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C-Almost Normality and L-Almost Normality

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Abstract. The main purpose of this paper is to introduce and study new topological properties called C-almost normality and L-almost normality. A space X is called a C-*almost normal* (resp. L-*almost normal*) space if there exist an almost normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact (resp. Lindelöf) subspace $A \subseteq X$. We investigate these properties and present some examples to illustrate the relationships among them with other kinds of topological properties.

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1. Introduction

The notions of epi-normality, C-normality and L-normality were introduced by Arhangel'skii during his visiting to Department of Mathematics in King Abdulaziz University, Saudi Arabia on 2012. C-normality has been studied by Alzahrani and Kalantan in 2017, see [10]. L-normality has been studied by Kalantan and Saeed in 2017, see [16]. Then, Alzahrani studied the notions of C-regularity, L-regularity, C-Tychonoff and L-Tychonoff in 2018, see [8, 9]. Thabit studied the notion of epi-partial normality in 2021, see [34]. At the end of 2021, Thabit and others studied the notion of epi-quasi normality [33]. Recently, Thabit and Alqurashi studied the notion of C-quasi normality and L-quasi normality in [32]. In this paper, we study two new properties which are C-almost normality and L-almost normality. We show that these new properties are different from each other, and they are different from C-normality, L-normality, C-regularity, L-regularity, epi-almost normality, C-quasi normality, L-quasi normality and so on. Some properties, counterexample and relationships of these properties are

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investigated.
Throughout this paper, a space X means a topological space. The set of all real numbers is denoted by $\mathbb{R}$, the set of all rational numbers is denoted by $\mathbb{Q}$ and the set of all irrational numbers is denoted by $\mathbb{P}$. For a subset A of a space X, $\overline{A}$ and $\text{int}(A)$ denote to the closure and the interior of A in X, respectively. We need to recall the following definitions: a subset A of a space X is said to be a *regularly-open* set or an *open domain* set if it is the interior of its own closure, or equivalently if it is the interior of some closed set [19]. A complement of an open domain subset is called a *closed domain* subset. A subset A of a space X is called a π-*closed* set if it is a finite intersection of closed domain sets [36]. Two sets A and B of a space X are said to be *separated* if there exist two disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$ [11, 12, 22]. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called a topology that is *coarser* than $\mathcal{T}$ and $\mathcal{T}$ is called *finer* [12]. A space X is said to be a *normal* space [12], if any pair of disjoint closed subsets A and B of X can be separated by two disjoint open subsets. A space X is said to be a π-*normal* space [14], if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated by two disjoint open subsets. A space X is said to be an *almost-normal* space [14, 26], if any pair of disjoint closed subsets A and B of X, one of which is closed domain, can be separated by two disjoint open subsets. A space X is said to be a *mildly normal* space [27], if any pair of disjoint closed domain subsets A and B of X can be separated by two disjoint open subsets. A space X is said to be a *partially normal* space [6], if any pair of disjoint closed subsets A and B of X, one of which is closed domain and the other is π-closed, can be separated by two disjoint open subsets. A space X is said to be a *completely regular* (resp. an *almost completely regular*) space if for each $x \in X$ and each closed (resp. closed domain) set F in X such that $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$ [12, 26]. A space X is said to be a *regular* (resp. an *almost regular*) space if for each $x \in X$ and each closed (resp. closed domain) set F in X such that $x \notin F$, there exist two disjoint open subsets U and V such that $x \in U$ and $F \subseteq V$ [25]. A space $(X, \mathcal{T})$ is said to be an *epi-normal* space [15], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_4 (T_1-normal). A space $(X, \mathcal{T})$ is said to be an *epi-mildly normal* space [17], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Hausdorff mildly normal. A space $(X, \mathcal{T})$ is said to be an *epi-almost normal* space [4], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Hausdorff almost normal. A space $(X, \mathcal{T})$ is said to be an *epi-regular* space [7], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_3 (regular and T_1). A space $(X, \mathcal{T})$ is said to be an *epi-partially normal* space [34], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Hausdorff partially normal. A space $(X, \mathcal{T})$ is said to be an *epi-quasi normal* space [33], if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Hausdorff quasi normal. A space X is said to be *Hausdorff* or a T_2-*space*, if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ are $U \cap V = \emptyset$ [12]. A space X is said to be *completely Hausdorff* or *Urysohn* [12, 29], if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A space X is said to be a *sub-metrizable* space [13], if there exists a metric d on X such that the topology $\mathcal{T}_d$ on

X generated by d is coarser than $\mathcal{T}$. A space X is called a *C-normal* [10], (resp. *C-regular* [8]; *C-Tychonoff* [9]) space if there exist a normal (resp. regular, Tychonoff) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A space X is called an *L-normal* [16] (resp. *L-regular* [8], *L-Tychonoff* [9]) space if there exist a normal (resp. regular, Tychonoff) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. A space X is called a *C-almost regular* (resp. *C-completely regular, C-almost completely regular, C_2-almost regular, C_2-almost completely regular*) space [31], if there exist an almost regular (resp. completely regular, almost completely regular, Hausdorff almost regular, Hausdorff almost completely regular) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A space X is called an *L-almost regular* (resp. *L-completely regular, L-almost completely regular, L_2-almost regular, L_2-almost completely regular*) space [3], if there exist an almost regular (resp. completely regular, almost completely regular, Hausdorff almost regular, Hausdorff almost completely regular) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. A space X is called a *C-quasi normal* (resp. *L-quasi normal*) space [32], if there exist a quasi normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact (resp. Lindelöf) subspace $A \subseteq X$. The topology on X generated by the family of all open domain subsets denoted by $\mathcal{T}_s$ is coarser than $\mathcal{T}$ and $(X, \mathcal{T}_s)$ is called the *semi regularization* of X. A space $(X, \mathcal{T})$ is called *semi-regular* if $\mathcal{T} = \mathcal{T}_s$ [21]. A space X is called an *H-closed* space [12], if X is closed in every Hausdorff space in which X can be embedded or X is Hausdorff almost compact [18, 23]. Let M be a non-empty subset of a space $(X, \mathcal{T})$. Define a topology $\mathcal{T}_{(M)}$ on X as follows: $\mathcal{T}_{(M)} = \{U \cup K : U \in \mathcal{T} \text{ and } K \subseteq X \setminus M\}$. Then, $(X, \mathcal{T}_{(M)})$ is called a discrete extension of $(X, \mathcal{T})$ denoted by X_M, where $\mathcal{T}_d \subseteq \mathcal{T} \subseteq \mathcal{T}_{(M)}$, see [2]. Let $(X, \mathcal{T})$ be a space and $p \notin X$. Put $X^p = X \cup \{p\}$. Define a topology $\mathcal{T}^*$ on X^p by: $\mathcal{T}^* = \{U \cup \{p\} : U \in \mathcal{T}\} \cup \{\emptyset\}$. The space $(X^p, \mathcal{T}^*)$ is called the closed extension of $(X, \mathcal{T})$, [29, Example 12], see [1].

2. Preliminaries

First, we give the main definitions of this work:

Definition 1. Let X be a space.

(1) A space X is called a *C-almost normal* space if there exist an almost normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

(2) A space X is called an *L-almost normal* space if there exist an almost normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$.

From Definition 1, it is clear that every almost normal space is both C-almost normal and L-almost normal, because $Y = X$ and the identity function $f : X \to X$ satisfy the requirements. Next, we present the following basic results:

Theorem 1. *Every C-normal space is C-almost normal.*

Proof. Let X be a C-normal space. Then, there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Since Y is a normal space, we have Y is almost normal. Therefore, X is C-almost normal.

The converse of Theorem 1 is not necessary to be true in general. For example, the finite complement topology $(\mathbb{R}, \mathcal{CF})$, Example 3, is a C-almost normal space, which is not C-normal. The deleted Tychonoff plank topology, Example 6, is an example of a C-almost normal space, which is not almost normal. The modified Dieudonné plank, Example 8, is an L-almost normal space, which is not almost normal. Since every sub-metrizable space is epi-normal and every epi-normal space is C-normal [10], we get the following corollary:

Corollary 1.

(1) Every sub-metrizable space is C-almost normal.

(2) Every epi-normal space is C-almost normal.

Now, we improve Corollary 1 by the following theorem:

Theorem 2. *Every epi-almost normal space is C-almost normal.*

Proof. Let X be an epi-almost normal space. Then, there exist a topology $\mathcal{T}'$ on X, which is coarser than $\mathcal{T}$, such that $(X, \mathcal{T}')$ is Hausdorff almost normal. Thus, the identity mapping $I_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is a bijective continuous function. Let D be any compact subspace of $(X, \mathcal{T})$. Then, $I_X(D)$ is a compact Hausdorff subspace of $(X, \mathcal{T}')$ as $I_X(D) = D$ is a compact Hausdorff subspace in both $(X, \mathcal{T})$ and $(X, \mathcal{T}')$. Therefore, the restriction of the identity function on D onto $I_X(D)$ is a homeomorphism, as *every continuous 1-1 function of a compact space onto a Hausdorff space is a homeomorphism* [12]. Hence, X is C-almost normal.

The converse of Theorem 2 is not necessarily true in general. For example, the particular point topology $(\mathbb{R}, \mathcal{T}_p)$, Example 1, is a C-almost normal space, which is neither epi-almost normal, epi-normal nor sub-metrizable being not Hausdorff. The countable complement topology, Example 4, is a C-almost normal space, which is neither epi-normal, epi-almost normal nor sub-metrizable. Some other counterexamples are given in Section 4. In view of the fact: "*If X is a T_1-space such that the only compact subsets of X are the finite subsets, then X is C-normal*" [10], we conclude.

Corollary 2. If X is a T_1-space such that the only compact subsets of X are the finite subsets, then X is C-almost normal.

Theorem 3. *Every compact C-almost normal space is almost normal.*

Proof. Let X be a compact C-almost normal space. Then, there exist an almost normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace C of X. Since X is compact, take $C = X$. Since f is bijective, we get $f : X \to Y$ is a homeomorphism. Since Y is an almost normal space, we get X is almost normal.

Corollary 3. If X is a compact non almost normal space, then X cannot be C-almost normal.

A space X is called a *locally compact* space if for each $x \in X$ and each open neighborhood V of x there exists an open neighborhood U of x such that $x \in U \subseteq \overline{U} \subseteq V$ and $\overline{U}$ is compact [12]. In view of the fact: "*Every Hausdorff locally compact space is a C-normal space*"[10], we get:

Corollary 4. Every Hausdorff locally compact space is C-almost normal.

The converse of Corollary 4 is not necessarily true in general. For example, the Dieudonné plank topology, Example 5, is a C-almost normal space, which is not locally compact.

Theorem 4. *C-almost normality is a topological property.*

Proof. Let X be a C-almost normal space and $X \cong Z$. Let Y be an almost normal space and $f : X \to Y$ be a bijective function such that the restriction function $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Let $g : Z \to X$ be a homeomorphism. Then, Y and $f \circ g : Z \to Y$ satisfy the requirements.

Theorem 5. *C-almost normality is an additive property.*

Proof. Let X_s be a C-almost normal space for each $s \in S$. Then, there exist an almost normal space Y_s and a bijective function $f_s : X_s \to Y_s$ such that the restriction function $f_s|_{C_s} : C_s \to f_s(C_s)$ is a homeomorphism for each compact subspace $C_s \subseteq X_s$. Since Y_s is an almost normal space for each $s \in S$, the sum $\underset{s\in S}{\oplus} Y_s$ is almost normal. Consider the function sum [12], $\underset{s\in S}{\oplus} f_s : \underset{s\in S}{\oplus} X_s \to \underset{s\in S}{\oplus} Y_s$ defined by $\underset{s\in S}{\oplus} f_s(x) = f_t(x)$ if $x \in X_t$ and $t \in S$. Now, the subspace $C \subseteq \underset{s\in S}{\oplus} X_s$ is compact if and only if the set $S_0 = \{s \in S : C \cap X_s \neq \emptyset\}$ is finite and $C \cap X_s$ is compact in X_s for each $s \in S_0$. If $C \subseteq \underset{s\in S}{\oplus} X_s$ is compact, then $(\underset{s\in S}{\oplus} f_s)|_C$ is a homeomorphism because $f_s|_{(C\cap X_s)}$ is a homeomorphism for each $s \in S_0$. Hence, $\underset{s\in S}{\oplus} X_s$ is C-almost normal.

Note that: if X is a C-almost normal space and $f : X \to Y$ is a witness of the C-almost normality of X, then f may not be continuous. For example, the countable complement topology, Example 4, is a C-almost normal space and the witness of the C-almost normality of X is not continuous. But it will be if X is Fréchet. A space X is called a *Fréchet* space if for any subset B of X and any $x \in \overline{B}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of points of B such that $a_n \longrightarrow x$ [12].

Theorem 6. *If X is a C-almost normal Fréchet space and $f : X \to Y$ is a witness of the C-almost normality of X, then f is continuous.*

Proof. Let X be a C-almost normal Fréchet space and $f : X \to Y$ be a witness of the C-almost normality of X. Take $B \subseteq X$ and pick $y \in f(\overline{B})$. Since f is bijective, there exists a unique element $x \in X$ such that $y = f(x) \in f(\overline{B})$. Thus, $x \in \overline{B}$. Since X is Fréchet, there exists a sequence $(a_n) \subseteq B$ such that $a_n \longrightarrow x$. Since the subspace $K = \{x\} \cup \{a_n : n \in \mathbb{N}\}$ of X is compact, the induced mapping $f|_K : K \to f(K)$ is a homeomorphism. Let $W \subseteq Y$ be any open neighborhood of y. Then, $W \cap f(K)$ is an open neighborhood of y in the subspace $f(K)$. Since $f|_K$ is a homeomorphism, we have $f^{-1}(W \cap f(K)) = f^{-1}(W) \cap K$ is an open neighborhood of x in a subspace K of X. Then, there exists an $m \in \mathbb{N}$ such that $a_n \in f^{-1}(W \cap f(K))$ for each $n \geq m$. Hence, $f(a_n) \in W \cap f(K)$ for each $n \geq m$ and thus $W \cap f(B) \neq \emptyset$. So, we get $y \in \overline{f(B)}$. Therefore, we obtain $f(\overline{B}) \subseteq \overline{f(B)}$. Thus, f is continuous.

Recall that: a space X is called a *k-space* if X is Hausdorff and it is a quotient image of a locally compact space [12]. In view of the facts that: "*A function f from a k-space X into a space Y is continuous if and only if $f|_Z : Z \to Y$ is continuous for each compact subspace Z of X*", every first countable space is Fréchet and every Hausdorff locally compact is a k-space [12], we obtain:

Corollary 5. If X is a C-almost normal first countable (resp. k-space, Hausdorff locally compact) space and $f : X \to Y$ is a witness of the C-almost normality of X, then f is continuous.

Next, we give the following results:

Proposition 1. *If X is a T_1 C-almost normal space, then a witness Y is a T_1-space.*

Proof. Let X be a T_1 C-almost normal space. Since X is a C-almost normal space, there exist an almost normal space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_C : C \to f(C)$ is a homeomorphism for each compact subset $C \subseteq X$. Suppose Y is not T_1. Then, there exist two distinct elements x and y in Y such that if U is an open neighborhood of x, then $y \in U$ or if V is an open neighborhood of y, then $x \in V$. Thus, the set $M = \{f^{-1}(x), f^{-1}(y)\}$ is a T_1-compact subspace of X. Then, $f|_M : M \to f(M)$ is a homeomorphism. But $f(M) = \{x, y\}$ cannot be T_1, which is a contradiction. Therefore, Y must be T_1.

Proposition 2. *If X is a T_1 L-almost normal space, then a witness Y is a T_1-space.*

Proof. Similar to the proof of Proposition 1.

Theorem 7. *If $(X, \mathcal{T})$ is a C-almost normal Fréchet (resp. first countable, k-space) such that the witness $(Y, \mathcal{T}')$ of the C-almost normality of X is Hausdorff, then $(X, \mathcal{T})$ is epi-almost normal.*

Proof. Let X be a C-almost normal Fréchet space (resp. first countable, k-space) and the witness Y of the C-almost normality of X be Hausdorff. Then, there exist a Hausdorff almost normal space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_C : C \to f(C)$ is a homeomorphism for each compact subset $C \subseteq X$. Since X is Fréchet (resp. first countable, k-space), we have f is continuous. Define a topology $\mathcal{T}^\star$ on X as follows: $\mathcal{T}^\star = \{f^{-1}(U) : U \in \mathcal{T}'\}$. Clearly, $\mathcal{T}^\star$ is a topology on X, which is coarser than $\mathcal{T}$, such that $f : (X, \mathcal{T}^\star) \to (Y, \mathcal{T}')$ is continuous. If $W \in \mathcal{T}^\star$, then $W = f^{-1}(U)$ for some open set U in $\mathcal{T}'$. So, $f(W) = f(f^{-1}(U)) = U$, which is an open set in $(Y, \mathcal{T}')$. Thus, $f : (X, \mathcal{T}^\star) \to (Y, \mathcal{T}')$ is open and hence a homoeomorphism. Since $(Y, \mathcal{T}')$ is Hausdorff almost normal and $(X, \mathcal{T}^\star) \cong (Y, \mathcal{T}')$, we obtain $(X, \mathcal{T}^\star)$ is Hausdorff almost normal. Since $\mathcal{T}^\star \subseteq \mathcal{T}$, we conclude: $(X, \mathcal{T})$ is epi-almost normal

Since every Hausdorff nearly compact space is epi-normal [17, Theorem 17], every epi-normal space is epi-almost normal and every epi-almost normal space is C-almost normal (Theorem 2), we conclude:

Corollary 6.

(1) Every Hausdorff nearly compact space is C-almost normal.

(2) Every Hausdorff nearly paracompact space is C-almost normal.

Theorem 8. *Every L-almost normal space is C-almost normal.*

Proof. Let X be an L-almost normal space. Then, there exist an almost normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace A of X. Since every compact subset is Lindelöf, we have each compact subspace C of X is a Lindelöf subspace of X. Thus, the restriction function $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace C of X. Therefore, X is C-almost normal.

The converse of Theorem 8 is not necessarily true in general. For example, the countable complement extension topology, Example 10, and the Smirnov's deleted sequence topology, Example 9, are C-almost normal space, which are not L-almost normal.

Theorem 9. *Every L-normal space is L-almost normal.*

Proof. It is similar to that of Theorem 1.

The converse of Theorem 9 is not necessary to be true. For example, the finite complement topology, Example 3, is an L-almost normal space, which is not L-normal. The rational sequence topology, Example 19, is an L-almost normal space, which is not L-normal.

Theorem 10. *Every Lindelöf L-almost normal space is almost normal.*

Proof. It is similar to that of Theorem 3.

Corollary 7. *Every Lindelöf non almost normal space cannot be L-almost normal.*

Some counterexamples are given in Section 4. Note that: if X is L-almost normal and $f : X \to Y$ is a witness of the L-almost normality of X, then f may not be continuous, see Example 4. But it will be if X is of a countable tightness. A space X is of a *countable tightness* if for each subset B of X and each $x \in \overline{B}$, there exists a countable subset B_0 of B such that $x \in \overline{B}_0$ [12]. Note that:

$$\text{first countable} \Longrightarrow \text{Fréchet} \Longrightarrow \text{sequential} \Longrightarrow \text{countable tightness}$$

Theorem 11. *If X is an L-almost normal space of a countable tightness and $f : X \to Y$ is a witness of the L-almost normality of X, then f is continuous.*

Proof. Let A be any non-empty subset of X. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that $y = f(x)$. Then, $x \in \overline{A}$. Pick a countable subset $A_0 \subseteq A$ such that $x \in \overline{A_0}$. Let $B = \{x\} \cup A_0$. Then, B is a Lindelöf subspace of X and hence $f|_B : B \to f(B)$ is a homoeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of y. Then, $V \cap f(B)$ is open in the subspace $f(B)$ containing y. Thus, $f^{-1}(V) \cap B$ is an open set in the subspace B containing x. Thus, $(f^{-1}(V) \cap B) \cap A_0 \neq \emptyset$. So, $(f^{-1}(V) \cap B) \cap A \neq \emptyset$. Hence, $\emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A)$. Hence, $y \in \overline{f(A)}$. Thus, we obtain $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous.

Corollary 8. If X is an L-almost normal Fréchet (resp. first countable, sequential) space and $f : X \to Y$ is a witness of the L-almost normality of X, then f is continuous.

Theorem 12. *If X is a T_3 separable, L-almost normal space and of a countable tightness, then X is almost normal and epi-almost normal.*

Proof. Let X be a T_3 separable, L-almost normal space and of a countable tightness. Let Y be an almost normal space and $f : X \to Y$ be a bijective witness of the L-almost normality of X. Since X is of a countable tightness, we have f is continuous. Let D be a countable dense subset of X. We show that f is closed. Let H be any proper closed subset of X. Suppose $f(p) = q \in Y \setminus f(H)$, then $p \notin H$. By regularity of X, there are two disjoint open sets U and V in X such that $p \in U$ and $H \subseteq V$. Then, $U \cap (D \cup \{p\})$ is open in the Lindelöf subspace $D \cup \{p\}$. So, $f(U \cap (D \cup \{p\}))$ is open in the subspace $f(D \cup \{p\})$ of Y containing q. Then, $f(U \cap (D \cup \{p\})) = f(U) \cap f(D \cup \{p\}) = W \cap f(D \cup \{p\})$ for some open subset W in Y with $q \in W$. We claim $W \cap f(H) = \emptyset$. Suppose $W \cap f(H) \neq \emptyset$. Then, there exists an $y \in W \cap f(H)$. Let $x \in H$ such that $f(x) = y$. Note that $x \in V$. Since D is dense in X and also dense in the open set V, we get $x \in \overline{V \cap D}$. Since W is open in Y and f is continuous, we have $f^{-1}(W)$ is open in X containing x and $f^{-1}(W) \cap (V \cap D) \neq \emptyset$. Choose $d \in f^{-1}(W) \cap (V \cap D)$. Then, $f(d) \in W \cap f(V \cap D) \subseteq W \cap f(D \cup \{p\}) = f(U \cap (D \cup \{p\}))$. So, $f(d) \in f(U) \cap f(V)$, which is a contradiction. Hence, it must be $W \cap f(H) = \emptyset$. It can be observed that $q \in W$ as $q \in Y \setminus f(H)$ was arbitrary, then $f(H)$ is closed. So, f is a homeomorphism. Since Y is an almost normal space, we have X is almost normal. Since X is Hausdorff almost normal, we get X is epi-almost normal.

Since every second countable space is a Lindelöf separable space [12], and every Lindelöf L-almost normal space is almost normal (Theorem 10), we get:

Corollary 9.

(1) Every Hausdorff second countable L-almost normal space is epi-almost normal.

(2) Every second countable L-almost normal space is almost normal.

Observed that: epi-almost normality and L-almost normality are different from each other. the countable complement topology, Example 4, and the finite complement topology, Example 3, are L-almost normal spaces, which are not epi-almost normal because they are not Hausdorff. The Smirnov's deleted sequence topology, Example 9, and the countable complement extension topology, Example 10, are epi-almost normal spaces, which are not L-almost normal.

Theorem 13. *L-almost normality is a topological property.*

Proof. It is similar to that of Theorem 4.

Theorem 14. *L-almost normality is an additive property.*

Proof. The proof is similar to that of Theorem 5.

Theorem 15. *If X is a C-almost normal space such that every Lindelöf subspace of X is contained in a compact subspace of X, then X is L-almost normal.*

Proof. Let X be a C-almost normal space such that if A is a Lindelöf subspace of X, there exists a compact subspace B of X such that $A \subseteq B$. Let Y be any almost normal space and $f : X \to Y$ be a bijective function such that $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace C of X. Now, let A be any Lindelöf subspace of X. Pick a compact subspace B of X such that $A \subseteq B$. Then, $f|_B : B \to f(B)$ is a homeomorphism. Thus, $f|_A : A \to f(A)$ is a homeomorphism as $(f|_B)|_A = f|_A$. Therefore, X is L-almost normal.

Since every almost compact Urysohn space is almost regular, every Hausdorff almost regular nearly compact (nearly paracompact) space is almost normal, every Hausdorff almost compact almost regular space is almost normal [20, 21], and every Hausdorff paracompact space is almost normal [26], we get:

Corollary 10.

(1) Every Hausdorff almost compact almost regular (almost completely regular) space is L-almost normal.

(2) Every Urysohn almost compact space is L-almost normal.

(3) Every Hausdorff nearly compact almost regular (almost completely regular) space is L-almost normal.

(4) Every Hausdorff nearly paracompact almost regular (almost completely regular) space is L-almost normal.

3. Properties and relationships of both C-almost normality and L-almost normality

In this section, we present some properties and relationships of C-almost normality and L-almost normality:

Theorem 16. *Every C-completely regular Fréchet (resp. first countable, k-space) Lindelöf space is C-almost normal.*

Proof. Let X be a C-completely regular Fréchet (resp. first countable, k-space) Lindelöf space. Then, there exist a completely regular space Y and a bijective function $f : X \to Y$ such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. Since X is Fréchet (resp. first countable, k-space), we have f is continuous. Since a continuous image of a Lindelöf space is Lindelöf [12], we conclude: Y is a Lindelöf space. Hence, Y is normal because any completely regular Lindelöf space is normal [12]. Thus, X is a C-normal space. Hence, X is C-almost normal.

Theorem 17. *Every C-regular Fréchet (resp. first countable, k-space) Lindelöf space is C-almost normal.*

Proof. Similar to the proof of Theorem 16.

Since every T_1 C-completely regular Fréchet (resp. first countable, k-space) Lindelöf space is epi-normal, see Theorem 2.26 in [31], we conclude:

Corollary 11.

(1) Every T_1 C-completely regular Fréchet (resp. first countable, k-space) Lindelöf space is epi-almost normal.

(2) Every T_1 C-regular Fréchet (resp. first countable, k-space) Lindelöf space is epi-almost normal.

Since every C-regular Fréchet Lindelöf space is C-normal [8], and every σ-compact (resp. second countable) space is Lindelöf [12], we conclude:

Corollary 12.

(1) Every C-regular Fréchet (resp. first countable, k-space) σ-compact space is C-almost normal.

(2) Every C-completely regular Fréchet (resp. first countable, k-space) σ-compact space is C-almost normal.

(3) Every C-regular Fréchet Lindelöf space is C-almost normal.

(4) Every C-completely regular Fréchet Lindelöf space is C-almost normal.

The next result is obvious:

Theorem 18.

(1) *Every C-almost regular (resp. C-almost completely regular) compact space is C-almost normal.*

(2) *Every L-almost regular (resp. L-almost completely regular) Lindelöf paracompact space is L-almost normal.*

The next lemma can be proved easily:

Lemma 1. *Every T_1-almost normal space is almost regular [27].*

Theorem 19. *If X is a T_1 C-almost normal space, then X is C-almost regular.*

Proof. Let X be a T_1 C-almost normal space. Then, there exist an almost normal space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. By Proposition 1, Y is a T_1 almost normal space. By Lemma 1 we get: the space Y is almost regular. Therefore, X is C-almost regular.

Theorem 20. *If X is a T_1 L-almost normal space, then X is L-almost regular.*

Proof. Similar to the proof of Theorem 19.

Theorem 21. *Every C_2-almost regular Fréchet (resp. sequential, first countable) paracompact space is epi-almost normal and hence C-almost normal.*

Proof. Let X be a C_2-almost regular Fréchet (resp. sequential, first countable) paracompact. Then, there exist a Hausdorff almost regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. Since X is Fréchet (resp. sequential, first countable), we get f is continuous. By using similar arguments to the proof of Theorem 7, we conclude: X is Hausdorff paracompact. Since every Hausdorff paracompact space is T_4, we get X is Hausdorff almost normal. Therefore, $(X, \mathcal{T})$ is epi-almost normal and hence C-almost normal.

The proof of the next results is similar to that of Theorem 21:

Theorem 22.

(1) *Every C_2-almost completely regular Fréchet (resp. sequential, first countable) paracompact space is epi-almost normal and hence C-almost normal.*

(2) *Every L_2-almost regular first countable paracompact space is epi-almost normal.*

(3) *Every L_2-almost completely regular first countable paracompact space is epi-almost normal.*

Theorem 23. *Every C_2-almost regular Fréchet (resp. sequential, first countable) Lindelöf space is epi-almost normal.*

Proof. Let X be a C_2-almost regular Fréchet (resp. sequential, first countable) Lindelöf space. Then, there exist a Hausdorff almost regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. Since X is Fréchet (resp. sequential, first countable), we get f is continuous. Thus, $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_s{}')$ is 1-1, onto and continuous function such that $(Y, \mathcal{T}_s{}')$ is a T_3-space, where $(Y, \mathcal{T}_s{}')$ is a semi-regularization of $(Y, \mathcal{T}')$. Since a continuous image of a Lindelöf space is Lindelöf [12], and X is a Lindelöf space, we get: $(Y, \mathcal{T}_s{}')$ is T_3-Lindelöf. Since every regular Lindelöf space is paracompact and hence almost normal [12, 26], we get: $(Y, \mathcal{T}_s{}')$ is a T_3-almost normal space. Now, define $\mathcal{T}^* = \{f^{-1}(U) : U \in \mathcal{T}_s{}'\}$. By using similar arguments to the proof of Theorem 7, we conclude: $\mathcal{T}^*$ is a topology coarser than $\mathcal{T}$ and $(X, \mathcal{T}^*) \cong (Y, \mathcal{T}_s{}')$. Since $(Y, \mathcal{T}_s{}')$ is a T_3-almost normal space, we have: $(X, \mathcal{T}^*)$ is T_3-almost normal. Since $\mathcal{T}^* \subseteq \mathcal{T}$, we obtain: $(X, \mathcal{T})$ is epi-almost normal.

The proof of the next result is similar to that of Theorem 23:

Theorem 24.

(1) *Every C_2-almost completely regular Fréchet (resp. sequential, first countable) Lindelöf space is epi-almost normal.*

(2) *Every L_2-almost regular Fréchet (resp. sequential, first countable) Lindelöf space is epi-almost normal.*

(3) *Every L_2-almost completely regular Fréchet (resp. sequential, first countable) Lindelöf space is epi-almost normal.*

Since every epi-almost normal space is C-almost normal (Theorem 2), every epi-almost normal space is epi-completely regular [5], and every C_2-paracompact space is C-normal [24], we conclude:

Corollary 13.

(1) Every C_2-almost regular first countable Lindelöf space is C-almost normal.

(2) Every C_2-almost completely regular first countable Lindelöf space is C-almost normal.

(3) Every L_2-almost regular first countable Lindelöf space is L-almost normal.

(4) Every L_2-almost completely regular first countable Lindelöf space is L-almost normal.

(5) Every epi-almost normal space is C-completely regular.

(6) Every C_2-paracompact space is C-almost normal.

Since any closed extension space $(X^p, \mathcal{T}^*)$ of a given space $(X, \mathcal{T})$ is π-normal [1, Theorem 9], we obtain:

Corollary 14. Every closed extension space $(X^p, \mathcal{T}^*)$ of a given space $(X, \mathcal{T})$ is both C-almost normal and L-almost normal.

Since every quotient space of an almost normal space is almost normal [26], the next result is obvious:

Theorem 25. *If $(X, \mathcal{T})$ is a C-almost normal (resp. an L-almost normal) first countable space, then there exists a topology $\mathcal{T}^*$ coarser than $\mathcal{T}$ such that $(X, \mathcal{T}^*)$ is almost normal.*

Note that: any Tychonoff space Y has a one-point compactification X, $X = Y \cup \{p\}$, $p \notin Y$, X is a Hausdorff compact space and $\{p\}$ is closed and open subset of X [2]. Now, we get the following result:

Theorem 26. *If Y is a Tychonoff space, then any discrete extension space X_M of any compactification X of Y is C-almost normal.*

Proof. Let Y be a Tychonoff space. Then, any compactification X of a Tychonoff space Y is a Hausdorff compact space and hence it is a T_4-space. Now, if X_M is a discrete extension space of X, then the topology on X is coarser than the topology on X_M. Then, X_M is an epi-normal space. Hence, X_M is epi-almost normal. By Theorem 2, X_M is C-almost normal.

Theorem 27. *Every C-almost regular first countable Lindelöf space is C-almost normal.*

Proof. Let X be a C-almost regular first countable Lindelöf space. Then, there exist an almost regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. Since X is first countable, we get f is continuous. Thus, $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_s{}')$ is 1-1, onto and continuous function such that $(Y, \mathcal{T}_s{}')$ is a regular space. Since a continuous image of a Lindelöf space is Lindelöf [12], and X is Lindelöf, we get: $(Y, \mathcal{T}_s{}')$ is a regular Lindelöf space. Since every regular Lindelöf space is paracompact and hence almost normal [12, 26], we get: $(Y, \mathcal{T}_s{}')$ is an almost normal space such that $f|_A : A \to f(A)$ is a homeomorphism for each compact subset $A \subseteq X$. Therefore, X is C-almost normal.

Corollary 15. *Every C-almost completely regular first countable Lindelöf space is C-almost normal.*

From Theorem 27 and every Hausdorff first countable space is C-almost completely regular [31], we conclude the next corollary:

Corollary 16.

(1) Every Hausdorff first countable Lindelöf space is C-almost normal.

(2) Every Hausdorff first countable Lindelöf space is C-almost normal.

4. Some counterexamples

It can be observed that: any uncountable indiscrete space is an example of an L-almost normal space, which is neither C-Tychonoff nor epi-almost normal being not Hausdorff. The following example is an L-almost normal space, which is not C-regular.

Example 1. *The particular point topology* [29, Example 10], $(\mathbb{R}, \mathcal{T}_p)$ is neither a C-regular nor C-normal space [8, 10]. Since the particular point topology $(\mathbb{R}, \mathcal{T}_p)$ is almost normal, we get $(\mathbb{R}, \mathcal{T}_p)$ is C-almost normal and L-almost normal. Therefore, $(\mathbb{R}, \mathcal{T}_p)$ is an example of a C-almost normal and L-almost normal space, which is neither C-regular nor C-normal.

Note that: C-almost normality (resp. L-almost normality) does not imply to C-almost regularity. Here is a counterexample.

Example 2. *The excluded point topology* [29, Example 15], Let X be an uncountable set and $p \in X$ be fixed. The excluded point topology on X is denoted by $\mathcal{E}_p$ and defined as: $U \in \mathcal{E}_p \iff U = X \ \vee \ p \notin U$. Then, $(X, \mathcal{E}_p)$ is a topological space, which is a T_0, compact, first countable, paracompact and normal space, and it is neither T_1, regular nor semi regular [29]. Since the only open set containing p is X itself, any closed domain set in X contains p and any singleton $\{x\}$, where $x \neq p$, is open. Thus, any closed domain set F in X such that $x \notin F$, we obtain: x and F cannot be separated. Hence, X is neither almost regular nor almost completely regular. Since X is compact Lindelöf normal space, which is neither almost regular nor almost completely regular, the space X is C-almost normal and L-almost normal space, which is neither C-almost regular, L-almost regular, C-almost completely regular nor L-almost completely regular [3]. Since X is not T_1, we obtain: X is neither epi-almost regular nor epi-almost normal. Therefore, the space $(X, \mathcal{E}_p)$ is an example of a Lindelöf C-almost normal and L-almost normal space, which is neither C-almost regular, C-regular, C-completely regular, epi-almost regular nor epi-almost normal.

Example 3. *The finite complement topology* [29, Example 19], $(\mathbb{R}, \mathcal{CF})$ is a T_1-compact space and every subspace of $(\mathbb{R}, \mathcal{CF})$ is compact [29]. Note that: $(\mathbb{R}, \mathcal{CF})$ is not a C-regular space [8]. Hence, it is not C-Tychonoff. Since $(\mathbb{R}, \mathcal{CF})$ is an almost normal space, we get $(\mathbb{R}, \mathcal{CF})$ is both C-almost normal and L-almost normal. Therefore, $(\mathbb{R}, \mathcal{CF})$ is an example of a C-almost normal and L-almost normal space, which is neither C-normal, C-regular nor epi-almost normal.

Example 4. *The countable complement topology* [29, Example 20], $(\mathbb{R}, \mathcal{CC})$ is a C-regular space that is not L-regular [8]. Since $(\mathbb{R}, \mathcal{CC})$ is an almost normal space, we have $(\mathbb{R}, \mathcal{CC})$ is both C-almost normal and L-almost normal. Therefore, $(\mathbb{R}, \mathcal{CC})$ is an example of a C-almost normal and L-almost normal space, which is neither L-regular, L-Tychonoff, L-normal, normal, regular, epi-regular nor epi-almost normal.

The following example is a Tychonoff C-almost normal space, which is not locally compact:

Example 5. *The Dieudonné plank topology* [29, Example 89], is a C-normal space [10]. Hence, it is C-almost normal. It is well-known that the Dieudonné plank is a Tychonoff non-normal space, which is not locally compact [29]. Therefore, the Dieudonné plank topology is an example of a C-almost normal Tychonoff space, which is neither locally compact nor normal.

Example 6. *The deleted Tychonoff plank* [29, Example 87], is a Hausdorff locally compact space. By Corollary 4, the deleted Tychonoff plank is C-almost normal. It is well-known that the deleted Tychonoff plank is neither almost-normal nor sub-metrizable [8, 10]. Therefore, the deleted Tychonoff plank topology is an example of a C-almost normal Tychonoff space, which is neither sub-metrizable nor almost normal.

Example 7. The left ray topology $(\mathbb{R}, \mathcal{L})$ and the right ray topology $(\mathbb{R}, \mathcal{R})$ are almost normal spaces because they are normal. Thus, $(\mathbb{R}, \mathcal{L})$ and $(\mathbb{R}, \mathcal{R})$ are C-almost normal and L-almost normal spaces, which are neither C-regular nor epi-almost normal [7].

Example 8. *The modified Dieudonné plank* [16, Example 2.2], is an L-normal space which is not mildly normal. Hence, the modified Dieudonné plank is an L-almost normal and C-almost normal space, which is not almost normal.

The following example is an L-quasi normal space, which is neither L-almost normal, almost normal nor L-Tychonoff.

Example 9. *The Smirnov's deleted sequence topology* [29, Example 64], is a Urysohn, Lindelöf and second countable space, which is neither regular, normal, semi regular nor compact [29]. Since $\mathcal{U} \subseteq \mathcal{T}$, we get: the space X is sub-metrizable. Thus, it is an epi-almost normal space. Hence, it is a C-almost normal space. Since the Smirnov's deleted sequence topology is a quasi normal and C-regular space [8, 33], which is neither normal nor almost normal, we obtain that: the Smirnov's deleted sequence topology is an L-quasi normal space, but it is neither L-almost normal nor L-regular. Thus, the Smirnov's deleted sequence topology is an example of an epi-almost normal, C-almost normal, C-quasi normal and L-quasi normal space, which is neither almost normal, L-Tychonoff nor L-almost normal.

The next example is an epi-almost normal space, which is not L-almost normal.

Example 10. *The countable complement extension topology* [29, Example 63], is a Hausdorff, Urysohn and Lindelöf space, which is neither regular, completely regular, normal, semi regular, compact nor first countable [29]. Since a subset A of X is compact if and only if it is finite [29], we get X is C-almost normal. Since X is sub-metrizable, we have X is epi-almost normal. It can be observed that: the space X is a quasi normal and almost completely regular space, which is neither normal nor almost normal [30]. Thus, X is an L-almost completely regular and L-quasi normal space, but it is neither L-almost normal nor L-regular. So, the countable complement extension topology is an example of an epi-almost normal, epi-regular, L-quasi normal and epi-completely regular space, which is neither L-almost normal nor L-regular.

The next example is an epi-quasi normal and L-quasi normal space, which is neither epi-almost normal, L-almost normal, almost normal nor C-regular.

Example 11. *The simplified arens square topology* [29, Example 81], is a Hausdorff, semi regular, Lindelöf, σ-compact, separable, second countable and first countable space, which is neither regular, completely Hausdorff, Urysohn, normal, compact, paracompact nor countably compact [29]. Since X is a semi regular non regular space, we get: X is not almost regular. Since X is a T_1 non almost regular space, we obtain: X is not almost normal. The simplified arens square topology is an epi-quasi normal and quasi normal space, which is not semi normal [33]. Hence, it is C-quasi normal and L-quasi normal. Since X is not Urysohn, it is neither epi-almost normal, epi-regular nor epi-completely regular. Since X is a Hausdorff Lindelöf first countable space that is not epi-almost normal, by Corollary 11 the space X is neither C-regular, C-completely regular nor C-Tychonoff. Since X is a Lindelöf space that is neither almost normal nor almost regular, we conclude that: the space X is neither L-almost normal, L-almost regular nor L-completely regular. Note that: the simplified arens square topology is C-almost completely regular, C-almost regular and C-almost normal (see Theorem 27). Therefore, The simplified arens square topology is an example of an epi-quasi normal, quasi normal, L-quasi normal, semi regular, C-quasi normal, C-almost completely regular and C-almost normal space, which is neither epi-almost normal, epi-regular, C-regular, L-almost regular, Urysohn nor C-normal.

The following example is a C-almost normal space, which is neither L-almost normal nor almost normal.

Example 12. *The irregular lattice topology* [29, Example 79], is a Hausdorff, Urysohn, countable, σ-compact, Lindelöf and second countable space, which is neither regular, normal, semi regular, completely regular, Tychonoff, compact nor paracompact [29]. Since the irregular lattice topology is mildly normal space, which is not partially normal [6], we obtain : it is an epi-mildly normal space, which is neither almost normal, epi-almost normal, epi-quasi normal, almost regular, almost completely regular, almost compact, nearly paracompact, epi-regular nor H-closed [33]. Since X is a Lindelöf space which is neither almost normal nor almost regular, by Corollary 7 we get X is neither L-almost normal nor L-almost regular. Since X is Hausdorff second countable non epi-regular space, we conclude X is neither C-regular, C-completely regular, C-normal nor C-Tychonoff [31]. By Theorem 27, we conclude that: the irregular lattice topology is C-almost normal. Therefore, the irregular lattice topology is epi-mildly normal, C-almost regular, C-almost completely regular and C-almost normal space which is neither almost normal, almost regular, C-regular, epi-regular, epi-almost normal, C-normal nor L-almost normal.

Here is an example of an L-almost normal space that is neither C-almost regular nor epi-almost normal.

Example 13. *The integer broom topology* [29, Example 121], is a T_0, normal, semi normal, compact, Lindelöf, separable, countable and paracompact space, which is neither T_1, regular, completely regular nor semi regular [29]. Thus, X is an L-almost normal space.

Since X is not T_1, it is neither epi-almost regular nor epi-almost normal. Since the only open neighborhood of the origin is X itself [29], we conclude: any non-empty closed domain subset of X contains the origin. So, if A is a closed domain and $x \notin A$, we get: x and A cannot be separated. Hence, X is neither almost regular nor almost completely regular. Since X is a Lindelöf normal non Hausdorff space, the integer broom topology is an L-normal space, which is not C-almost regular. Therefore, the integer broom topology is an example of an L-almost normal compact (Lindelöf, paracompact) space, which is neither epi-almost normal, epi-almost regular nor C-almost regular.

Here is an example of an epi-almost completely regular, paracompact, Lindelöf space, which is neither C-almost normal nor C-regular.

Example 14. *The maximal compact topology:* [29, Example 99], is a compact, T_1, separable, paracompact, locally compact and Lindelöf space, which is neither Hausdorff, regular, completely regular, normal, first countable nor second countable [29]. X is a semi normal space, which is not normal [26, Example 2.3]. Since every semi normal mildly normal space is normal [28], and X is a semi normal non normal space, we obtain: X is not mildly normal. Hence, X is not almost normal. Since X is a T_1-semi normal space, we get: X is semi regular because every T_1-semi normal space is semi regular [26]. Since every semi regular almost regular space is regular [25], and X is a semi regular non regular space, we obtain: X is not almost regular. Since X is compact space which is neither almost regular, almost normal nor Hausdorff, we conclude: X is neither C-almost regular, C-almost normal, epi-regular nor epi-almost normal. Therefore, the maximal compact topology is an example of an epi-almost completely regular, paracompact, Lindelöf space, which is neither C-almost regular, epi-almost normal nor C-almost normal.

Example 15. *The modified fort space* [29, Example 27]: Let $X = \mathbb{N} \cup \{x_1, x_2\}$. Any subset $A \subseteq \mathbb{N}$ is open. Any set U such that $x_1 \in U$ or $x_2 \in U$ is open if and only if $\mathbb{N} - U$ is finite. Then, X is a T_1, compact and paracompact space, which is neither Hausdorff, regular, semi regular, normal, completely regular, separable nor first countable. The closure of any open set U containing x_1 contains x_2 [29]. Thus, if A is a non-empty closed domain subset of X such that $x_1 \in A$, then $x_2 \in A$. So, if A is a closed domain containing x_1 such that $y \notin A$, where $x_1 \neq y \neq x_2$, then y and A can be separated. Hence, the space X is almost regular and almost completely regular. Since X is not Hausdorff, the space X is not epi-almost normal. Since X is compact almost regular, we have X is an almost normal space which is not semi normal. Therefore, the modified fort space is an L-almost normal and L-almost completely regular space. Since X is a Lindelöf space that is neither regular, normal nor Hausdorff, the modified fort space is neither C-regular, C-completely regular, C-normal, C_2-almost regular nor C_2-almost completely regular [31]. Hence, it is an L-almost normal space, which is neither C-regular, C-normal, epi-almost normal nor C_2-almost regular.

Example 16. *The odd-even topology* [29, Example 6], is a regular, completely regular, normal, Lindelöf and locally compact, but it is neither T_0, compact nor semi regular [29]. Thus, the odd-even topology is a C-regular, C-completely regular, C-normal and C-almost normal space, which is neither epi-almost regular nor epi-almost normal because it is not T_1. The odd-even topology is also an L-normal, L-regular and L-almost normal space. Since every C-Tychonoff first countable space is Hausdorff (hence Urysohn), and X is a first countable non Hausdorff space, by Corollary 2 in [9], we obtain: X is not C-Tychonoff. Therefore, the odd-even topology is an example of a C-completely regular, C-almost normal and L-almost normal space, which is neither C-Tychonoff nor L-Tychonoff.

Example 17. *The relatively prime integer topology and the prime integer topology* [29, Example 60, 61], are Hausdorff, semi regular, Lindelöf, first countable spaces that are neither Urysohn, paracompact, almost normal, quasi normal, almost regular nor regular [33, Example 2.9]. The two spaces are epi-mildly normal spaces which are neither almost completely regular, almost regular, epi-almost normal, epi-regular nor epi-completely regular. Since the two spaces are Hausdorff Lindelöf first countable space which are not epi-almost normal, by Corollary 11 we get: the two spaces are neither C-regular nor C-completely regular. Hence, they are neither L-regular, L-completely regular nor L-Tychonoff. Since the two spaces are Lindelöf spaces which are neither almost normal nor almost regular, we conclude that they are neither L-almost normal nor L-almost regular. Since the two spaces are C-almost regular first countable Lindelöf spaces, by Theorem 27 we get: the two spaces are C-almost normal. Therefore, the relatively prime integer topology and the prime integer topology are C-almost completely regular and C-almost normal spaces which are neither L-almost normal, almost normal nor epi-almost normal.

Example 18. Let $X = \{a_{ij}, b_{ij}, c_i, a, b : i, j \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, and each $n \in \mathbb{N}$, define $U^n(c_i) = \{c_i, a_{ij}, b_{ij} : j \geq n\}$. For each $n \in \mathbb{N}$, define $U^n(a) = \{a, a_{ij} : i \geq n, j = 1, 2, 3, \dots\}$. And for each $n \in \mathbb{N}$, define $U^n(b) = \{b, b_{ij} : i \geq n, j = 1, 2, 3, \dots\}$. Now, for each $i, j \in \mathbb{N}$, we declare that the singletons $\{a_{ij}\}$ and $\{b_{ij}\}$ are open, i.e., each a_{ij} and b_{ij} are isolated points. A basic open neighborhood of a point c_i, where $i \in \mathbb{N}$, is of the form $U^n(c_i)$, where $n \in \mathbb{N}$. A basic open neighborhood of the point a is of the form $U^n(a)$, where $n \in \mathbb{N}$, and a basic open neighborhood of the point b is of the form $U^n(b)$, where $n \in \mathbb{N}$. Then, X is a semi regular space, which is not semi normal [26, Example 2.4]. X is also Hausdorff space, which is neither Urysohn, regular, mildly normal, compact, paracompact nor epi-mildly normal [17, Example 16]. Hence, X is neither almost normal, epi-almost normal, epi-normal, epi-regular nor epi-completely regular. Since X is a countable space, it is a Lindelöf second countable space. So, X is a Hausdorff Lindelöf second countable C-paracompact space, which is not C_2-paracompact [24, Example 2.25]. Since X is a Lindelöf non almost normal space, it is not L-almost normal. Since X is T_1-space, it is epi-almost completely regular. By Theorem 2.44 in [31], we obtain: X is C-almost completely regular. Since X is Lindelöf space, which is neither almost regular, almost completely regular nor almost normal, we get: X is neither L-almost regular, L-regular, L-completely regular nor L-almost normal. Since X is C-almost regular first countable Lindelöf space, by Theorem 27 we conclude that: the space X is C-almost normal. Therefore, the space X is an example of a Hausdorff countable C-almost normal space, which is neither almost normal, L-almost normal nor epi-almost normal.

The next example is an L-almost normal space which is neither L-normal.

Example 19. *The rational sequence topology* [29, Example 65], is a first countable, zero-dimensional, Tychonoff, locally compact, separable space which is neither paracompact, normal nor Lindelöf [29]. Also, $(\mathbb{R}, \mathcal{RS})$ is a regular, semi regular and almost normal space, which is not normal [35]. Observed that: in the rational sequence topology, we have the usual topology $\mathcal{U}$ on $\mathbb{R}$ is a topology coarser than $\mathcal{RS}$, i.e. $\mathcal{U} \subseteq \mathcal{RS}$. Since $(\mathbb{R}, \mathcal{U})$ is a Hausdorff normal space, we get: the rational sequence topology is epi-normal sub-metrizable space. Hence, it is an epi-almost normal space. Since X is Hausdorff almost normal space, we get: it is both C-almost normal and L-almost normal. Since every Tychonoff first countable separable L-normal space is normal [16], X is Tychonoff first countable separable and not normal, we get: X is not L-normal. Since every Hausdorff locally compact space is C-normal, we get the rational sequence topology is C-normal. Therefore, the rational sequence topology is an example of an epi-normal, epi-almost normal, C-almost normal, L-almost normal, C-normal, L-Tychonoff space which is not L-normal.

At the end of this research, we present the next remarks:

Remark 1. It can be observed that:

(1) C-almost normality (resp. L-quasi normality) does not imply to L-almost normality, and any epi-almost normal space is not necessary to be L-almost normal. For example: the Smirnov's deleted sequence topology, Example 9, and the countable complement extension topology, Example 10, are L-quasi normal, epi-almost normal and C-almost normal spaces, which are neither almost normal nor L-almost normal.

(2) L-almost normality and C-almost normality do not imply to epi-almost normality. For example: the integer broom topology, Example 13, is both a C-almost normal and L-almost normal space which is not epi-almost normal. The finite complement topology, Example 3, and the countable complement topology, Example 4, are both C-almost normal and L-almost normal spaces, which are not epi-almost normal.

(3) L-almost normality and C-almost normality do not imply to almost normality. For example: the modified Dieudonné plank topology, Example 8, is an L-almost normal and C-almost normal space, which is not almost normal.

(4) Epi-quasi normality and quasi normality do not imply to L-almost normality. For example, the simplified arens square topology, Example 11, is an epi-quasi normal, quasi normal, C-quasi normal and L-quasi normal space which is neither L-almost normal nor almost normal.

(5) C-almost normality does not imply almost normality or sub-metrizability. For example, the deleted Tychonoff plank, Example 6, is a C-almost normal space, which is neither sub-metrizable nor almost normal.

(6) L-almost normality does not imply to epi-almost normality nor C-almost regularity. For example: the excluded point topology, Example 2, is an L-almost normal and C-almost normal space, which is neither epi-almost normal nor C-almost regular.

(7) The space $\omega_1 \times (\omega_1 + 1)$ is both C-normal and L-normal space, which is not almost normal [16, Example 2.4]. Thus, the space $\omega_1 \times (\omega_1 + 1)$ is both C-almost normal and L-almost normal space, which is not almost normal.

(8) Every Hausdorff C-paracompact first countable Lindelöf space is not necessarily epi-almost normal. For example, the space presented in Example 18 is a Hausdorff C-paracompact first countable Lindelöf space, which is not epi-almost normal being not Urysohn.

(9) Almost normality is not preserved by a discrete extension space. For example, the space presented in [2, Example 15], $X = (\omega_1 \times Y) \cup \{p\}$, $p \notin \omega_1 \times Y$, be a one-point compactification of $\omega_1 \times Y$. Then, X is an almost normal space being Hausdorff compact space. But the discrete extension $X_{(\omega_1 \times Y)} = (\omega_1 \times Y) \cup \{p\}$, $\{p\}$ is closed and open subset of $X_{(\omega_1 \times Y)}$, is not almost normal being not mildly normal.

(10) Any T_1-compact space is not necessary to be C-almost normal. For example, the maximal compact topology, Example 14, is a T_1-compact space which is not C-almost normal.

(11) L-almost normality and C-almost normality are different from L-normality and C-normality respectively. For example, the rational sequence topology, Example 19, is an L-almost normal space, which is not L-normal. The finite complement topology, Example 3, is both L-almost normal and C-almost normal space which is neither C-normal nor L-normal.

(12) Every C-almost normal T_1 compact space is not necessarily epi-almost normal or Urysohn. For example: the modified fort space, Example 15, is a C-almost normal T_1-compact space, which is neither epi-almost normal nor Urysohn.

(13) If X is C-almost completely regular first countable Lindelöf space is not necessary to be epi-almost normal, almost normal nor L-almost normal. For example: the irregular lattice topology, Example 12, and the relatively prime integer topology and the prime integer topology, Example 17, are C-almost completely regular (C-almost regular) first countable Lindelöf spaces, which are neither epi-almost normal, almost normal nor L-almost normal.

(14) A C-almost normal first countable Lindelöf k-space is not necessary to be epi-almost normal. For example, the odd-even-topology, Example 16, is a C-almost normal first countable Lindelöf k-space, which is not epi-almost normal.

(15) Every Hausdorff epi-almost completely regular first countable Lindelöf k-space is not necessary to be epi-almost normal, almost normal nor L-almost normal. For example: the space presented in Example 18, is a Hausdorff epi-almost completely regular first countable Lindelöf k-space, which is neither epi-almost normal nor L-almost normal.

(16) If X is a Hausdorff C-almost normal space, then a witness Y is not necessary to be Hausdorff. For example: the relatively prime integer topology and the prime integer topology, Example 17, is a C-almost normal Hausdorff first countable space, and the witness Y of the C-almost normality of X cannot be Hausdorff because if Y is Hausdorff then X will be epi-almost normal which is a contradiction.

The following problems are still open until now in this work:

Problems:

(1) Is there an example of a Hausdorff locally compact space, which is not L-almost normal?.

(2) Is every C-almost normal T_1 first countable space, Hausdorff?.

(3) Is there an example of a Hausdorff space, which is not C-almost normal?.

(4) Are C-almost normality and L-almost normality preserved by discrete extension spaces?.

5. Conclusion

New versions of normality, called C-almost normality and L-almost normality, have been studied. Some results, properties, relationships and counterexamples have been presented and given.

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