag{32}
 \end{aligned}$$

It easy see that

$$\phi_i s_i(t-a+\eta, \eta) + k_i(\sigma_i j_i(t-a+\eta, \eta) + \delta_i v_i(t-\eta+a, \eta) + (1-\phi_i) s_i(t-a+\eta, \eta) \leq \Lambda_i [\phi_i + k_i(1-\phi_i)]$$

By using inequality (28) and Fatou's lemma, we have

$$\lim_{t \rightarrow +\infty} \lambda_i(t) \leq e^{Rc} \mathfrak{R}_0^i(\rho_i) \limsup_{t \rightarrow +\infty} \lambda_i(t).$$

Since  $e^{Rc} \mathfrak{R}_0^i(\rho_i) < 1$ ,  $\Rightarrow \limsup_{t \rightarrow +\infty} \lambda_i(t) = 0 \Rightarrow$

$$\begin{cases} \lim_{t \rightarrow +\infty} i_i(t, a) = \lim_{t \rightarrow +\infty} j_i(t, a) = \lim_{t \rightarrow +\infty} l_i(t, a) = 0 \\ \lim_{t \rightarrow +\infty} s_i(t, a) = s_i^0(a) \lim_{t \rightarrow +\infty} v_i(t, a) = \frac{\Lambda_i - s_i^0(a) + \rho_{j5} v_j^0(a)}{1 - \rho_{i5}} \end{cases}$$

For this disease can disappear without any form of intervention, according to this result we must ensure that there is no new infected and the infectious rate does not reach a certain spread.

### 6. Existence of an endemic state

**Theorem 4.** *An interior endemic equilibrium of the form*

$E^* = (s_1^*(a), l_1^*(a), i_1^*(a), j_1^*(a), v_1^*(a), s_2^*(a), l_2^*(a), i_2^*(a), j_2^*(a), v_2^*(a))$  whenever  $\mathfrak{R}^1(\psi_1) > 1$  and  $\mathfrak{R}^2(\psi_2) > 1$ . which corresponds to case when the disease persists in the two sub-populations.

*Proof.*  $E^* = (s_1^*(a), l_1^*(a), i_1^*(a), j_1^*(a), v_1^*(a), s_2^*(a), l_2^*(a), i_2^*(a), j_2^*(a), v_2^*(a))$  satisfies the following equations

$$\left\{ \begin{aligned} \frac{d}{da} s_1^*(a) &= \alpha_i b_1(a) - [\beta_1(a)c_1(a)g_1(a)\Gamma_1^* + \psi_1(a) + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{11}]s_1^*(a) + \rho_{21}s_2^*(a) \\ \frac{d}{da} l_1^*(a) &= \beta_1(a)c_1(a)g_1(a)\Gamma_1^*[(1 - \phi_1)s_1^*(a) + \delta_1 v_1^*(a) + \sigma_1 j_1^*(a)] \\ &\quad - (b(a) + k_1 - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{12})l_1^*(a) + \rho_{22}l_2^*(a) \\ \frac{d}{da} i_1^*(a) &= k_1 l_1^*(a) + \phi_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1^* s_1^*(a) - [r_1 + b(a) + \mu_1(a) - \mu_1(a)i_1^*(a) - \mu_1(a)i_2^*(a) + \rho_{13}]i_1^*(a) + \rho_{23}i_2^*(a) \\ \frac{d}{da} j_1^*(a) &= r_1 i_1^*(a) - (\sigma_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1^* + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{14})j_1^*(a) + \rho_{24}j_2^*(a) \\ \frac{d}{da} v_1^*(a) &= \psi_1(a)s_1^*(a) + \rho_{25}v_2^*(a) - (\delta_1 \beta_1(a)c_1(a)g_1(a)\Gamma_1^* + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{15})v_1^*(a) \\ \frac{d}{da} s_2^*(a) &= \alpha_2 b_2(a) - [\beta_2(a)c_2(a)g_2(a)\Gamma_2^* + \psi_2(a) + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{21}]s_2^*(a) + \rho_{11}s_1^*(a) \\ \frac{d}{da} l_2^*(a) &= \beta_2(a)c_2(a)g_2(a)\Gamma_2^*[(1 - \phi_2)s_2^*(a) + \delta_2 v_2^*(a) + \sigma_2 j_2^*(a)] \\ &\quad - (b(a) + k_2 - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{22})l_2^*(a) + \rho_{12}l_1^*(a) \\ \frac{d}{da} i_2^*(a) &= k_2 l_2^*(a) + \phi_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2^* s_2^*(a) - [r_2 + b(a) + \mu_2(a) - \mu_1(a)i_1^*(a) - \mu_1(a)i_2^*(a) + \rho_{23}]i_2^*(a) + \rho_{13}i_1^*(a) \\ \frac{d}{da} j_2^*(a) &= r_2 i_2^*(a) - (\sigma_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2^* + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{24})j_2^*(a) + \rho_{14}j_1^*(a) \\ \frac{d}{da} v_2^*(a) &= \psi_2(a)s_2^*(a) + \rho_{15}v_1^*(a) - (\delta_2 \beta_2(a)c_2(a)g_2(a)\Gamma_2^* + b(a) - \mu_1(a)i_1^*(a) - \mu_2(a)i_2^*(a) + \rho_{25})v_2^*(a) \\ \Gamma_i^* &= \int_0^a \widehat{\beta}_i(a)i_i^*(a)da \end{aligned} \right. \tag{33}$$

$$s_i^*(0) = \Lambda_i; \quad l_i^*(0) = i_i^*(0) = j_i^*(0) = v_i^*(0) = 0 \tag{34}$$

$$s_i^*(a) = \Lambda_i e^{-\int_0^a [\beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma_i^* + b(\tau) - \mu_1(\tau)i_1^*(\tau) - \mu_2(\tau)i_2^*(\tau) + \rho_{i1} - \widehat{\rho}_{j1}]d\tau}$$

$$+ \alpha_i \int_0^a b_i(\eta) e^{-\int_\eta^a [\beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma_i^* + b(\tau) - \mu_1(\tau)i_1^*(\tau) - \mu_2(\tau)i_2^*(\tau) + \rho_{i1} - \widehat{\rho}_{j1}]d\tau} d\eta \tag{35}$$

Let

$$h_i^*(\eta, \Gamma_i^*) = (1 - \phi_i)s_i^*(\eta) + \delta_i v_i^*(\eta) + \sigma_i j_i^*(\eta) \tag{36}$$

$$l_i^*(a) = \Gamma_i^* \int_0^a \beta_i(\eta)c_i(\eta)g_i(\eta)h_i(\eta, \Gamma_i^*)e^{-\int_\eta^a (b(\tau) - \mu_1(\tau)i_1^*(\tau) - \mu_2(\tau)i_2^*(\tau) + \rho_{i2} - \widehat{\rho}_{j2})d\tau} d\eta \tag{37}$$

$$i_i^*(a) = \int_0^a [k_i l_i^*(\eta) + \phi_i \beta_i(\eta)c_i(\eta)g_i(\eta)\Gamma_i^* s_i^*(\eta)]e^{-\int_\eta^a (b(\tau) + r_i + \mu_i(\tau) - \mu_1(\tau)i_1^*(\tau) - \mu_2(\tau)i_2^*(\tau) + \rho_{i3} - \widehat{\rho}_{j3})d\tau} d\eta \tag{38}$$

$$j_i^*(a) = r_i \int_0^a i_i^*(\eta) e^{-\int_\eta^a (\sigma_i \beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma_i^* + b(\tau) - \mu_1(\tau)i_1^*(\tau) - \mu_2(\tau)i_2^*(\tau) + \rho_{i4} - \widehat{\rho}_{j4})d\tau} d\eta \tag{39}$$

$$v_i^*(a) = \int_0^a (\psi_i(\eta) s_i^*(\eta)) e^{-\int_\eta^a (\delta_i \beta_i(\tau) c_i(\tau) g_i(\tau) \Gamma_i^* + b(\tau) - \mu_i(\tau) i_i^*(\tau) - \mu_j(\tau) i_j^*(\tau) + \rho_{i5} - \widehat{\rho}_{j5}) d\tau} d\eta \quad (40)$$

By injecting (37) in (38), we obtain :

$$i_i^*(a) = \Gamma_i^* \int_0^a \beta_i(\eta) c_i(\eta) g_i(\eta) e^{-\int_\eta^a (b(\tau) - \mu_1(\tau) i_1^*(\tau) - \mu_2(\tau) i_2^*(\tau) - r_i + \mu_i(\tau) + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \times \\ [\phi_i s_i^*(\eta) + k_i h_i(\eta, \Gamma_i^*) \int_\eta^a e^{-\int_\xi^\eta (r_i + \mu_i(\tau) - k_i) - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} d\xi] d\eta \quad (41)$$

By injecting (41) in the expression of  $\Gamma_i^*$ , and dividing by  $\Gamma_i^*$  (since  $\Gamma_i^* \neq 0$ ) we obtain :

$$1 = \int_0^{a+} \widehat{\beta}_i(a) \int_0^a \beta_i(\eta) c_i(\eta) g_i(\eta) e^{-\int_\eta^a (b(\tau) - \mu_1(\tau) i_1^*(\tau) - \mu_2(\tau) i_2^*(\tau) - r_i + \mu_i(\tau) + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \times \\ [\phi_i s_i^*(\eta) + k_i h_i(\eta, \Gamma_i^*) \int_\eta^a e^{-\int_\xi^\eta (r_i + \mu_i(\tau) - k_i) - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} d\xi] d\eta da \quad (42)$$

Let  $H_{ij}$ , the fonction define by :

$$H_{ij}(\Gamma_i^*) = \int_0^{a+} \widehat{\beta}_i(a) \int_0^a \beta_i(\eta) c_i(\eta) g_i(\eta) e^{-\int_\eta^a (b(\tau) - \mu_1(\tau) i_1^*(\tau) - \mu_2(\tau) i_2^*(\tau) - r_i + \mu_i(\tau) + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \times \\ [\phi_i s_i^*(\eta) + k_i h_i(\eta, \Gamma_i^*) \int_\eta^a e^{-\int_\xi^\eta (r_i + \mu_i(\tau) - k_i) - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} d\xi] d\eta da \quad (43)$$

Since  $h_i(\eta, 0) = N_{\psi_i}(\eta)$  i.e when  $\Gamma_i^* = 0$ ,  $s_i^*(a) = s_i^0(a)$  and  $v_i^*(a) = v_i^0(a)$ , so the net reproductive number is given by

$H_{ij}(0) = \mathfrak{R}^i(\psi_i, \rho_{ij})$ , i.e

$$\mathfrak{R}^i(\psi_i, \rho_{ij}) = \int_0^{a+} \widehat{\beta}_i(a) \int_0^a e^{-\int_\eta^a (b(\tau) + \mu_i(\tau) + r_i + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} \beta_i(\eta) c_i(\eta) g_i(\eta) \\ [\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta) \int_\eta^a e^{-\int_x^\eta (\mu_i(\tau) + r_i - k_i - \rho_{i2} + \widehat{\rho}_{j2} + \rho_{i3} - \widehat{\rho}_{j3}) d\tau} dx] d\eta da \quad (44)$$

We now see that an endemic steady state exists if equation (42) has a positive solution.

Since

$H_i(0) = \mathfrak{R}^i(\psi_i)$ , hence  $H_i(0) > 1$ . We know that  $s_i^*(a) + l_i^*(a) + i_i^*(a) + v_i^*(a) + j_i^*(a) = \Lambda_i < 1$ . Hence

$$i_i^*(a) < 1 \quad (45)$$

Since  $\Gamma_i^* > 0$ , from (43) and (45) we obtain :

$$\Gamma_i^* H_i(\Gamma_i^*) = \int_0^{a+} \widehat{\beta}_i(a) \int_0^a \Gamma_i^* \beta_i(\eta) c_i(\eta) g_i(\eta) e^{-\int_\eta^a (\beta(\tau) - \mu_1(\tau) i_1^*(\tau) - \mu_2(\tau) i_2(\tau) + r_i + \mu_i(\tau)) d\tau} \times$$

$$[\phi_i s_i^*(\eta) + k_i h_i(\eta, \Gamma_i^*) \int_\eta^a e^{-\int_\xi^\eta ((r_i + \mu_i(\tau) - k_i) d\tau} d\xi] d\eta da < \int_0^{a+} \widehat{\beta}_i(a) da = \beta_i^+.$$

In particular, for  $\Gamma_i^* = \beta_i^+$ , we have  $H_i(\beta_i^+) < 1$ , but  $H_i(0) > 1$ . Since  $H_i$  is continuous function of  $\Gamma_i^*$ , we conclude that  $H(\Gamma_i^*) = 1$ , has a positive solution  $\widehat{\Gamma}_i^*$  on  $]0; \beta_i^+[$ . This solution may not be unique since  $H_i$  may not be monotone ( $H_i(\Gamma_i^*)$  depends on  $h(\alpha, \Gamma_i^*)$  which is defined implicitly). It follows that when  $\Re^i(\psi_i) > 1$ , there exists an endemic steady state distribution which is given by the unique solution of equation (42) corresponding to  $\widehat{\Gamma}_i^*$

### 7. Simulation

In this section, the numerical method used in our simulations is based on the finite difference method. Forward in time-Backward in age numerical scheme is used as in [36]. Each equation in system (5) can be rewritten as

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) f(t, a) = g(t, a)$$

can be approximated by

$$\frac{f(t_{k+1}, a_i) - f(t_k, a_i)}{\Delta t} + \frac{f(t_k, a_i) - f(t_k, a_{i-1})}{\Delta a} = g(t_k, a_i)$$

In view of the influence of migration on the dynamics of tuberculosis modeled by (5), we considered patch2 in an endemic situation fig2 and patch1 under global stability conditions. In absence of migration, we obtain the fig 3a; 4a; 5a, which translate the overall stability of the patch. By varying the migration rates, we obtain the figure of the type fig 3b; 4b; 5b, which indicate that the migration disrupts the stability and could lead the patch a dramatic situation.



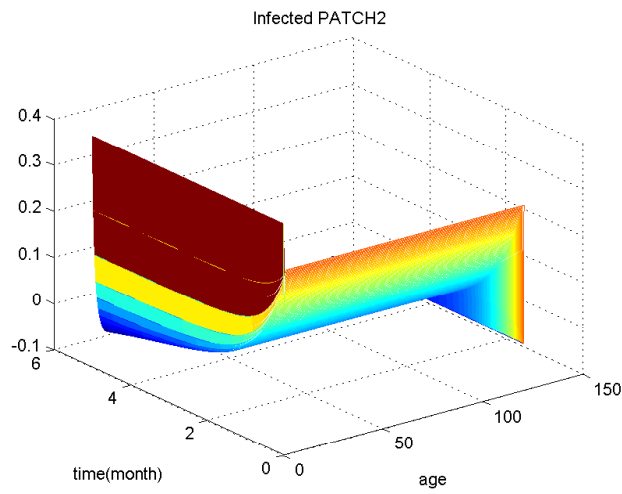
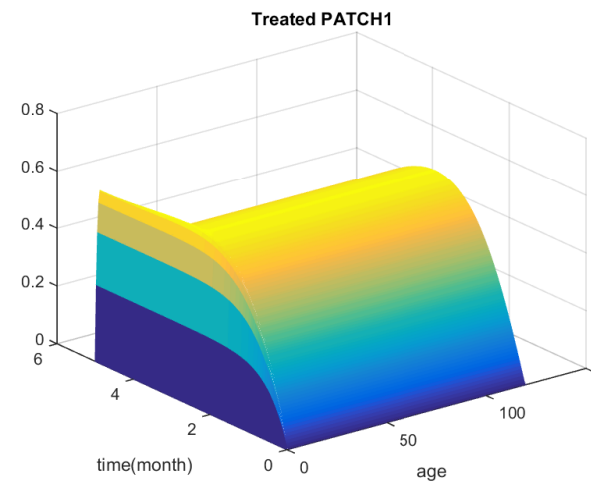
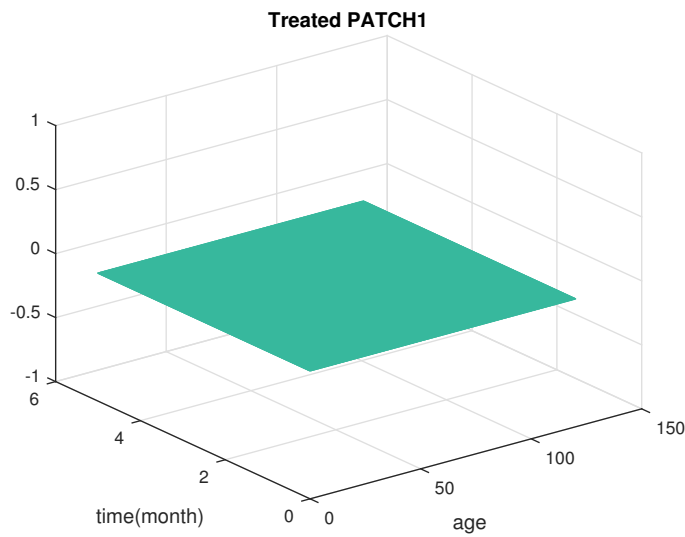


fig.2. Infectious individuals from path 2  $\Re^2(\psi, \rho) > 1$



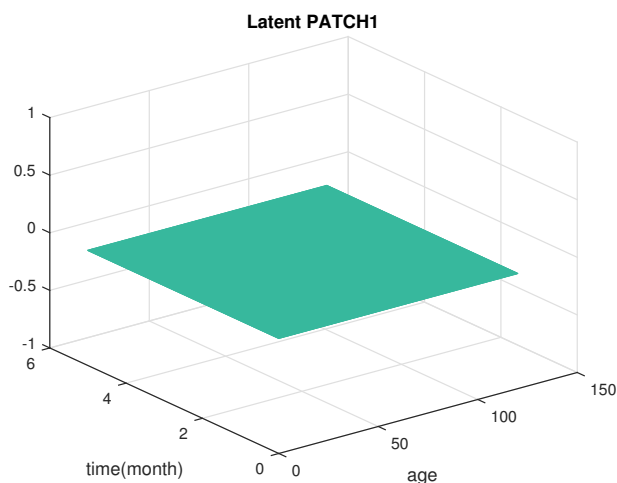
**fig.3a.** Evolution of  $J_1$  when  $\Re_0^1(\rho) < 1$ , with  $\rho = 0$     **fig.3b.** Evolution of  $J_1$  when  $\Re_0^1(\rho) < 1$ , with  $\rho \neq 0$



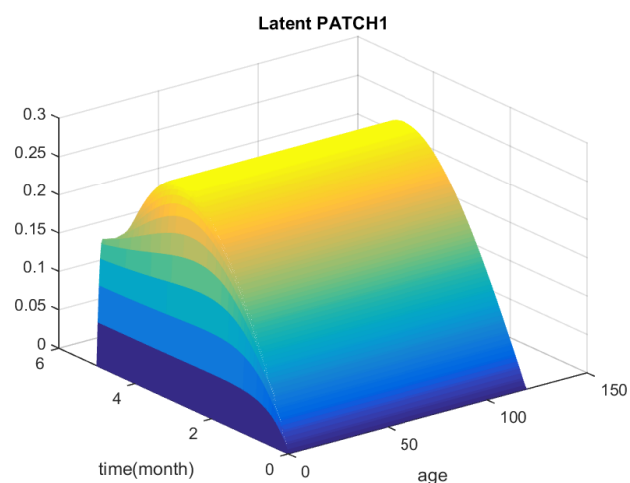
**fig.4a.** Evolution of  $I_1$  when  $\mathfrak{R}_0^1(\rho) < 1$ , with  $\rho = 0$



**fig.4b.** Evolution of  $I_1$  when  $\mathfrak{R}_0^1(\rho) < 1$ , with  $\rho \neq 0$



**fig.5a.** Evolution of  $L_1$  when  $\mathfrak{R}_0^1(\rho) < 1$ , with  $\rho = 0$



**fig.5b.** Evolution of  $L_1$  when  $\mathfrak{R}_0^1(\rho) < 1$ , with  $\rho \neq 0$

## 8. Conclusion

In this paper, we analyzed a two-patch age-structured model applied to tuberculosis in a context where migration is not controlled. The aim is to see the impact of migration on the spread of tuberculosis. Thus, we allowed migration to all individuals regardless of their epidemiological status. This study shows that migration has a negative impact on the spread of TB. The results of this analysis compared to those obtained in [19], allow us to affirm that the control of the migration makes it possible to avoid the propagation of

this disease, and reduce the exorbitant cost as well as human that the governments deploy for the eradication of tuberculosis. We intend to extend this model by introducing the compartments of the loss of seen otherwise, the individuals who leave the treatment and the resistant to all the forms of antituberculous drugs that we had, we also envisage to study the stability of the endemic equilibrium as well as the possibilities of existence of Hopf bifurcation using the method developed by P. Magal, S. Ruan [15].

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