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Structural and Spectral Properties of $k-$Quasi Class $Q(N)$ and $k-$Quasi Class $Q^*(N)$ Operators

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce two new classes of operators: $k-$quasi class $Q(N)$ and $k-$quasi class $Q^*(N)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is of $k-$quasi class $Q(N)$ for a fixed real number $N \geq 1$ and k a natural number, if T satisfies

$$N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^k x\|^2,$$

for all $x \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is of $k-$quasi class $Q^*(N)$ for a fixed real number $N \geq 1$ and k a natural number, if T satisfies

$$N\|T^*T^k x\|^2 \leq \|T^{k+2}x\|^2 + \|T^k x\|^2,$$

for all $x \in \mathcal{H}$. We study structural and spectral properties of these classes of operators. Also we compare this new classes of operators with other known classes of operators.

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1. Introduction

In this paper let $\mathcal{L}(\mathcal{H})$ stand for the C^* algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by ker$T$ the null space, by $T(\mathcal{H})$ the range of T. By $\sigma(T)$ we write the spectrum of T, the $r(T)$ is the spectral radius of operator T which is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The $\sigma_a(T)$ is the approximate point spectrum of operator T and it is proved that if $\lambda \in \sigma_a(T)$, then there exist the sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to \infty$. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$. The operator T is normaloid if $r(T) = \|T\|$ and it is quasinilpotent if $r(T) = 0$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be:

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- $k-$quasi$-$paranormal (see [9]) if

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^k x\|,$$

for all $x \in \mathcal{H}$, where k is a natural number.

- $k-$quasi$-*-$paranormal (see [5]) if

$$\|T^*T^k x\|^2 \leq \|T^{k+2}x\|\|T^k x\|,$$

for all $x \in \mathcal{H}$, where k is a natural number.

- $k-$quasi$-$class $\mathcal{A}$ operator (see [2]) if

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k.$$

- $k-$quasi$-*-$class $\mathcal{A}$ operator (see [10]), if

$$T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k.$$

- $k-$quasi class Q (see [4]) if

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^k x\|^2),$$

for all $x \in \mathcal{H}$, where k is a natural number. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the $k-$quasi class Q if

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O.$$

- $k-$quasi class Q^* (see [6]) if

$$\|T^*T^k x\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^k x\|^2\right),$$

for all $x \in \mathcal{H}$, where k is a natural number. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the $k-$quasi class Q^* if

$$T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \geq O.$$

- class $Q(N)$ (see [7]) if

$$N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2,$$

for all $x \in \mathcal{H}$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the class $Q(N)$ if

$$T^{*2}T^2 - NT^*T + I \geq O.$$

- Class $Q^*(N)$ (see [7]) if
$$N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2,$$
for all $x \in \mathcal{H}$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the class $Q^*(N)$ if
$$T^{*2}T^2 - NTT^* + I \geq O.$$

The following definitions describes the classes of operators we will study in this paper.

Definition 1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of $k-$quasi class $Q(N)$, for a fixed real number $N \geq 1$ and k a natural number, if T satisfies*
$$N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2,$$
for all $x \in \mathcal{H}$.

Definition 2. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q^*(N)$, for a fixed real number $N \geq 1$ and k a natural number, if T satisfies*
$$N\|T^*T^kx\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2,$$
for all $x \in \mathcal{H}$.

In this paper we give basic properties of $k-$quasi class $Q(N)$ and $k-$quasi class $Q^*(N)$ operators. We discuss some inclusion relations, the structural and spectral properties and also we obtained a matrix representation of these new classes of operators.

2. Inclusion Relations and Basic Properties

First, we state a proposition which gives necessary and sufficient conditions for an operator T to be of $k-$quasi class $Q(N)$.

Proposition 1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of $k-$quasi class $Q(N)$ if and only if*
$$T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq O,$$
for a fixed real number $N \geq 1$ and k a natural number.

Proof. Since T is of $k-$quasi class $Q(N)$, for a fixed real number $N \geq 1$ and k a natural number then
$$N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2,$$
for all $x \in \mathcal{H}$. Then,
$$\begin{aligned}
&(T^{k+2}x|T^{k+2}x) - N(T^{k+1}x|T^{k+1}x) + (T^kx|T^kx) \geq 0 \\
&\Leftrightarrow (T^{*(k+2)}T^{k+2}x|x) - N(T^{*(k+1)}T^{k+1}x|x) + (T^{*k}T^kx|x) \geq 0 \\
&\Leftrightarrow ((T^{*(k+2)}T^{k+2} - NT^{*(k+1)}T^{k+1} + T^{*k}T^k)x|x) \geq 0 \\
&\Leftrightarrow T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0.
\end{aligned}$$

From the definition of the $k-$quasi class $Q(N)$, we see that this new class of operators could compare to several classes of operators.

Proposition 2. *The following assertions hold.*

(i) class $Q(N) \subseteq k-$quasi class $Q(N)$.

(ii) $k-$quasi class $Q = k-$quasi class $Q(2)$.

(iii) $k-$quasi$-$paranormal $\subseteq k-$quasi class $Q(N)$ for $N \in [1, 2]$.

(iv) $k-$quasi$-$class $\mathcal{A} \subseteq k-$quasi class $Q(N)$ for $N \in [1, 2]$.

(v) $k-$quasi class $Q(N) \subseteq k-$quasi class $Q(N-1)$ for $N \geq 2$.

Proof.

(i) From the definition of the class $Q(N)$ operator

$$T^{*2}T^2 - NT^*T + I \geq O,$$

and the Proposition 1 we see that every operator of the class $Q(N)$ is also an operator of the $k-$quasi class $Q(N)$. Thus, we have the following implication:

$$\text{class } Q(N) \subseteq k-\text{quasi class } Q(N).$$

(ii) It is clear from definitions.

(iii) From the definition of $k-$quasi$-$paranormal we have:

$$\begin{aligned} \|T^{k+1}x\|^2 &\leq \|T^{k+2}x\| \|T^k x\| \\ &\leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^k x\|^2) \\ &\leq \frac{1}{N}(\|T^{k+2}x\|^2 + \|T^k x\|^2), \end{aligned}$$

for a real number $N \in [1, 2]$. This proves the result.

(iv) Since T belongs to $k-$quasi$-$class $\mathcal{A}$, we have

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2 T^k.$$

Let $x \in \mathcal{H}$. Then

$$\begin{aligned} 2\|T^{k+1}x\|^2 = & \\ & 2\langle T^{*(k+1)}T^{k+1}x, x\rangle \ = \\ & 2\langle T^{*k}|T|^2 T^k x, x\rangle \ \leq \\ & 2\langle T^{*k}|T^2|T^k x, x\rangle \ \leq \end{aligned}$$

$$2\||T^2|T^kx\| \cdot \|T^kx\| =$$
$$2\|T^{k+2}x\| \cdot \|T^k\| \leq$$
$$\|T^{k+2}x\|^2 + \|T^kx\|^2$$

Therefore

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^kx\|^2) \leq \frac{1}{N}(\|T^{k+2}x\|^2 + \|T^kx\|^2),$$

for a real number $N \in [1, 2]$. This proves the result.

(v) Since T is of $k-$quasi class $Q(N)$, for a fixed real number $N \geq 2$ and k a natural number then

$$N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2,$$

for all $x \in \mathcal{H}$. Then,

$$(N-1)\|T^{k+1}x\|^2 \leq N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2.$$

Hence, T is an operator of $k-$quasi class $Q(N-1)$. It follows that,

$$k-\text{quasi class } Q(N) \subseteq k-\text{quasi class } Q(N-1) \subseteq \ldots \subseteq k-\text{quasi class } Q(3) \subseteq k-\text{quasi class } Q(2).$$

Similarly, we state a proposition giving necessary and sufficient conditions for an operator T to be of $k-$quasi class $Q^*(N)$ and also the proposition where we compare this class of operator with other existing classes of operators.

Since the techniqes of the proofs of the resultes for both classes are almost the same we omit the proofs of the results of $k-$quasi class $Q^*(N)$.

Proposition 3. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of $k-$quasi class $Q^*(N)$, if and only if*

$$T^{*k}(T^{*2}T^2 - NTT^* + I)T^k \geq O,$$

for a fixed real number $N \geq 1$ and k a natural number.

Proposition 4. *The following assertions hold.*

(i) *class $Q^*(N) \subseteq k-$quasi class $Q^*(N)$.*

(ii) *$k-$quasi class $Q^* = k-$quasi class $Q^*(2)$.*

(iii) *$k-$quasi$-*-$paranormal $\subseteq k-$quasi class $Q^*(N)$for $N \in [1, 2]$.*

(iv) *$k-$quasi$-*-$class $\mathcal{A} \subseteq k-$quasi class $Q^*(N)$ for $N \in [1, 2]$.*

(v) *$k-$quasi class $Q^*(N) \subseteq k-$quasi class $Q^*(N-1)$ for $N \geq 2$.*

In the following we state a proposition which gives conditions for an operator T of the $k-$quasi class $Q(N)$ to be an operator of the class $Q(N)$.

Proposition 5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of the $k-$quasi class $Q(N)$. If T has dense range, then T is an operator of the class $Q(N)$.*

Proof. Since T^k has dense range then, $\overline{T^k(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $T^k(x_n) \to y$, $n \to \infty$. Since T is an operator of the $k-$quasi class $Q(N)$, then

$$\left\langle (T^{*k}(T^{*2}T^2 - NT^*T + I)T^k)x_n, x_n \right\rangle \geq 0,$$
$$\langle (T^{*2}T^2 - NT^*T + I)T^k x_n, T^k x_n \rangle \geq 0, \text{ for all } n \in \mathbb{N}.$$

By the continuity of the inner product, we have

$$\langle (T^{*2}T^2 - NT^*T + I)y, y \rangle \geq 0$$

Therefore T is an operator of the class $Q(N)$.

Similarly, we state a proposition which gives conditions for an operator T of the $k-$quasi class $Q^*(N)$ to be an operator of the class $Q^*(N)$.

Proposition 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of the $k-$quasi class $Q^*(N)$. If T^k has dense range, then T is an operator of the class $Q^*(N)$.*

In the following we give a proposition which gives conditions for an operator T of the $k-$quasi class $Q(N)$ to be an $k-$quasi$-$paranormal operator.

Proposition 7. *If T is a $k-$quasi class $Q(N)$ operator and T^2 is an isometry, then T is $k-$quasi$-$paranormal operator for $N \geq 2$.*

Proof. Let T be a $k-$quasi class $Q(N)$ operator. Then

$$N\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^k x\|^2 = (\|T^{k+2}x\| - \|T^k x\|)^2 + 2\|T^{k+2}x\|\|T^k x\|.$$

Suppose that T^2 is isometry, then

$$\|T^2 x\| = \|x\| \Rightarrow \|T^4 x\| = \|T^2 x\| \Rightarrow \cdots \Rightarrow \|T^{k+2}x\| = \|T^k x\|$$

for all $x \in \mathcal{H}$. From this we have

$$\|T^{k+1}x\|^2 \leq \frac{2}{N}\|T^{k+2}x\|\|T^k x\| \leq \|T^{k+2}x\|\|T^k x\|,$$

for $N \geq 2$. This proves the result.

Similarly, we give a proposition which gives conditions for an operator T of the $k-$quasi class $Q^*(N)$ to be an $k-$quasi$- * -$paranormal operator.

Proposition 8. *If T is a $k-$quasi class $Q^*(N)$ operator and T^2 is an isometry, then T is $k-$quasi$-*-$paranormal operator for $N \geq 2$.*

Now we will prove some properties of these new classes of operators.

Proposition 9. *Let T be an operator of $k-$quasi class $Q(N)$.*

(i) If T commutes with an isometric operator S, then TS is an operator of the $k-$quasi class $Q(N)$.

(ii) If S is unitarily equivalent to operator T, then S is an operator of the $k-$quasi class $Q(N)$.

Proof. Similarly as Proposition 2.4. in [7].

Proposition 10. *Let T be an operator of $k-$quasi class $Q^*(N)$.*

(i) If T commutes with an unitary operator S, then TS is an operator of the $k-$quasi class $Q^*(N)$.

(ii) If S is unitarily equivalent to operator T, then S is an operator of the $k-$quasi class $Q^*(N)$.

Proposition 11. *Let $T \in L(H)$. If $\|T\| \leq \dfrac{1}{\sqrt{N}}$, then T is an operator of $k-$quasi class $Q(N)$.*

Proof. From $\|T\| \leq \dfrac{1}{\sqrt{N}}$, we have

$$\|T\|^2 \leq \frac{1}{N}.$$

Then,

$$\begin{aligned}
&\|Tx\|^2 \leq \frac{1}{N}\|x\|, \text{for all} x \in H \\
&(Tx, Tx) - \frac{1}{N}(x, x) \leq 0, \text{for all} x \in H \\
&((I - NT^*T)x, x) \geq 0, \text{for all} x \in H \\
&I - NT^*T \geq 0 \\
&T^{*2}T^2 - NT^*T + I \geq 0 \\
&T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0
\end{aligned}$$

so T is an operator of $k-$quasi class $Q(N)$.

Proposition 12. *Let $T \in L(H)$. If $\|T^*\| \leq \dfrac{1}{\sqrt{N}}$, then T is an operator of $k-$quasi class $Q^*(N)$.*

Proposition 13. *Let M be a closed invariant subspace of $\mathcal{H}$ for the operator T of $k-$quasi class $Q(N)$. Then, the restriction $T_{|T^k(M)}$ is of class $Q(N)$.*

Proof. Let $u \in M$. Then

$$\begin{aligned}
&T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0\\
&(T^{*k}(T^{*2}T^2 - NT^*T + I)T^k u, u) \geq 0\\
&((T^{*2}T^2 - NT^*T + I)T^k u, T^k u) \geq 0\\
&((T^{*2}T^2 - NT^*T + I)y, y) \geq 0, y = T^k u \in M.
\end{aligned}$$

It follows that the restriction $T_{|T^k(M)}$ is of class $Q(N)$.

Proposition 14. *Let M be a closed invariant subspace of $\mathcal{H}$ for the operator T of $k-$quasi class $Q^*(N)$. Then, the restriction $T_{|T^k(M)}$ is of class $Q^*(N)$.*

Proposition 15. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of $k-$quasi class $Q(N)$ if and only if TST^{-1} is of $k-$quasi class $Q(N)$.*

Proof. Let S be an operator of $k-$quasi class $Q(N)$. Then

$$S^{*k}(S^{*2}S^2 - NS^*S + I)S^k \geq 0.$$

From this we have

$$TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^* \geq 0.$$

Now, we prove that operator TT^* commutes with operator $TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*$. Since operator S commutes with operator T^*T, operator S^* also commutes with operator T^*T. From this we have

$$\begin{aligned}
&TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*[TT^*]\\
&= TS^{*k}(S^{*2}S^2 - NS^*S + I)S^k[T^*T]T^*\\
&= T[T^*T]S^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*\\
&= [TT^*]TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*
\end{aligned}$$

Thus operator TT^* commutes with operator $TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*$. Then, operator $[TT^*]^{-1}$ also commutes with operator $TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*$. Since the operator $[TT^*]^{-1}$ and $TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*$ are positive, then

$$TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*[TT^*]^{-1} \geq 0.$$

Since operator S commutes with operator T^*T, we get

$$
\begin{aligned}
&(TST^{-1})^{*k} = (TST^{-1})^*(TST^{-1})^* \dots (TST^{-1})^* \\
&= T^{*-1}S^*T * T^{*-1}S^*T^* \dots T^{*-1}S^*T^* = T^{*-1}S^{*k}T^* \\
&(TST^{-1})^k = TS^kT^{-1} \\
&(TST^{-1})^*(TST^{-1}) = T^{*-1}S^*T^*TST^{-1} = TS^*ST^{-1} \\
&(TST^{-1})^{*2}(TST^{-1})^2 = T^{*-1}S^{*2}T^*TS^2T^{-1} = TS^{*2}S^2T^{-1}
\end{aligned}
$$

To prove that operator $TST^{-1} = M$ is an operator of $k-$quasi class $Q(N)$, we substitute last equations in the above expression and obtain

$$
\begin{aligned}
&M^{*k}(M^{*2}M^2 - NM^*M + I)M^k \\
&T^{*-1}S^{*k}T^*[TS^{*2}S^2T^{-1} - NTS^*ST^{-1} + I]TS^kT^{-1} \\
&= TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^{-1}
\end{aligned}
$$

Now we prove that the last expression is positive. Since

$$
\begin{aligned}
&TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*[TT^*]^{-1} \geq 0 \\
&TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^*T^{*-1}T^{-1} \geq 0 \\
&TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^{-1} \geq 0
\end{aligned}
$$

Hence, operator TST^{-1} is an operator of $k-$quasi class $Q(N)$.

Conversely, let $TST^{-1} = M$ be an operator of $k-$quasi class $Q(N)$. Then

$$M^{*k}(M^{*2}M^2 - NM^*M + I)M^k \geq 0.$$

Similarly, we have that

$$
\begin{aligned}
&TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^{-1} \geq 0 \\
&T^*TS^{*k}(S^{*2}S^2 - NS^*S + I)S^kT^{-1}T \geq 0 \\
&[T^*T]S^{*k}(S^{*2}S^2 - NS^*S + I)S^k \geq 0.
\end{aligned}
$$

Operator T^*T commutes with operator S and hence with operator $[T^*T]S^{*k}(S^{*2}S^2 - NS^*S+I)S^k$. Therefore, operator $[T^*T]^{-1}$ also commutes with operator $[T^*T]S^{*k}(S^{*2}S^2 - NS^*S + I)S^k$. Since these operators are positive, we have

$$[T^*T]^{-1}[T^*T]S^{*k}(S^{*2}S^2 - NS^*S + I)S^k \geq 0.$$

Therefore,

$$S^{*k}(S^{*2}S^2 - NS^*S + I)S^k \geq 0.$$

What does it mean that operator S is of $k-$quasi class $Q(N)$.

Proposition 16. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of $k-$quasi class $Q^*(N)$ if and only if TST^{-1} is of $k-$quasi class $Q^*(N)$.*

The next proposition give necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be an operator of this class of operators.

Proposition 17. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of $k-$ quasi class $Q(N)$ if and only if*

$$|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - N|\alpha_{n+k}|^2 + 1 \geq 0.$$

for every n and k natural numbers.

Proof. Since T is a weighted shift, its adjoint T^* is also a weighted shift and defined by $T(e_n) = |a_n|e_{n+1}, T^*(e_n) = |a_{n-1}|e_{n-1}$. Thus, we have

$$\begin{aligned}
&(T^*T)(e_n) = |a_n|^2 e_n \\
&(T^{*2}T^2)(e_n) = |a_n|^2|a_{n+1}|^2 e_n \\
&T^k(e_n) = |a_n||a_{n+1}|\ldots|a_{n+k-1}e_{n+k} \\
&(T^*T)(e_{n+k}) = |a_{n+k}|^2 e_{n+k} \\
&(T^{*2}T^2)(e_{n+k}) = |a_{n+k}|^2|a_{n+k+1}|^2 e_{n+k} \\
&T^{*k}(e_{n+k}) = |a_{n+k-1}||a_{n+k-2}|\ldots|a_n|e_n
\end{aligned}$$

Now, since T is an operator of $k-$ quasi class $Q(N)$, we have

$$T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0.$$

From

$$\begin{aligned}
&T^{*k}(T^{*2}T^2 - NT^*T + I)T^k(e_n) \\
&= |a_n||a_{n+1}|\ldots|a_{n+k-1}|T^{*k}(T^{*2}T^2 - NT^*T + I)(e_{n+k}) \\
&= |a_n||a_{n+1}|\ldots|a_{n+k-1}|(|a_{n+k}|^2|a_{n+k+1}|^2 - N|a_{n+k}|^2 + 1)T^{*k}(e_{n+k}) \\
&= |a_n|^2|a_{n+1}|^2\ldots|a_{n+k-1}|^2(|a_{n+k}|^2|a_{n+k+1}|^2 - N|a_{n+k}|^2 + 1)(e_n) \geq o.
\end{aligned}$$

we have inequality

$$|a_{n+k}|^2|a_{n+k+1}|^2 - N|a_{n+k}|^2 + 1 \geq 0.$$

Example 1. *Consider the operator $T : l^2 \to l^2$ defined by*

$$T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$$

where $\alpha_n = \dfrac{N}{2^N}$ for a natural number $N \geq 1$.

This operator T is of $k-$quasi class $Q(N)$ and quasi nilpotent but it is not quasi-hyponormal.

To prove that operator T is of $k-$quasi class $Q(N)$ and quasi nilpotent it is similarly as Example 2.5 in [6]. This operator is not quasi-hyponormal from the fact that:

$$\alpha_n \not\leq \alpha_{n+1}$$

(see Proposition 3.4 in [8]).

Similarly, the next proposition give necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be an operator of this class of operators.

Proposition 18. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of $k-$ quasi class $Q^*(N)$ if and only if*

$$|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - N|\alpha_{n+k-1}|^2 + 1 \geq 0$$

for every n.

In following propositions we give the inclusion of approximate point spectrum of these classes of operators.

Proposition 19. *Let $T \in L(H)$ be a regular $k-$quasi class $Q(N)$ operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} \leq |\lambda| \leq \|T\|\}.$$

Proof. Let T be a regular $k-$quasi class $Q(N)$ operator. For every unit vector x in Hilbert space $\mathcal{H}$, we have:

$$\begin{aligned}
&\|x\|^2 = \|(T^{k+1})^{-1} \cdot (T^{k+1})x\|^2 \\
\leq &\|(T^{k+1})^{-1}\|^2 \cdot \|T^{k+1}x\|^2 \\
\leq &\|(T^{k+1})^{-1}\|^2 \cdot \frac{1}{N} \cdot (\|T^{k+2}x\|^2 + \|T^k x\|^2) \\
\leq &\frac{1}{N} \cdot \|(T^{k+1})^{-1}\|^2 \cdot (\|T^{k+1}\|^2 \cdot \|Tx\|^2 + \|T^{k-1}\|^2 \cdot \|Tx\|^2).
\end{aligned}$$

So,

$$N \leq \|Tx\|^2 \cdot \|(T^{k+1})^{-1}\|^2 \cdot (\|T^{k+1}\|^2 + \|T^{k-1}\|^2),$$

where we have

$$\|Tx\| \geq \frac{\sqrt{N}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}}.$$

Now, assume that $\lambda \in \sigma_a(T)$, then there exists a sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to \infty$.

From the last inequation we have:

$$\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda| \cdot \|x_n\| \geq \frac{\sqrt{N}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} - |\lambda|.$$

Now, when $n \to \infty$ we have

$$|\lambda| \geq \frac{\sqrt{N}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}}.$$

So, we have

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} \leq |\lambda| \leq \|T\|\}.$$

Therefore the proof is completed.

Proposition 20. *Let $T \in L(H)$ be a regular $k-$quasi class $Q^*(N)$ operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|(T^*T^k)^{-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} \leq \lambda \leq \|T\|\}.$$

Let $T = U|T|$ be the polar decomposition of operator T. The Aluthge transformation of operator T given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced by Aluthge (see [1]). The adjoint of Aluthge transformation, the $*-$Aluthge transformation is defined by Yamazaki (see [11]) as $\tilde{T}^{(*)} \stackrel{def}{=} (\tilde{T^*})^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$.

The following propositions give the equivalence between Aluthge transformation and $*-$Aluthge transformation of these new classes of operators.

Proposition 21. *Let $T \in L(H)$. Then, $\tilde{T}$ is an operator of quasi class $Q(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of quasi class $Q(N)$.*

Proof. Similarly as Theorem 2.14 in [7].

Proposition 22. *Let $T \in L(H)$. Then, $\tilde{T}$ is an operator of quasi class $Q^*(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of quasi class $Q^*(N)$.*

3. A Matrix Representation

In this section we give some results for the matrix representation of these classes of operators.

Proposition 23. *Let $T \in \mathcal{L}(\mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},$$

B is any operator. If A is operator of class $Q(N)$ then T is an operator of $k-$quasi class $Q(N)$.

Proof. Let be $D = A^{*2}A^2 - NA^*A + I$. A simple calculation shows that:

$$\begin{aligned}
&T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}, \\
&T^{*(k+2)} = \begin{pmatrix} A^{*(k+2)} & 0 \\ B^*A^{*(k+1)} & 0 \end{pmatrix}, \\
&T^{(k+2)} = \begin{pmatrix} A^{(k+2)} & A^{(k+1)}B \\ 0 & 0 \end{pmatrix}, \\
&T^{*(k+2)}T^{(k+2)} = \begin{pmatrix} A^{*(k+2)}A^{(k+2)} & A^{*(k+2)}A^{(k+1)}B \\ B^*A^{*(k+1)}A^{(k+2)} & B^*A^{*(k+1)}A^{(k+1)}B \end{pmatrix}. \\
&T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \\
=&T^{*(k+2)}T^{(k+2)} - NT^{*(k+1)}T^{(k+1)} + T^{*k}T^k \\
=&\begin{pmatrix} A^{*k}DA^k & A^{*k}DA^{(k-1)}B \\ B^*A^{*(k-1)}DA^k & B^*A^{*(k-1)}DA^{(k-1)}B \end{pmatrix}
\end{aligned}$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (T^{*(k+2)}T^{(k+2)} - NT^{*(k+1)}T^{(k+1)} + T^{*k}T^k)u, u\rangle \\
&= \langle A^{*k}DA^kx, x\rangle + \langle A^{*k}DA^{(k-1)}By, x\rangle \\
&+ \langle B^*A^{*(k-1)}DA^kx, y\rangle + \langle B^*A^{*(k-1)}DA^{(k-1)}By, y\rangle \\
&= \langle DA^kx, A^kx\rangle + \langle DA^{(k-1)}By, A^kx\rangle \\
&+ \langle DA^kx, A^{(k-1)}By\rangle + \langle DA^{(k-1)}By, A^{(k-1)}By\rangle \\
&= \langle D(A^kx + A^{(k-1)}By), (A^kx + A^{(k-1)}By)\rangle \geq 0
\end{aligned}$$

because A is operator of class $Q(N)$ then, $D = A^{*2}A^2 - NA^*A + I \geq O$, so T is operator of $k-$quasi class $Q(N)$.

Proposition 24. *Suppose that T^k does not have a dense range, then the following statements are equivalent:*

(i) Operator T is a $k-$quasi class $Q(N)$ operator;

(ii) $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *on* $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, *where A is an operator of the class $Q(N)$ on $\overline{T^k(\mathcal{H})}, C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}, B$ is any operator.*

Proof. (1) $\Rightarrow$ (2) Suppose that $T \in \mathcal{L}(\mathcal{H})$ is an operator of $k-$quasi class $Q(N)$. Since that T^k does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Since T is an operator of $k-$quasi class $Q(N)$, we have

$$T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0.$$

Therefore, after some calculation similar as in Proposition 9 we get:

$$\langle (T^{*2}T^2 - NT^*T + I)x, x\rangle = \langle (A^{*2}A^2 - NA^*A + I)x, x\rangle \geq 0,$$

for all $x \in \overline{T^k(\mathcal{H})}$.

Hence

$$A^{*2}A^2 - NA^*A + I \geq 0.$$

This shows that A is an operator of the class $Q(N)$ on $\overline{T^k(\mathcal{H})}$.

Let P be the orthogonal projection of H onto $\overline{T^k(\mathcal{H})}$. For any

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Then

$$\langle C^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-P)x\rangle = 0.$$

Thus $T^{*k} = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [3, Corollary 7]. Since $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where A is an operator of the class $Q(N)$ on $\overline{T^k(\mathcal{H})}$, and $C^k = O$.

A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix},$$

$$T^*T = \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B + C^*C \end{pmatrix},$$

$$T^{*k} = \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* & 0 \end{pmatrix},$$
$$T^k = \begin{pmatrix} A^k & (\sum_{j=0}^{k-1} A^j BC^{k-1-j}) \\ 0 & 0 \end{pmatrix},$$

Then, we have

$$\begin{aligned} &T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \\ &= \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* & 0 \end{pmatrix} \\ &\times \begin{pmatrix} D & A^{*2}AB + A^{*2}BC \\ B^*A^*A^2 + C^*B^*A^2 - 2B^*A & |AB + BC|^2 + |C^2|^2 - N(B^*B + C^*C) + I \end{pmatrix} \\ &\times \begin{pmatrix} A^k & \sum_{j=0}^{k-1} A^j BC^{k-1-j} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{*k}DA^k & A^{*k}D\sum_{j=0}^{k-1} A^j BC^{k-1-j} \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* DA^k & M \end{pmatrix}, \end{aligned}$$

where $D = A^{*2}A^2 - NA^*A + I, M = (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* D \sum_{j=0}^{k-1} A^j BC^{k-1-j}$. Let $v = x \oplus y$ be a vector in $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where $x \in \overline{T^k(\mathcal{H})}$ and $y \in \ker T^{*k}$. Then,

$$\begin{aligned} &\left\langle T^{*k}(T^{*2}T^2 - NT^*T + I)T^k v, v \right\rangle \\ &= \left\langle A^{*k}DA^k x, x \right\rangle \\ &+ \left\langle A^{*k}D\sum_{j=0}^{k-1} A^j BC^{k-1-j} y, x \right\rangle \\ &+ \left\langle (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* DA^k x, y \right\rangle \\ &+ \left\langle (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* D \sum_{j=0}^{k-1} A^j BC^{k-1-j} y, y \right\rangle \\ &= \left\langle D(A^k x + \sum_{j=0}^{k-1} A^j BC^{k-1-j} y), A^k x + \sum_{j=0}^{k-1} A^j BC^{k-1-j} y \right\rangle. \end{aligned}$$

Since A is an operator of the class $Q(N)$, we have that $D = A^{*2}A^2 - NA^*A + I \geq 0$. Therefore,

$$\left\langle T^{*k}(T^{*2}T^2 - NT^*T + I)T^k v, v \right\rangle \geq 0$$

for all $v \in \mathcal{H}$. Hence,

$$T^{*k}(T^{*2}T^2 - NT^*T + I)T^k \geq 0$$

So we have that T is a $k-$quasi class $Q(N)$ operator.

Proposition 25. *If T^k does not have a dense range, then the following statements are equivalent:*

(i) $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *on* $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, *where* $A^{*2}A^2 - N(AA^* + BB^*) + I \geq 0, C^k = O$ *and* $\sigma(T) = \sigma(A) \cup \{0\}, B$ *is any operator;*

(ii) Operator T is a $k-$quasi class $Q^*(N)$ operator.

Proof.

(1) $\Rightarrow$ (2) Suppose that $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where $A^{*2}A^2 - N(AA^* + BB^*) + I \geq 0$ and $C^k = O$.

A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix},$$

$$TT^* = \begin{pmatrix} AA^* + BB^* & BC^* \\ CB^* & CC^* \end{pmatrix},$$

$$T^{*k} = \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* & 0 \end{pmatrix},$$

$$T^k = \begin{pmatrix} A^k & (\sum_{j=0}^{k-1} A^j BC^{k-1-j}) \\ 0 & 0 \end{pmatrix},$$

Then, we have

$$\begin{aligned} &T^{*k}(T^{*2}T^2 - NTT^* + I)T^k \\ &= \begin{pmatrix} A^{*k}DA^k & A^{*k}D\sum_{j=0}^{k-1} A^j BC^{k-1-j} \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* DA^k & M \end{pmatrix}, \end{aligned}$$

where $D = A^{*2}A^2 - N(AA^* + BB^*) + I, M = (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* D \sum_{j=0}^{k-1} A^j BC^{k-1-j}$. Let $v = x \oplus y$ be a vector in $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where $x \in \overline{T^k(\mathcal{H})}$ and $y \in \ker T^{*k}$. Then,

$$\begin{aligned} &\left\langle T^{*k}(T^{*2}T^2 - NTT^* + I)T^k v, v \right\rangle \\ &= \left\langle A^{*k}DA^k x, x \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \left\langle A^{*k} D \sum_{j=0}^{k-1} A^j B C^{k-1-j} y, x \right\rangle \\
&+ \left\langle (\sum_{j=0}^{k-1} A^j B C^{k-1-j})^* D A^k x, y \right\rangle \\
&+ \left\langle (\sum_{j=0}^{k-1} A^j B C^{k-1-j})^* D \sum_{j=0}^{k-1} A^j B C^{k-1-j} y, y \right\rangle \\
&= \left\langle D(A^k x + \sum_{j=0}^{k-1} A^j B C^{k-1-j} y), A^k x + \sum_{j=0}^{k-1} A^j B C^{k-1-j} y \right\rangle .
\end{aligned}
$$

Since $D = A^{*2}A^2 - N(AA^* + BB^*) + I \geq 0$, we have

$$\left\langle T^{*k}(T^{*2}T^2 - NTT^* + I)T^k v, v \right\rangle \geq 0$$

for all $v \in \mathcal{H}$. Hence,

$$T^{*k}(T^{*2}T^2 - NTT^* + I)T^k \geq 0$$

So we have that T is a $k-$quasi class $Q^*(N)$ operator.

(2) $\Rightarrow$ (1) Suppose that $T \in \mathcal{L}(\mathcal{H})$ is an operator of $k-$quasi class $Q^*(N)$. Since that T^k does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \;\; \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Since T is an operator of $k-$quasi class $Q^*(N)$, we have

$$T^{*k}(T^{*2}T^2 - NTT^* + I)T^k \geq 0.$$

Let P be the projection onto $\overline{T^k(\mathcal{H})}$. Than we have

$$P(T^{*2}T^2 - NTT^* + I)P \geq 0.$$

Therefore,

$$A^{*2}A^2 - N(AA^* + BB^*) + I \geq 0.$$

On the other hand, for any $x = (x_1, x_2) \in \mathcal{H}$, we have

$$\langle C^k x_2, x_2 \rangle = \langle T^k(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^{*k}(I-P)x \rangle = 0,$$

which implies $C^k = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [3, Corollary 7] and $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$.

Corollary 1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $k-$quasi class $Q^*(N)$ operator, the range of T^k not to be dense, then*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad on \;\; \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

where A is an operator of the class $Q^*(N)$ on $\overline{T^k(\mathcal{H})}$, $C^k = O$.

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