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Upper and lower almost contra-(Λ, sp)-continuity

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Abstract. Our main purpose is to introduce the concepts of upper and lower almost contra-(Λ, sp)-continuous multifunctions. Moreover, several characterizations of upper and lower almost contra-(Λ, sp)-continuous multifunctions are investigated.

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1. Introduction

In 1996, Dontchev [8] introduced and studied the concept of contra-continuous functions. In 1999, Dontchev and Noiri [10] considered a slightly weaker form of contra-continuity called contra-semicontinuity and investigated the class of strongly S-closed spaces. In 2001, Caldas and Jafari [7] introduced and investigated the concept of contra-β-continuous functions. In 2002, Jafari and Noiri [16] introduced and studied a new form of functions called contra-precontinuous functions. In 2004, Ekici [11] introduced and investigated almost contra-precontinuity as a new generalization of regular set-connectedness [9], contra-precontinuity [16], contra-continuity [8], almost s-continuity [19] and perfect continuity [18]. In 2005, Nasef [17] defined a new class of functions called contra-γ-continuous functions which lies between classes of contra-semicontinuous functions and contra-β-continuous functions. The first initiation of the concept of contra-continuous multifunctions has been done by Ekici et al. [12]. In 2009, Ekici et al. [13] introduced and studied a new generalization of contra-continuous multifunctions called almost contra-continuous multifunctions. In 2010, Ekici et al. [14] introduced and studied two new concepts namely contra-precontinuous multifunctions and almost contra-precontinuous multifunctions which are containing the class of contra-continuous multifunctions and contained in the class of weakly precontinuous multifunctions. In 2018, Boonpok et al. [6] introduced and studied the notions of upper and lower almost (τ_1, τ_2)-precontinuous multifunctions.

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Abd El-Monsef et al. [15] introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets [1]. Noiri and Hatir [20] introduced the concept of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The concept of (Λ, sp)-continuous multifunctions was introduced and investigated in [3]. The purpose of the present paper is to introduce the notions of upper and lower almost contra-(Λ, sp)-continuous multifunctions. In particular, several characterizations of upper and lower almost contra-(Λ, sp)-continuous multifunctions are discussed.

2. Preliminaries

Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [15] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [20] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [20] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda, sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda, sp)}$.

Lemma 1. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda, sp)}$ and $[A^{(\Lambda, sp)}]^{(\Lambda, sp)} = A^{(\Lambda, sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda, sp)} \subseteq B^{(\Lambda, sp)}$.

(3) $A^{(\Lambda, sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A = A^{(\Lambda, sp)}$.

Lemma 2. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-interior, the following properties hold:*

(1) $A_{(\Lambda, sp)} \subseteq A$ and $[A_{(\Lambda, sp)}]_{(\Lambda, sp)} = A_{(\Lambda, sp)}$.

(2) If $A \subseteq B$, then $A_{(\Lambda, sp)} \subseteq B_{(\Lambda, sp)}$.

(3) $A_{(\Lambda, sp)}$ is (Λ, sp)-open.

(4) *A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.*

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

A subset A of a topological space (X, τ) is said to be $s(\Lambda, sp)$-*open* (resp. $p(\Lambda, sp)$-*open*, $\beta(\Lambda, sp)$-*open*, $\alpha(\Lambda, sp)$-*open*, $r(\Lambda, sp)$-*open*) if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$) [3]. The family of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $p\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$, $\alpha\Lambda_{sp}O(X, \tau)$, $r\Lambda_{sp}O(X, \tau)$). The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) set is said to be $s(\Lambda, sp)$-*closed* (resp. $p(\Lambda, sp)$-*closed*, $\beta(\Lambda, sp)$-*closed*, $\alpha(\Lambda, sp)$-*closed*, $r(\Lambda, sp)$-*closed*). The family of all $s(\Lambda, sp)$-closed (resp. $p(\Lambda, sp)$-closed, $\beta(\Lambda, sp)$-closed, $\alpha(\Lambda, sp)$-closed, $r(\Lambda, sp)$-closed) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}C(X, \tau)$ (resp. $p\Lambda_{sp}C(X, \tau)$, $\beta\Lambda_{sp}C(X, \tau)$, $\alpha\Lambda_{sp}C(X, \tau)$, $r\Lambda_{sp}C(X, \tau)$). Let A be a subset of a topological space (X, τ). The intersection of all $s(\Lambda, sp)$-closed (resp. $p(\Lambda, sp)$-closed, $\alpha(\Lambda, sp)$-closed) sets containing A is called the $s(\Lambda, sp)$-*closure* [23] (resp. $p(\Lambda, sp)$-*closure*, $\alpha(\Lambda, sp)$-*closure* [5, 22]) of A and is denoted by $A^{s(\Lambda,sp)}$ (resp. $A^{p(\Lambda,sp)}$, $A^{\alpha(\Lambda,sp)}$).

Throughout this paper, the spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A}F(x)$. Moreover, $F : X \to Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [21].

3. On upper and lower almost contra-(Λ, sp)-continuous multifunctions

We begin this section by introducing the concepts of upper and lower almost contra-(Λ, sp)-continuous multifunctions.

Definition 1. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) *lower almost contra-(Λ, sp)-continuous at $x \in X$ if, for each $r(\Lambda, sp)$-closed set K of Y with $x \in F^-(K)$, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^-(K)$;*

(ii) *upper almost contra-(Λ, sp)-continuous at $x \in X$ if, for each $r(\Lambda, sp)$-closed set K of Y with $x \in F^+(K)$, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^+(K)$;*

(iii) *lower (upper) almost contra-(Λ, sp)-continuous if F has this property at each point of X.*

Theorem 1. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost contra-(Λ, sp)-continuous;

(2) $F^+(K)$ is (Λ, sp)-open in X for every $r(\Lambda, sp)$-closed set K of Y;

(3) $F^-(V)$ is (Λ, sp)-closed in X for every $r(\Lambda, sp)$-open set V of Y;

(4) $F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X for every (Λ, sp)-open set V of Y;

(5) $F^+([K_{(\Lambda,sp)}]^{(\Lambda,sp)})$ is (Λ, sp)-open in X for every (Λ, sp)-closed set K of Y;

(6) for each $x \in X$ and for each $s(\Lambda, sp)$-open set V of Y with $F(x) \subseteq V$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$;

(7) $F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every $s(\Lambda, sp)$-open set V of Y.

Proof. (1) ⇒ (2): Let K be any $r(\Lambda, sp)$-closed set of Y and $x \in F^+(K)$. Since F is upper almost contra (Λ, sp)-continuous, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^+(K)$. Thus, $F^+(K)$ is (Λ, sp)-open in X.

(2) ⇒ (1): The proof is obvious.

(2) ⇔ (3): It follows from the fact that $F^+(Y - K) = X - F^-(K)$ for every subset K of Y.

(3) ⇔ (4): Let V be any (Λ, sp)-open set of Y. Then $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open in Y and by (3), $F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X. The converse is obvious.

(4) ⇔ (5): It follows from the fact that $F^+(Y - K) = X - F^-(K)$ for every subset K of Y.

(5) ⇔ (2): It similar to that (3) ⇔ (4).

(6) ⇒ (7): Let V be any $s(\Lambda, sp)$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. By (6), there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$. Thus, $x \in U \subseteq F^+(V^{(\Lambda,sp)})$ and hence $x \in [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$. This shows that $F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(7) ⇒ (2): Let K be any $r(\Lambda, sp)$-closed set of Y. Then K is $s(\Lambda, sp)$-open in Y. By (7), we have $F^+(K) \subseteq [F^+(K)]_{(\Lambda,sp)}$ and hence $F^+(K)$ is (Λ, sp)-open in X.

(2) ⇒ (6): Let $x \in X$ and V be any $s(\Lambda, sp)$-open set of Y with $F(x) \subseteq V$. Since $V^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed and by (2), $F^+(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X. Then, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^+(V^{(\Lambda,sp)})$. Thus, $F(U) \subseteq V^{(\Lambda,sp)}$.

Theorem 2. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost contra-(Λ, sp)-continuous;

(2) $F^-(K)$ is (Λ, sp)-open in X for every $r(\Lambda, sp)$-closed set K of Y;

(3) $F^+(V)$ is (Λ, sp)-closed in X for every $r(\Lambda, sp)$-open set V of Y;

(4) $F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* (Λ, sp)*-open set* V *of* Y*;*

(5) $F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* (Λ, sp)*-closed set* K *of* Y*;*

(6) for each $x \in X$ *and for each* $s(\Lambda, sp)$*-open set* V *of* Y *with* $F(x) \cap V \neq \emptyset$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(z) \cap V^{(\Lambda,sp)} \neq \emptyset$ *for each* $z \in U$*;*

(7) $F^-(V) \subseteq [F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 1.

Definition 2. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called almost contra-*(Λ, sp)*-continuous if, for each* $x \in X$ *and each* $r(\Lambda, sp)$*-closed set* K *of* Y *containing* $f(x)$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $f(U) \subseteq K$*.*

Corollary 1. *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) f *is almost contra-*(Λ, sp)*-continuous;*

(2) $f^{-1}(K)$ *is* (Λ, sp)*-open in* X *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*;*

(3) $f^{-1}(V)$ *is* (Λ, sp)*-closed in* X *for every* $r(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* (Λ, sp)*-open set* V *of* Y*;*

(5) $f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* (Λ, sp)*-closed set* K *of* Y*;*

(6) for each $x \in X$ *and for each* $s(\Lambda, sp)$*-open set* V *of* Y *containing* $f(x)$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V^{(\Lambda,sp)}$*;*

(7) $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Lemma 3. [4] *Let* V *be a subset of a topological space* (X, τ)*. If* $V \in \beta\Lambda_{sp}O(X, \tau)$*, then* $V^{(\Lambda,sp)} \in r\Lambda_{sp}C(X, \tau)$*.*

Theorem 3. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper almost contra-*(Λ, sp)*-continuous;*

(2) $F^+(V^{(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $F^+(V^{(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $s(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* $p(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\beta(\Lambda, sp)$-open set of Y. By Lemma 3, $V^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed and by Theorem 1, $F^+(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let V be any $p(\Lambda, sp)$-open set of Y. Then $Y - [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed and $s(\Lambda, sp)$-open. By (3), we have

$$\begin{aligned} X - F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)}) &= F^+(Y - [V^{(\Lambda,sp)}]_{(\Lambda,sp)}) \\ &= F^+([Y - [V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \end{aligned}$$

is (Λ, sp)-open and hence $F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X.

(4) $\Rightarrow$ (1): Let V be any $r(\Lambda, sp)$-open set of Y. Then V is $p(\Lambda, sp)$-open in Y and by (4), $F^-(V) = F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X. Thus, by Theorem 1, F is upper almost contra-(Λ, sp)-continuous.

Theorem 4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost contra-(Λ, sp)-continuous;

(2) $F^-(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X for every $\beta(\Lambda, sp)$-open set V of Y;

(3) $F^-(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X for every $s(\Lambda, sp)$-open set V of Y;

(4) $F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X for every $p(\Lambda, sp)$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost contra-(Λ, sp)-continuous;

(2) $f^{-1}(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X for every $\beta(\Lambda, sp)$-open set V of Y;

(3) $f^{-1}(V^{(\Lambda,sp)})$ is (Λ, sp)-open in X for every $s(\Lambda, sp)$-open set V of Y;

(4) $f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X for every $p(\Lambda, sp)$-open set V of Y.

Lemma 4. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ [5, 22].

(2) $A^{s(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ [23].

(3) $A^{p(\Lambda,sp)} = A \cup [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Lemma 5. *For a subset V of a topological space (X, τ), the following properties hold:*

(1) $V^{\alpha(\Lambda,sp)} = V^{(\Lambda,sp)}$ for every $V \in \beta\Lambda_{sp}O(X, \tau)$.

(2) $V^{p(\Lambda,sp)} = V^{(\Lambda,sp)}$ *for every* $V \in s\Lambda_{sp}O(X,\tau)$.

(3) $V^{s(\Lambda,sp)} = V^{(\Lambda,sp)}$ *for every* $V \in p\Lambda_{sp}O(X,\tau)$.

Theorem 5. *For a multifunction* $F : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) F *is upper almost contra-*(Λ, sp)*-continuous;*

(2) $F^{+}(V^{\alpha(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $F^{+}(V^{p(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $s(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $F^{-}([V^{s(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* $p(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. This is an immediate consequence of Theorem 3 and Lemma 5.

Theorem 6. *For a multifunction* $F : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) F *is lower almost contra-*(Λ, sp)*-continuous;*

(2) $F^{-}(V^{\alpha(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $F^{-}(V^{p(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $s(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $F^{+}([V^{s(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* $p(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. This is an immediate consequence of Theorem 4 and Lemma 5.

Corollary 3. *For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) f *is almost contra-*(Λ, sp)*-continuous;*

(2) $f^{-1}(V^{\alpha(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $f^{-1}(V^{p(\Lambda,sp)})$ *is* (Λ, sp)*-open in* X *for every* $s(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $f^{-1}([V^{s(\Lambda,sp)}]_{(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every* $p(\Lambda, sp)$*-open set* V *of* Y*.*

Theorem 7. *For a multifunction* $F : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) F *is upper almost contra-*(Λ, sp)*-continuous;*

(2) $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(3) $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}(V^{s(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any (Λ, sp)-open set of Y. Then $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open in Y. By Theorem 1, $F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X. Since $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $F^-(V) \subseteq F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ and hence $[F^-(V)]^{(\Lambda,sp)} \subseteq F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$.

(2) $\Rightarrow$ (1): Let V be any $r(\Lambda, sp)$-open set of Y. Then V is (Λ, sp)-open in Y. By (2), we have $[F^-(V)]^{(\Lambda,sp)} \subseteq F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)}) = F^-(V)$ and hence $F^-(V)$ is (Λ, sp)-closed in X. Thus, by Theorem 1, F is upper almost contra-(Λ, sp)-continuous.

(2) $\Leftrightarrow$ (3): It follows from Lemma 4.

Theorem 8. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost contra-(Λ, sp)-continuous;

(2) $[F^+(V)]^{(\Lambda,sp)} \subseteq F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(3) $[F^+(V)]^{(\Lambda,sp)} \subseteq F^+(V^{s(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y.

Proof. The proof is similar to that of Theorem 7.

Corollary 4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost contra-(Λ, sp)-continuous;

(2) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(3) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}(V^{s(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y.

Let A be a subset of a topological space (X, τ). A point $x \in X$ is said to be in the $\theta s(\Lambda, sp)$-*closure* of A, denoted by $A^{\theta s(\Lambda,sp)}$, if $A \cap U^{(\Lambda,sp)} \neq \emptyset$ for each $s(\Lambda, sp)$-open set U of X containing x. A subset A of a topological space (X, τ) is called $\theta s(\Lambda, sp)$-*closed* if $A = A^{\theta s(\Lambda,sp)}$. The complement of a $\theta s(\Lambda, sp)$-closed set is called $\theta s(\Lambda, sp)$-*open*.

Theorem 9. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost contra-(Λ, sp)-continuous;

(2) $F^-(V)$ is (Λ, sp)-open in X for every $\theta s(\Lambda, sp)$-open set V of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\theta s(\Lambda, sp)$-closed set K of Y;

(4) $[F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{s(\Lambda,sp)})$ for every subset B of Y;

(5) $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+(B^{\theta s(\Lambda,sp)})$ for every subset B of Y;

(6) $F(A^{(\Lambda,sp)}) \subseteq [F(A)]^{\theta s(\Lambda,sp)}$ for every subset A of X.

Proof. (1) ⇒ (2): Let V be any $\theta s(\Lambda, sp)$-open set of Y. There exists a family of $r(\Lambda, sp)$-closed sets $\{K_\gamma \mid \gamma \in \Gamma\}$ such that $V = \cup\{K_\gamma \mid \gamma \in \Gamma\}$. It follows from Theorem 2 that $F^-(V) = \cup\{F^-(K_\gamma) \mid \gamma \in \Gamma\}$ is (Λ, sp)-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let B be any subset of Y. Then $[B^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open and hence $[B^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $\theta s(\Lambda, sp)$-open in Y. By (3), $F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})$ is (Λ, sp)-closed in X. Thus, $[F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} = F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)}) \subseteq F^+(B^{s(\Lambda,sp)})$.

(4) ⇒ (5): Let B be any subset of Y. For any $r(\Lambda, sp)$-open set V with $B \subseteq V$, we have $[F^+(B)]^{(\Lambda,sp)} \subseteq [F^+(V)]^{(\Lambda,sp)} = [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(V^{s(\Lambda,sp)}) = F^+(V)$. Thus, $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+(\cap\{V \in r\Lambda_{sp}O(X, \tau) \mid B \subseteq V\}) = F^+(B^{\theta s(\Lambda,sp)})$.

(5) ⇒ (1): Let V be any $s(\Lambda, sp)$-open set of Y. By (5),

$$\begin{aligned} X - [F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)} &= [F^+(Y - V^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &\subseteq F^+([Y - V^{(\Lambda,sp)}]^{\theta s(\Lambda,sp)}) \\ &= F^+(Y - V^{(\Lambda,sp)}) \\ &= X - F^-(V^{(\Lambda,sp)}) \end{aligned}$$

and hence $F^-(V) \subseteq F^-(V^{(\Lambda,sp)}) \subseteq [F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$. By Theorem 2, F is lower almost contra-(Λ, sp)-continuous.

(5) ⇒ (6): Let A be any subset of X and $B = F(A)$. Then $A \subseteq F^+(B)$ and by (5), $A^{(\Lambda,sp)} \subseteq [F^+(B)]^{(\Lambda,sp)} \subseteq F^+(B^{\theta s(\Lambda,sp)})$. Thus, $F(A^{(\Lambda,sp)}) \subseteq F(F^+(B^{\theta s(\Lambda,sp)})) \subseteq B^{\theta s(\Lambda,sp)} = [F(A)]^{\theta s(\Lambda,sp)}$.

(6) ⇒ (5): Let B be any subset of Y. By (6), we have

$$F([F^+(B)]^{(\Lambda,sp)}) \subseteq [F(F^+(B))]^{\theta s(\Lambda,sp)} \subseteq B^{\theta s(\Lambda,sp)}$$

and hence $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+(B^{\theta s(\Lambda,sp)})$.

Corollary 5. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost contra-(Λ, sp)-continuous;

(2) $f^{-1}(V)$ is (Λ, sp)-open in X for every $\theta s(\Lambda, sp)$-open set V of Y;

(3) $f^{-1}(K)$ is (Λ, sp)-closed in X for every $\theta s(\Lambda, sp)$-closed set K of Y;

(4) $[f^{-1}([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{s(\Lambda,sp)})$ for every subset B of Y;

(5) $[f^{-1}(B)]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\theta s(\Lambda,sp)})$ for every subset B of Y;

(6) $f(A^{(\Lambda,sp)}) \subseteq [f(A)]^{\theta s(\Lambda,sp)}$ for every subset A of X.

Definition 3. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be upper strongly $s(\Lambda, sp)$-continuous if, for each $x \in X$ and each $s(\Lambda, sp)$-open set V of Y such that $F(x) \subseteq V$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V$.*

Theorem 10. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper strongly $s(\Lambda, sp)$-continuous;

(2) $F^+(V)$ is (Λ, sp)-open in X for every $s(\Lambda, sp)$-open set V of Y;

(3) $F^-(K)$ is (Λ, sp)-closed in X for every $s(\Lambda, sp)$-closed set K of Y;

(4) $[F^-(B)]^{(\Lambda,sp)} \subseteq F^-(B^{s(\Lambda,sp)})$ for every subset B of Y;

(5) $F^+(B_{s(\Lambda,sp)}) \subseteq [F^+(B)]_{(\Lambda,sp)}$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $s(\Lambda, sp)$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. Since F is upper strongly $s(\Lambda, sp)$-continuous, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^+(V)$. Thus, $F^+(V) \subseteq [F^+(V)]_{(\Lambda,sp)}$ and hence $F^+(V)$ is (Λ, sp)-open in X.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $B^{s(\Lambda,sp)}$ is $s(\Lambda, sp)$-closed and by (3), $F^-(B^{s(\Lambda,sp)})$ is (Λ, sp)-closed in X. Thus, $[F^-(B)]^{(\Lambda,sp)} \subseteq [F^-(B^{s(\Lambda,sp)})]^{(\Lambda,sp)} = F^-(B^{s(\Lambda,sp)})$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), $X - [F^+(B)]_{(\Lambda,sp)} = [X - F^+(B)]^{(\Lambda,sp)} = [F^-(Y - B)]^{(\Lambda,sp)} \subseteq F^-([Y - B]^{s(\Lambda,sp)}) = F^-(Y - B_{s(\Lambda,sp)}) = X - F^+(B_{s(\Lambda,sp)})$. Therefore, $F^+(B_{s(\Lambda,sp)}) \subseteq [F^+(B)]_{(\Lambda,sp)}$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $s(\Lambda, sp)$-open set of Y such that $F(x) \subseteq V$. By (5), we have $F^+(V) \subseteq [F^+(V)]_{(\Lambda,sp)}$ and hence $F^+(V)$ is (Λ, sp)-open in X. Put $U = F^+(V)$, then U is a (Λ, sp)-open set of X containing x such that $F(U) \subseteq V$. This shows that F is upper strongly $s(\Lambda, sp)$-continuous.

Definition 4. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower strongly $s(\Lambda, sp)$-continuous if, for each $x \in X$ and each $s(\Lambda, sp)$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

Theorem 11. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower strongly $s(\Lambda, sp)$-continuous;

(2) $F^-(V)$ is (Λ, sp)-open in X for every $s(\Lambda, sp)$-open set V of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $s(\Lambda, sp)$-closed set K of Y;

(4) $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+(B^{s(\Lambda,sp)})$ for every subset B of Y;

(5) $F^-(B_{s(\Lambda,sp)}) \subseteq [F^-(B)]_{(\Lambda,sp)}$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 10.

Definition 5. *A topological space (X, τ) is called strongly $s(\Lambda, sp)$-regular if, for each $s(\Lambda, sp)$-closed set K and each $x \in X - K$, there exists a $r(\Lambda, sp)$-closed set F containing x such that $F \cap K = \emptyset$.*

Lemma 6. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is strongly $s(\Lambda, sp)$-regular;

(2) for each $s(\Lambda, sp)$-open set W of X and each $x \in W$, there exists a $s(\Lambda, sp)$-open set V such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq W$;

(3) for each $s(\Lambda, sp)$-open set W of X and each $x \in W$, there exists a $r(\Lambda, sp)$-closed set F such that $x \in F \subseteq W$;

(4) $A^{s(\Lambda,sp)} = A^{\theta s(\Lambda,sp)}$ for every subset A of X;

(5) every $s(\Lambda, sp)$-open set of X is $\theta s(\Lambda, sp)$-open.

Theorem 12. *Let (Y, σ) be a strongly $s(\Lambda, sp)$-regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower strongly $s(\Lambda, sp)$-continuous;

(2) $F^+(B^{\theta s(\Lambda,sp)})$ is (Λ, sp)-closed in X for every subset B of Y;

(3) F is lower almost contra-(Λ, sp)-continuous.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 6, $B^{\theta s(\Lambda,sp)}$ is $s(\Lambda, sp)$-closed and by Theorem 11, $F^+(B^{\theta s(\Lambda,sp)})$ is (Λ, sp)-closed.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), we have

$$[F^+(B)]^{(\Lambda,sp)} \subseteq [F^+(B^{\theta s(\Lambda,sp)})]^{(\Lambda,sp)} = F^+(B^{\theta s(\Lambda,sp)})$$

and by Theorem 9, F is lower almost contra-(Λ, sp)-continuous.

(3) $\Rightarrow$ (1): Let V be any $s(\Lambda, sp)$-open set of Y. Since (Y, σ) is strongly $s(\Lambda, sp)$-regular, by Lemma 6, V is $\theta s(\Lambda, sp)$-open. By Theorem 9, $F^-(V)$ is (Λ, sp)-open in X. Thus, by Theorem 11, F is lower strongly $s(\Lambda, sp)$-continuous.

Theorem 13. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper strongly $s(\Lambda, sp)$-continuous multifunction and $G : (Y, \sigma) \to (Z, \eta)$ is an upper almost contra-(Λ, sp)-continuous multifunction, then $G \circ F : (X, \tau) \to (Z, \eta)$ is upper almost contra-(Λ, sp)-continuous.*

Proof. Let K be any $r(\Lambda, sp)$-closed set of Z. We have $(G \circ F)^+(K) = F^-(G^+(K))$. Since G is lower almost contra-(Λ, sp)-continuous, by Theorem 1, $G^+(K)$ is (Λ, sp)-open in X and hence $G^+(K)$ is $s(\Lambda, sp)$-open. Since F is lower strongly $s(\Lambda, sp)$-continuous, by Theorem 10, $F^+(G^+(K))$ is (Λ, sp)-open. Thus, $G \circ F$ is upper almost contra-(Λ, sp)-continuous.

Theorem 14. *If $F : (X, \tau) \to (Y, \sigma)$ is a lower strongly $s(\Lambda, sp)$-continuous multifunction and $G : (Y, \sigma) \to (Z, \eta)$ is a lower almost contra-(Λ, sp)-continuous multifunction, then $G \circ F : (X, \tau) \to (Z, \eta)$ is lower almost contra-(Λ, sp)-continuous.*

Proof. The proof is similar to that of Theorem 13.

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