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Fuzzification of the Dual B-algebra

Ma. Rizal V. Dicen[1,*], Katrina E. Belleza-Fuentes[1]

[1] *Department of Computer, Information Sciences and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines*

Abstract. This paper introduces the fuzzification of the dual B-algebra and presented some of its related properties. A characterization of a fuzzy (normal) dual B-algebra is also investigated.

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1. Introduction

In 1965, after the publication of Lotfi Zadeh's article entitled "Fuzzy Sets", the era of fuzzy mathematics had started. Fuzzy mathematics is the branch of mathematics which includes fuzzy set theory and fuzzy logic that deals with partial inclusion of elements in a set on a spectrum, as opposed to simple binary "yes" or "no" (0 or 1) inclusion [8]. A fuzzy set is a class of objects with a continuum of grades of membership [8]. The term fuzzification refers to the process of mapping the numerical input of a system to fuzzy sets with some degree of membership. This degree of membership can be anywhere within the closed interval from 0 to 1 [4]. Fuzzy technology is very productive in many applications especially in the field of computer science and control engineering, expert system, management science, robotics, to name a few [7].

In 1999, Y.B. Jun,et. al introduced the *fuzzification* of B-algebras and at the same time investigated the characterization of *fuzzy* normal B-algebras [5]. By 2020, M. Ahmed and E. Ahmed investigated the properties of *fuzzy* BCK-algebras. In their research,they applied the concept of fuzzy sets and gave properties to some algebraic structures such as an ideal, semilattice, lower semilattice, lattice and sub-algebra [1].

In 2019, K. Belleza and J. Vilela, introduced and gave the characterization of the notion of dual B-algebra and its initial properties [2]. In their research, they investigated the relationships between the dual B-algebra and the BCK-algebra.

This paper considers the fuzzification of the dual B-algebra and its related properties. This also aims to give a characterization of a fuzzy dual B-algebra.

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*Corresponding author.
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Email addresses: `marizverdadicen@gmail.com` (M. Dicen), `kebelleza@usc.edu.ph` (K. Belleza-Fuentes)

2. Preliminaries

An algebra of type $(2, 0)$ is an algebra with a binary operation and a constant element.

Definition 1. [3] A **binary operation** "$*$" on a set S is a function mapping $S\times S\to S$. For each element $(a, b) \in S \times S$, we will denote $*((a, b)) \in S$ by $a * b$. That is, $*$ is a binary operation on S if and only if for all $a, b \in S$, $a * b \in S$.

Definition 2. [2] A **Dual** B**-algebra** X is a triple $(X, \circ, 1)$ where X is a non-empty set with a binary operation "$\circ$" and a constant "1" satisfying the following axioms for all x, y, z in X:

$$\text{(DB1) } x \circ x = 1; \quad \text{(DB2) } 1 \circ x = x; \quad \text{(DB3) } x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z.$$

A non-empty subset N of a dual B-algebra X is called a dual B-subalgebra of X if $x \circ y \in N$ for any $x, y \in N$. A non-empty subset N of a dual B-algebra X is said to be normal if $(x \circ f) \circ (y \circ g) \in N$ whenever $x \circ y \in N$ and $f \circ g \in N$. Note that any normal subset N of a dual B algebra X is a B-subalgebra of X, the converse need not be true [6]. A non-empty subset N of a dual B-algebra X is called a normal B-subalgebra of X if it is both a dual B-subalgebra and normal.

Lemma 1. [2] Let X be a dual B-algebra. Then for any $x,\ y \in X$, we have:

$$\text{(i) } x \circ y = 1; \quad \text{(ii) } (x \circ 1) \circ 1 = x.$$

3. Fuzzy Dual B-Algebra

In what follows, let X denotes a dual B-algebra unless otherwise specified.

Definition 3. A fuzzy set μ in X is called a fuzzy dual B-algebra if it satisfies the following inequality for all $x, y \in X$:

$$\mu(x \circ y) \geq \min\{\mu(x), \mu(y)\}.$$

Example 1. Let $X = \{1, A, B, C, D, E\}$ be the set with the following table:

$\circ$	1	A	B	C	D	E
1	1	A	B	C	D	E
A	B	1	A	D	E	C
B	A	B	1	E	C	D
C	C	D	E	1	A	B
D	D	E	C	B	1	A
E	E	C	D	A	B	1

Then $(X, \circ, 1)$ is a dual B-algebra by routine calculations.
Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(1)$=$\mu(C)$= 0.8 >0.3 = $\mu(x)$ for all $x \in X\backslash\{1, C\}$. Then, μ is a fuzzy dual B-algebra.

Proposition 1. Every fuzzy dual B- algebra μ satisfies the inequality $\mu(1) \geq \mu(x)$ for all $x \in X$.

Proof. Note that by (DB1) $x \circ x = 1$ for all $x \in X$, which implies that,

$$\mu(1) = \mu(x \circ x) \geq \min\ \{\mu(x), \mu(x)\} = \mu(x).$$

This proves that $\mu(1) \geq \mu(x)$ for all $x \in X$.

The following corollary follows immediately from Proposition 1.

Corollary 1. Let X be a fuzzy dual B-algebra. Then for all $x \in X$, $\mu(x) = \mu(x \circ 1)$.

Proof. Note that $x = (x \circ 1) \circ 1$ [by Lemma 1(ii)]. If μ is a fuzzy dual B-algebra, then,

$$\begin{aligned} \mu(x) &= \mu\big((x \circ 1) \circ 1\big) && \text{[by Lemma 1(ii)]} \\ &\geq \min\{\mu(x \circ 1), \mu(1)\} && \text{[by Definition 3]} \\ &\geq \min\{\min\{\mu(x), \mu(1)\}, \mu(1)\} \\ &\geq \min\{\mu(x), \mu(1)\} && \text{[by Proposition 1]} \\ &= \mu(x \circ 1). \end{aligned}$$

That is, $\mu(x) = \mu(x \circ 1)$ for all $x \in X$.

For any elements x and y of X, let us write $\prod^{n} x \circ y$ for $((((x \circ y) \circ y) \circ y)...)$ where y occurs n times.

Proposition 2. Let a fuzzy set μ in X be a fuzzy dual B-algebra and let $n \in \mathbb{N}$. Then for all $x \in X$,

(i) $\mu\big(\prod^{n} x \circ x\big) \geq \mu(x)$ whenever n is odd, and

(ii) $\mu\big(\prod^{n} x \circ x\big) = \mu(x)$ whenever n is even.

Proof. Let $x \in X$.
(i) Suppose that n is odd. Then $n = 2k-1$ for some positive integer k. Note that by DB1 and Proposition 1, $\mu(x \circ x) = \mu(1) \geq \mu(x)$ when $k = 1$.

Assume that $\mu\left(\prod^{2k-1} x \circ x\right) \geq \mu(x)$ for a positive integer k. Then,

$$\begin{aligned}
\mu\left(\prod^{2(k+1)-1} x \circ x\right) &= \mu\left(\prod^{2k+1} x \circ x\right) \\
&= \mu\left(\prod^{2k-1} ((x \circ x) \circ x) \circ x\right) \\
&= \mu\left(\prod^{2k-1} (1 \circ x) \circ x\right) && \text{[by (DB1)]} \\
&= \mu\left(\prod^{2k-1} x \circ x\right) && \text{[by (DB2)]} \\
&= \mu(1) \\
&\geq \mu(x). && \text{[by Proposition 1]}
\end{aligned}$$

Hence, the above implies that (i) holds.

(ii) Suppose that n is even. Then $n = 2k$ for some positive integer k. For $k = 1, n = 2$, we have, $\mu((x \circ x) \circ x) = \mu(1 \circ x) = \mu(x)$ by (DB1) and (DB2).

Suppose that $\mu\left(\prod^{2k} x \circ x\right) = \mu(x)$ for a positive integer k. Then,

$$\begin{aligned}
\mu\left(\prod^{2(k+1)} x \circ x\right) &= \mu\left(\prod^{2k+2} x \circ x\right) \\
&= \mu\left(\prod^{2k} ((x \circ x) \circ x) \circ x\right) \\
&= \mu\left(\prod^{2k} (1 \circ x) \circ x\right) && \text{[by (DB1)]} \\
&= \mu\left(\prod^{2k} (x \circ x)\right) && \text{[by (DB2)]} \\
&= \mu(x).
\end{aligned}$$

Hence, the above implies that (ii) holds.

Proposition 3. If a fuzzy set μ in X is a fuzzy dual B-algebra, then for all $x, y \in X$,

$$\text{(fDB1) } \mu(x \circ 1) \geq \mu(x); \quad \text{(fDB2) } \mu((y \circ 1) \circ x) \geq min\{\mu(x), \mu(y)\}.$$

Proof. Suppose X is a fuzzy dual B-algebra and μ is a fuzzy set for all $x, y \in X$.

(fDB1): By Definition 3, $\mu(x \circ 1) \geq \min\{\mu(x), \mu(1)\} \geq \mu(x)$, by Proposition 1. Hence, (fDB1) holds.

(fDB2): By Definition 3,

$$\begin{aligned} \mu((y \circ 1) \circ x) &\geq \min\{\mu(y \circ 1), \mu(x)\} \\ &\geq \min\{\min\{\mu(y), \mu(1)\}, \mu(x)\} \\ &\geq \min\{\mu(y), \mu(x)\} && \text{[by Proposition 1]} \\ &\geq \min\{\mu(x), \mu(y)\}. \end{aligned}$$

Hence, (fDB2) holds.

The next theorem is a characterization of a fuzzy dual B-algebra which is the converse of Proposition 3.

Theorem 1. *If a fuzzy set μ in X satisfies* (fDB1) *and* (fDB2), *then μ is a fuzzy dual B-algebra.*

Proof. Suppose that a fuzzy set μ in X satisfies the conditions in (fDB1) and (fDB2). Let $x, y \in X$. Then we have,

$$\begin{aligned} \mu(x \circ y) &= \mu(((x \circ 1) \circ 1) \circ y) && \text{[by Lemma 1(i)]} \\ &\geq \min\{\mu((x \circ 1) \circ 1), \mu(y)\} \\ &\geq \min\{\min\{\mu(x \circ 1), \mu(1)\}, \mu(y)\} \\ &\geq \min\{\mu(x \circ 1), \mu(y)\} && \text{[by (fDB2)]} \\ &= \min\{\mu(x), \mu(y)\}. && \text{[by (fDB1)]} \end{aligned}$$

Hence, μ is a fuzzy dual B-algebra.

In what follows is an example of the above theorem.

Example 2. The fuzzy set μ discussed in Example 1 satisfies (fDB1) and (fDB2). Considering the example below:

Let $x = A$ and $y = B$.

Note that $\mu(A \circ B) = \mu(((A \circ 1) \circ 1) \circ B) = \mu(A) = 0.3$. Then we have

$$\mu(A \circ B) = \mu(((A \circ 1) \circ 1) \circ B)$$

$$\begin{aligned} &\geq \min\{\mu((A \circ 1) \circ 1), \mu(B)\} \\ &\geq \min\{\mu(B), \mu(B)\} \\ &= 0.3. \end{aligned}$$

Hence, μ is indeed a fuzzy dual B-algebra.

Definition 4. A fuzzy set μ in X is said to be *fuzzy normal* if it satisfies the following inequality for all $f, g, x, y \in X$:

$$\mu((f \circ x) \circ (g \circ y)) \geq \min\{\mu(f \circ g), \mu(x \circ y)\}.$$

Example 3. Refer to the dual B-algebra from Example 1. Define a fuzzy set $\delta : X \to [0, 1]$ by $\delta(1) = \delta(A) = \delta(B) = 0.6$ and $\delta(C) = \delta(D) = \delta(E) = 0.1$, then δ is a fuzzy normal set in X by routine calculations.

Example 4. Let $X = \{1, a, b, c\}$ be a set with the following table:

$\circ$	1	a	b	c
1	1	a	b	c
a	c	1	a	b
b	b	c	1	a
c	a	b	c	1

Then $(X, \circ, 1)$ is a dual B-algebra by routine calculations. Suppose we define a map $\mu : X \to [0, 1]$ by $\mu(1) > \mu(b) > \mu(a) = \mu(c)$, then μ is a fuzzy normal set in X. And also, if we define a map $\varphi : X \to [0, 1]$ by $\varphi(1) = \varphi(b) > \varphi(a) = \varphi(c)$, then φ is also a fuzzy normal set in X.

Theorem 2. *Every fuzzy normal set μ in X is a fuzzy dual B-algebra.*

Proof. Since μ is fuzzy normal, recall Definition 4. For any $x, y \in X$, we have,

$$\begin{aligned} \mu((x \circ y) &= \mu((1 \circ 1) \circ (x \circ y)) \\ &\geq min\{\mu(1 \circ x), \mu(1 \circ y)\} \\ &= min\{\mu(x), \mu(y)\}. \qquad \text{[by (DB2)]} \end{aligned}$$

Hence, μ is a fuzzy dual B-algebra.

Remark 1. The converse of Theorem 2 is not always true.

Example 1 above is the correct representation of this remark.

Now, note that $\mu((C \circ D) \circ (1 \circ B)) = \mu(B \circ A) = \mu(A) = 0.3$. Then we have,

$$\begin{aligned} \mu((C \circ D) \circ (1 \circ B)) &= \min\{\mu(C \circ 1), \mu(D \circ B)\} \\ &= \min\{\mu(C), \mu(C)\} \\ &< 0.8. \end{aligned}$$

Manipulation above shows that the fuzzy dual B-algebra is not fuzzy normal. Thus, the remark holds true.

Definition 5. A fuzzy set μ in X is called *fuzzy normal* dual B-algebra if it is a fuzzy dual B-algebra which is fuzzy normal.

Example 5. The fuzzy sets discussed in Example 3 and Example 4 are indeed fuzzy normal dual B-algebras by routine calculations.

Proposition 4. If a fuzzy set μ in X is a fuzzy normal dual B-algebra, then $\mu(x \circ y) = \mu(y \circ x)$ for all $x, y \in X$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu(x \circ y) &= \mu(1 \circ (x \circ y)) && \text{[by DB2]} \\ &= \mu((y \circ y) \circ (x \circ y)) && \text{[by DB1]} \\ &\geq \min\{\mu(y \circ x), \mu(y \circ y)\} && \text{[since } \mu \text{ is fuzzy normal]} \\ &= \min\{\mu(y \circ x), \mu(1)\} && \text{[by DB1]} \\ &\geq \mu(y \circ x). && \text{[by Proposition 1]} \end{aligned}$$

Interchanging x with y, we obtain

$$\begin{aligned} \mu(y \circ x) &= \mu(1 \circ (y \circ x)) && \text{[by DB2]} \\ &= \mu((x \circ x) \circ (y \circ x)) && \text{[by DB1]} \\ &\geq \min\{\mu(x \circ y), \mu(x \circ x)\} && \text{[since } \mu \text{ is fuzzy normal]} \\ &= \min\{\mu(x \circ y), \mu(1)\} && \text{[DB1]} \\ &\geq \mu(x \circ y). && \text{[by Proposition 1]} \end{aligned}$$

By the manipulation above, we conclude that $\mu(x \circ y) = \mu(y \circ x)$ for all $x, y \in X$.

The following result will be used in the characterization of fuzzy normal dual B-algebra.

Theorem 3. *Let μ be a fuzzy normal dual B-algebra. Then the following set is a normal dual B-subalgebra of X,*

$$X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}.$$

Proof. It is sufficient to show that X_μ is normal since μ is a fuzzy normal dual B-algebra. Let $f, g, x, y \in X$ be such that $x \circ y \in X_\mu$ and $f \circ g \in X_\mu$. Then $\mu(x \circ y) = \mu(1) = \mu(f \circ g)$. Considering μ is fuzzy normal, it follows that

$$\mu((x \circ f) \circ (y \circ g)) \geq min\{\mu(x \circ y), \mu(f \circ g)\} \geq min\{\mu(1), \mu(1)\} = \mu(1).$$

Now, by Proposition 1,

$$\mu(1) \geq min\{\mu(x \circ y), \mu(f \circ g)\} \geq \mu((x \circ f) \circ (y \circ g)).$$

Therefore, by the manipulation above, $\mu((x \circ f) \circ (y \circ g)) = \mu(1)$*, which implies that* $(x \circ f) \ \circ \ (y \circ g) \in X_{\mu}$.

Theorem 4. *The intersection of any set of a fuzzy normal dual* B*-algebras is also fuzzy normal dual* B*-algebra.*

Proof. *Let* $\{\mu_\omega \mid \omega \in \mathcal{I}\}$ *be a family of fuzzy normal dual* B*-algebras and let* $f, g, x, y \in X$*. Then by the greatest lower bound of fuzzy set, definition of fuzzy normal, and by the minimum or infimum property, we have:*

$$\begin{aligned}
\Big(\bigcap_{\omega\in\mathcal{I}} \mu_\omega\Big)((x \circ f) \circ (y \circ g)) &= \inf_{\omega\in\mathcal{I}} \mu_\omega((x \circ f) \circ (y \circ g)) \\
&\geq \inf_{\omega\in\mathcal{I}} \{ min\{\mu_\omega(x \circ y), \mu_\omega(f \circ g)\}\} \\
&= min\{\inf_{\omega\in\mathcal{I}} (x \circ y), \inf_{\omega\in\mathcal{I}} (f \circ g)\} \\
&= min\Big\{\Big(\bigcap_{\omega\in\mathcal{I}} \mu_\omega\Big)(x \circ y), \Big(\bigcap_{\omega\in\mathcal{I}} \mu_\omega\Big)(f \circ g)\Big\},
\end{aligned}$$

which implies that $\bigcap_{\omega\in\mathcal{I}} \mu_\omega$ *is a fuzzy normal set in* X*. By Theorem 2, therefore* $\bigcap_{\omega\in\mathcal{I}} \mu_\omega$ *is a fuzzy normal dual* B*-algebra.*

Remark 2. The union of any set of fuzzy dual B-algebras may not be a fuzzy dual B-algebra as shown in the next example.

Example 6. Referring to Example 1. Define a fuzzy set $\delta : X \to [0, 1]$ by $\delta(1) = \delta(D) = 0.6 > 0.1 = \delta(A) = \delta(B) = \delta(C) = \delta(E)$, then it is also a fuzzy dual B-algebra. Now,

$$(\mu \cup \delta)(C \circ D) = (\mu \cup \delta)(A) = 0.3 \qquad \text{[by union property of fuzzy set]}$$

$$\text{and } \min\{(\mu \cup \sigma)(C), (\mu \cup \sigma)(D)\} = \min\{0.8, 0.6\} = 0.6.$$

Notice that $(\mu \cup \delta)(C \circ D) < \min\{(\mu \cup \sigma)(C), (\mu \cup \sigma)(D)\}$ which does not satisfy the definition. Thus, $\mu \cup \delta$ is not a fuzzy dual B-algebra. Recall from Theorem 2 that every fuzzy normal dual B-algebra is a fuzzy dual B-algebra. However, in view of Remark 2, the union of fuzzy normal dual B-algebras may not be a fuzzy normal B-algebra.

4. Conclusion

In this paper, the notion of a fuzzy dual B-algebra is presented together with some of its properties and characterizations. Here it is considered that every fuzzy normal set is a fuzzy dual B algebra but the converse is not always true. While undergoing an operation in fuzzy sets, the union of any set of fuzzy dual B-algebra may not be a fuzzy dual B-algebra. However, the intersection of any set of a fuzzy normal dual B-algebra is also a fuzzy normal dual B-algebra. The difference between a fuzzy dual B-algebra and fuzzy normal dual B-algebra is also presented.

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