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Toeplitz Determinants for the Class of Functions with Bounded Turning

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Abstract. In this paper, we obtain the upper bounds of the Toeplitz determinants for the class of functions with bounded turning. We also present some consequences of our main results. Some estimates obtained on Toeplitz determinants are sharp.
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1. Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. We denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E.

Let P denote the class of positive real part functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2}$$

which satisfy $\mathrm{Re}\, p(z) > 0$ for $z \in E$. This class is also known as the class of Carathéodory functions.

Let $G(\alpha, \delta)$ be the class of normalized functions $f(z) \in \mathcal{A}$ satisfying the condition $\mathrm{Re}\left(e^{i\alpha} f'(z)\right) > \delta$, $z \in E$, where $|\alpha| < \pi$, $0 \leqslant \delta < 1$, and $\cos\alpha > \delta$. This class was introduced by Mohamad [10].

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Remark 1. *For the specific values of the parameters α and δ, we obtain the special cases of $G(\alpha, \delta)$ as follows:*

(i) If we let $\alpha = \delta = 0$, then we have the class $G(0,0) \equiv R$ which satisfies $\operatorname{Re} f'(z) > 0$. The functions from R are said to be of bounded turning.

(ii) If we let $\alpha = 0$, then we have the class $G(0, \delta) \equiv R(\delta)$ which satisfies $\operatorname{Re}(f'(z)) > \delta$. The class $R(\delta)$ is called the class of bounded turning functions of order δ.

(iii) If we let $\delta = 0$, then we have the class $G(\alpha, 0) \equiv R(\alpha)$ which satisfies $\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$.

Goel and Mehrok [8], Macgregor [9], and Silverman and Silvia [17] were among the first researchers to study the classes R, $R(\delta)$, and $R(\alpha)$. Recently, the investigation into the class of bounded turning functions and coefficient problems such as the Hankel determinant for the higher order has been extensively studied by other researchers, see for example [3, 4]. We may point interested readers to recent advances in the class of bounded turning functions connected to a three-leaf-shaped domain and Bernoulli's lemniscate as well as their coefficient problems like the Hankel determinant, logarithmic coefficients, and the Hankel determinant with logarithmic coefficients, which point in a different direction than the current study, see [16, 23].

Finding estimates on the functional involving coefficients of $f(z) \in \mathcal{A}$ has been a major research area in geometric function theory since the development of the Bieberbach conjecture. Toeplitz determinant, for example, whose elements are the coefficients of $f(z) \in \mathcal{A}$ has been appealing to many researchers because it is related to the coefficient problems. Toeplitz determinant appeared in all branches of pure and applied mathematics, statistics and probability, image processing, quantum mechanics, queuing networks, signal processing, and time series analysis (see Ye and Lim [24] and references therein). Here we consider the symmetric Toeplitz determinant and it is defined by [21]

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \; a_1 = 1.$$

The estimates of the Toeplitz determinant were obtained for different classes of univalent functions. For instance, Ali et al. [2] studied Toeplitz matrices whose elements are the coefficients of bounded turning, starlike, close-to-convex, and univalent functions, Radhika et al. [12] obtained sharp bounds for Toeplitz determinants for the class of bounded turning functions, Zhang et al. [25] considered Toeplitz determinants of starlike functions connected with the sine function, and Zulfiqar et al. [26] investigated the fourth-order Toeplitz determinant for convex functions connected with the sine function. Much of the recent history of the development of this problem can also be found in [1, 5, 11, 13–15, 18–20, 22]. Thus, inspired by these works, in this paper, we aim to investigate the upper bounds of the second, third, and fourth-order Toeplitz determinants for functions of the class $G(\alpha, \delta)$. In particular, we find the bounds for the following determinants:

$$T_2(n) = \begin{vmatrix} a_n & a_{n+1} \\ a_{n+1} & a_n \end{vmatrix}, \ n \geqslant 2, \tag{3}$$

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \tag{4}$$

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_2 & a_2 \end{vmatrix}, \tag{5}$$

$$T_3(3) = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_3 & a_4 \\ a_5 & a_4 & a_3 \end{vmatrix}, \tag{6}$$

and

$$T_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & 1 & a_4 & a_3 \\ a_3 & a_4 & 1 & a_2 \\ a_4 & a_3 & a_2 & 1 \end{vmatrix}, \tag{7}$$

where the elements are the coefficients of the functions $f(z)$ of the form (1) in $G(\alpha, \delta)$. Besides, we point out several special cases and the consequences of our results.

We shall need the following lemmas in order to prove our main results.

Lemma 1. ([6]) *Let $p(z) \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$. Then*

$$|p_n| \leqslant 2, \ n \geqslant 1.$$

The inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2. ([7]) *Let $p(z) \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\}, \ 1 \leqslant k \leqslant n-1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

2. Main Results

In this section, we state and prove the main results of our present investigation.

Theorem 1. *Let $f(z) \in G(\alpha, \delta)$ be of the form* (1). *Then*

$$|T_2(n)| \leqslant 4{t_{\alpha\delta}}^2 \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right],$$

where $t_{\alpha\delta} = \cos\alpha - \delta$. The inequality is sharp.

Proof. Let a function $f(z) \in G(\alpha, \delta)$ given by (1). Then there exists a function $p(z) \in P$ of the form (2) such that [10]

$$\frac{e^{i\alpha} f'(z) - i \sin\alpha - \delta}{t_{\alpha\delta}} = p(z), \tag{8}$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.
Rearranging (8) and hence using the series representations for $f'(z)$ and $p(z)$, we get

$$1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots = e^{-i\alpha}\left[t_{\alpha\delta}\left(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots\right) + i \sin\alpha + \delta\right]. \tag{9}$$

Equating the coefficients of like powers of z^n, $n \geqslant 1$ yields

$$a_n = \frac{t_{\alpha\delta} e^{-i\alpha} p_{n-1}}{n}, \; n \geqslant 2. \tag{10}$$

Then, applying Lemma 1, we get

$$|a_n| = \frac{t_{\alpha\delta}\,|p_{n-1}|}{n} \leqslant \frac{2t_{\alpha\delta}}{n} \tag{11}$$

and so

$$|a_{n+1}| = \frac{t_{\alpha\delta}\,|p_n|}{n+1} \leqslant \frac{2t_{\alpha\delta}}{n+1}. \tag{12}$$

Clearly from (3) leads to

$$|T_2(n)| = \left|{a_n}^2 - {a_{n+1}}^2\right| \leqslant \left|{a_n}^2\right| + \left|{a_{n+1}}^2\right|. \tag{13}$$

Thus, making use of (11) and (12) gives the desired inequality. The inequality is sharp for the function $\frac{e^{i\alpha} f'(z) - i\sin\alpha - \delta}{t_{\alpha\delta}} = \frac{1+iz}{1-iz}$.

Theorem 2. *Let $f(z) \in G(\alpha, \delta)$ be of the form* (1). *Then*

$$|T_3(1)| \leqslant \frac{1}{9}\left(9 + 18{t_{\alpha\delta}}^2 + 4{t_{\alpha\delta}}^2 \sqrt{9{t_{\alpha\delta}}^2 - 6t_{\alpha\delta}\cos\alpha + 1}\right),$$

where $t_{\alpha\delta} = \cos\alpha - \delta$. The inequality is sharp.

Proof. By making use of (10) for $n = 2, 3$, from (4), we obtain

$$\begin{aligned} T_3(1) &= 1 - 2{a_2}^2 + 2{a_2}^2 a_3 - {a_3}^2 \\ &= 1 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^2 + 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right) - \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2 \\ &= \frac{1}{18}\left(18 - 9{t_{\alpha\delta}}^2 e^{-2i\alpha}{p_1}^2 - 2{t_{\alpha\delta}}^2 e^{-2i\alpha}{p_2}^2 + 3{t_{\alpha\delta}}^3 e^{-3i\alpha}{p_1}^2 p_2\right). \end{aligned} \tag{14}$$

Further, we can rearrange (14) as

$$|T_3(1)| = \frac{1}{18}\left|18 - 9{t_{\alpha\delta}}^2 e^{-2i\alpha}{p_1}^2 - 2{t_{\alpha\delta}}^2 e^{-2i\alpha}p_2\left(p_2 - \mu{p_1}^2\right)\right|, \tag{15}$$

where $\mu = \frac{3t_{\alpha\delta}e^{-i\alpha}}{2}$.
Thus, by the triangle inequality along with Lemma 1 and Lemma 2, we get

$$|T_3(1)| \leqslant \frac{1}{9}\left(9 + 18{t_{\alpha\delta}}^2 + 4{t_{\alpha\delta}}^2\sqrt{9{t_{\alpha\delta}}^2 - 6t_{\alpha\delta}\cos\alpha + 1}\right). \tag{16}$$

This inequality is sharp for the function $\frac{e^{i\alpha}f'(z) - i\sin\alpha - \delta}{t_{\alpha\delta}} = \frac{1+iz}{1-iz}$.

Theorem 3. *Let $f(z) \in G(\alpha, \delta)$ be of the form (1). Then*

$$|T_3(2)| \leqslant \frac{7{t_{\alpha\delta}}^3}{3},$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Using (10) for $n = 2, 3, 4$, from (5), it follows that

$$\begin{aligned} T_3(2) &= {a_2}^3 - 2a_2{a_3}^2 + 2{a_3}^2 a_4 - a_2{a_4}^2 \\ &= \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^3 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2 + 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right) \\ &\quad - \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2 \\ &= \frac{{t_{\alpha\delta}}^3 e^{-3i\alpha}}{288}\left(36{p_1}^3 - 32p_1{p_2}^2 + 16{p_2}^2 p_3 - 9p_1{p_3}^2\right). \end{aligned} \tag{17}$$

Rearranging the terms in (17) and hence applying the triangle inequality, then we can rewrite it as

$$|T_3(2)| \leqslant \frac{{t_{\alpha\delta}}^3}{288}\left[36|p_1|^3 + 32|p_1||p_2|^2 + 16|p_3|\left|p_4 - \eta_1{p_2}^2\right| + 16|p_3||p_4 - \eta_2 p_1 p_3|\right], \tag{18}$$

where $\eta_1 = 1$ and $\eta_2 = \frac{9}{16}$.
Further, by implementing Lemma 1 and Lemma 2, thus we obtain

$$|T_3(2)| \leqslant \frac{7{t_{\alpha\delta}}^3}{3}.$$

This concludes the proof.

Theorem 4. *Let $f(z) \in G(\alpha, \delta)$ be of the form* (1). *Then*

$$|T_3(3)| \leqslant \frac{112 {t_{\alpha\delta}}^3}{135},$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Using the values of a_3, a_4, and a_5 from (10) and in view of (6), it can be seen that

$$\begin{aligned} T_3(3) &= {a_3}^3 - 2a_3{a_4}^2 + 2{a_4}^2a_5 - a_3{a_5}^2 \\ &= \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^3 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2 + 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_4}{5}\right) \\ &\quad - \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_4}{5}\right)^2 \\ &= \frac{{t_{\alpha\delta}}^3e^{-3i\alpha}}{5400}\left(200{p_2}^3 - 225p_2{p_3}^2 + 135{p_3}^2p_4 - 72p_2{p_4}^2\right). \end{aligned} \tag{19}$$

After rearranging the terms and using triangular inequalities, (19) yields

$$|T_3(3)| \leqslant \frac{{t_{\alpha\delta}}^3}{5400}\left[200|p_2|^3 + 225\,|p_2|\,|p_3|^2 + 135\,|p_4|\,\left|p_6 - \eta_1{p_3}^2\right| + 135\,|p_4|\,|p_6 - \nu p_2p_4|\right], \tag{20}$$

where $\eta_1 = 1$ and $\nu = \frac{72}{135}$.
Finally, by applying Lemma 1 and Lemma 2, we get

$$|T_3(3)| \leqslant \frac{112{t_{\alpha\delta}}^3}{135}.$$

This completes the proof.

Theorem 5. *Let $f(z) \in G(\alpha, \delta)$ be of the form* (1). *Then*

$$|T_4(1)| \leqslant \frac{1}{1296}\left(1296 + 4392{t_{\alpha\delta}}^2 + 3456{t_{\alpha\delta}}^3 + 1921{t_{\alpha\delta}}^4\right),$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. From the expansion of (7) and using the values of a_2, a_3, and a_4 from (10), we get

$$\begin{aligned}
T_4(1) &= 1 - 2{a_2}^2 + {a_2}^4 - 2{a_3}^2 + {a_3}^4 - 2{a_4}^2 + {a_4}^4 - 2{a_2}^2{a_3}^2 - 2{a_2}^2{a_4}^2 \\
&\quad -2{a_3}^2{a_4}^2 + 8a_2a_3a_4 \\
&= 1 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^2 + \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^4 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2 + \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^4 \\
&\quad -2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2 + \left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^4 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2 \\
&\quad -2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2 - 2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)^2\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right)^2 \\
&\quad +8\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_1}{2}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_2}{3}\right)\left(\frac{t_{\alpha\delta}e^{-i\alpha}p_3}{4}\right) \\
&= \frac{1}{20736}\big(20736 - 10368{t_{\alpha\delta}}^2e^{-2i\alpha}{p_1}^2 + 1296{t_{\alpha\delta}}^4e^{-4i\alpha}{p_1}^4 - 4608{t_{\alpha\delta}}^2e^{-2i\alpha}{p_2}^2 \\
&\quad +256{t_{\alpha\delta}}^4e^{-4i\alpha}{p_2}^4 - 2592{t_{\alpha\delta}}^2e^{-2i\alpha}{p_3}^2 + 81{t_{\alpha\delta}}^4e^{-4i\alpha}{p_3}^4 - 1152{t_{\alpha\delta}}^4e^{-4i\alpha}{p_1}^2{p_2}^2 \\
&\quad -648{t_{\alpha\delta}}^4e^{-4i\alpha}{p_1}^2{p_3}^2 - 288{t_{\alpha\delta}}^4e^{-4i\alpha}{p_2}^2{p_3}^2 + 6912{t_{\alpha\delta}}^3e^{-3i\alpha}p_1p_2p_3\big).
\end{aligned} \tag{21}$$

Rearranging the terms in (21) and applying the triangle inequality, as well as some calculations, we can rewrite it in the following expression:

$$\begin{aligned}
|T_4(1)| \leqslant \frac{1}{20736}\Big[&20736 + 10368{t_{\alpha\delta}}^2|p_1|^2 + 256{t_{\alpha\delta}}^4|p_2|^4 + 81{t_{\alpha\delta}}^4|p_3|^4 + 2592{t_{\alpha\delta}}^2|p_3|^2 \\
&+648{t_{\alpha\delta}}^4|p_1|^2|p_3|^2 + 288{t_{\alpha\delta}}^4|p_2|^2|p_3|^2 + 1296{t_{\alpha\delta}}^4|p_1|^2\left|p_2 - \eta_1{p_1}^2\right| \\
&+4608{t_{\alpha\delta}}^2|p_2|\left|p_2 - \upsilon_1{p_1}^2\right| + 6912{t_{\alpha\delta}}^3|p_1||p_2||p_3 - \upsilon_2p_1p_2|\Big],
\end{aligned} \tag{22}$$

where $\eta_1 = 1$, $\upsilon_1 = \frac{9{t_{\alpha\delta}}^2e^{-2i\alpha}}{32}$, and $\upsilon_2 = \frac{t_{\alpha\delta}e^{-i\alpha}}{6}$.
Now, with the help of Lemma 1 and Lemma 2, we obtain

$$|T_4(1)| \leqslant \frac{1}{1296}\left(1296 + 4392{t_{\alpha\delta}}^2 + 3456{t_{\alpha\delta}}^3 + 1921{t_{\alpha\delta}}^4\right).$$

Thus, this completes the proof.

3. Corollaries and Consequences

In this section, we shall give the consequences of our main results.
For $\alpha = 0$ and $\delta = 0$ in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we get the estimates for the class R.

Corollary 1. *Let $f(z) \in R$. Then* *(i)* $|T_2(n)| \leqslant 4\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$.

(ii) $|T_3(1)| \leqslant \frac{35}{9}$.

(iii) $|T_3(2)| \leqslant \frac{7}{3}$.

(iv) $|T_3(3)| \leqslant \frac{112}{135}$.

(v) $|T_4(1)| \leqslant \frac{11065}{1296}$.

For $\alpha = 0$ in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we obtain the estimates for the class $R(\delta)$.

Corollary 2. *Let $f(z) \in R(\delta)$. Then*

(i) $|T_2(n)| \leqslant 4(1-\delta)^2\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$.

(ii) $|T_3(1)| \leqslant \frac{1}{9}\left(9+18(1-\delta)^2+4(1-\delta)^2(3\delta-2)\right)$.

(iii) $|T_3(2)| \leqslant \frac{7(1-\delta)^3}{3}$.

(iv) $|T_3(3)| \leqslant \frac{112(1-\delta)^3}{135}$.

(v) $|T_4(1)| \leqslant \frac{1}{1296}\left(1296+4392(1-\delta)^2+3456(1-\delta)^3+1921(1-\delta)^4\right)$.

By choosing $\delta = 0$ in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we obtain the estimates for the class $R(\alpha)$.

Corollary 3. *Let $f(z) \in R(\alpha)$. Then*

(i) $|T_2(n)| \leqslant 4\cos^2\alpha\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$.

(ii) $|T_3(1)| \leqslant \frac{1}{9}\left(9+18\cos^2\alpha+4\cos^2\alpha\sqrt{3\cos^2\alpha+1}\right)$.

(iii) $|T_3(2)| \leqslant \frac{7\cos\alpha^3}{3}$.

(iv) $|T_3(3)| \leqslant \frac{112\cos^3\alpha}{135}$.

(v) $|T_4(1)| \leqslant \frac{1}{1296}\left(1296+4392\cos^2\alpha+3456\cos^3\alpha+1921\cos^4\alpha\right)$.

We remark that the inequalities in Corollary 1(i), Corollary 1(ii), and Corollary 1(iii) coincide with the results of Ali et al. [2]. It is also shown in [2] that the results in Corollary 1(i) and Corollary 1(ii) were sharp. In the existing literature, no bounds for $|T_2(n)|$, $n \geqslant 2$, $|T_3(n)|$, $n = 1, 2, 3$, and $|T_4(1)|$ for the classes $G(\alpha, \delta)$, $R(\delta)$, and $R(\alpha)$ were obtained. Additionally, the results on $|T_3(3)|$ and $|T_4(1)|$ for functions in the class R had never been studied before.

4. Conclusions

In the present paper, we have considered the Toeplitz determinants whose elements are coefficients of univalent functions. We have obtained the upper bounds of $|T_2(n)|$, $n \geqslant 2$, $|T_3(n)|$, $n = 1, 2, 3$, and $|T_4(1)|$ not only for functions of the class $G(\alpha, \delta)$, but also for some classes of functions with bounded turning namely R, $R(\delta)$, and $R(\alpha)$. Some results obtained are reduced to the estimates proven in [2] for specific choices of parameters α and δ. For the class $G(\alpha, \delta)$, we have determined the sharp estimates for $|T_2(n)|$, $n \geqslant 2$ and $|T_3(1)|$. The results obtained perhaps give an opportunity for researchers to further investigate inequalities problems for functions of the class $G(\alpha, \delta)$ as well as other subclasses of S.

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