EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 336-362
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On some forms of closed sets and related topics

Chawalit Boonpok[1], Chokchai Viriyapong[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

Abstract. The purpose of the present article is to introduce the notion of (Λ, s)-closed sets. Especially, some properties of generalized (Λ, s)-closed sets are obtained. Several characterizations of some low separation axioms are given. Characterizations of (Λ, s)-extremally disconnected spaces are investigated. Furthermore, some characterizations of almost (Λ, s)-continuous functions are discussed.

2020 Mathematics Subject Classifications: 54A05, 54D10, 54G05

Key Words and Phrases: (Λ, s)-closed set, generalized (Λ, s)-closed set, (Λ, s)-R_0 space, (Λ, s)-extremally disconnected space, almost (Λ, s)-continuous function

1. Introduction

General topology plays an important role in many fields of applied sciences as well as branches of mathematics. The concepts of maximality and submaximality of general topological spaces were introduced by Hewitt [16]. He discovered a general way of constructing maximal topologies. The existence of a maximal space that is Tychonoff is nontrivial and due to van Douwen [28]. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel'skiǐ and Collins [2]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is σ-discrete [2]. Gillman and Jerison [14] introduced the notion of extremally disconnected topological spaces. Thompson [26] introduced the notion of S-closed spaces. Herrman [15, 16] showed that every S-closed weakly Hausdorff (or almost regular) space is extremally disconnected. Cameron [6] proved that every maximally S-closed space is extremally disconnected. In [21], the present author introduced the concept of locally S-closed spaces which is strictly weaker than that of S-closed spaces. Noiri [22] showed that every locally S-closed weakly Hausdorff (or almost regular) space is extremally disconnected. Sivaraj [25] has obtained some characterizations of extremally disconnected spaces by utilizing semi-open sets due

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4582

Email addresses: chawalit.b@msu.ac.th (C. Boonpok), chokchai.v@msu.ac.th (C. Viriyapong)

to Levine [17]. In [23], the present author obtained several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets.

The notion of R_0 topological spaces was first introduced by Shanin [24]. Davis [8] introduced the notion of a separation axiom called R_1. This notions were further investigated by Naimpally [20], Dube [10] and Dorsett [9]. Cammaroto and Noiri [7] have defined a weak separation axioms m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [19]. Levine [18] introduced the concept of generalized closed sets of a topological space and a class of topological spaces called $T_{\frac{1}{2}}$-spaces. Dunham [12] and Dunham and Levine [13] further studied some properties of generalized closed sets and $T_{\frac{1}{2}}$-spaces. In [17], the present author offered a new concept to the field of generalized topology by introducing semi-open sets, i.e., a subset of a topological space which is contained in the closure of the interior of its closure. Caldas and Dontchev [5] introduced the notions of Λ_s-sets and generalized Λ_s-sets and studied some characterizations of semi-$T_{\frac{1}{2}}$-spaces. Buadong et al. [4] introduced and investigated some separation axioms in generalized topology and minimal structure spaces. Dungthaisong et al. [11] investigated several characterizations of pairwise μ-$T_{\frac{1}{2}}$-spaces. Torton et al. [27] introduced and studied the concepts of $\mu_{(m,n)}$-regular spaces and $\mu_{(m,n)}$-normal spaces.

The article is organized as follows. In Section 3, we introduce the notion of (Λ, s)-closed sets. Moreover, some properties of (Λ, s)-closed sets are discussed. In Section 4, we introduce the notion of (Λ, s)-R_0 spaces and investigate some characterizations of (Λ, s)-R_0 spaces. In Section 5, we introduce the notion of generalized (Λ, s)-closed sets and investigate several fundamental properties of generalized (Λ, s)-closed sets. Furthermore, some characterizations of (Λ, s)-normal spaces are explored. In Section 6, we introduce the notion of (Λ, s)-extremally disconnected spaces and investigate several characterizations of such spaces. In Section 7, we introduce the notion of almost (Λ, s)-continuous functions and investigate several characterizations of such functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is called *semi-open* [17] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called *semi-closed.* By $SO(X, \tau)$ and $SC(X, \tau)$ we denote the family of all semi-open sets and the family of all semi-closed sets in a topological space (X, τ), respectively. The *semi-closure* of a set A, denoted by $s\mathrm{Cl}(A)$, is the intersection of all semi-closed sets containing A. The *semi-interior* of a set A, denoted by $s\mathrm{Int}(A)$, is the union of all semi-open sets contained in A. A subset A^{Λ_s} [5] (resp. A^{Λ_s}) is defined as follows: $A^{\Lambda_s} = \cap\{U \mid U \supseteq A,\ A \in SO(X, \tau)\}$ (resp. $A^{V_s} = \cup\{F \mid F \subseteq A,\ X - F \in SO(X, \tau)\}$).

Lemma 1. [5] *For subsets A, B and $A_\gamma (\gamma \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq A^{\Lambda_s}$.

(2) *If* $A \subseteq B$, *then* $A^{\Lambda_s} \subseteq B^{\Lambda_s}$.

(3) $(A^{\Lambda_s})^{\Lambda_s} = A^{\Lambda_s}$.

(4) $[\bigcup_{\gamma\in\nabla} A_\gamma]^{\Lambda_s} = \bigcup_{\gamma\in\nabla} A_\gamma^{\Lambda_s}$.

(5) *If* $A \in SO(X,\tau)$, *then* $A = A^{\Lambda_s}$.

(6) $(X - A)^{\Lambda_s} = X - A^{V_s}$.

(7) $A^{V_s} \subseteq A$.

(8) *If* $A \in SC(X,\tau)$, *then* $A = A^{V_s}$.

(9) $[\bigcap_{\gamma\in\nabla} A_\gamma]^{\Lambda_s} \subseteq \bigcap_{\gamma\in\nabla} A_\gamma^{\Lambda_s}$.

(10) $[\bigcup_{\gamma\in\nabla} A_\gamma]^{V_s} \supseteq \bigcup_{\gamma\in\nabla} A_\gamma^{V_s}$.

Definition 1. [5] *A subset* A *of a topological space* (X,τ) *is called a* Λ_s*-set (resp.* V_s*-set) if* $A = A^{\Lambda_s}$ *(resp.* $A = A^{V_s}$*).*

Lemma 2. [5] *For a topological space* (X,τ), *the following properties hold:*

(1) The subsets $\emptyset$ *and* X *are* Λ_s*-sets and* V_s*-sets.*

(2) Every union of Λ_s*-sets (resp.* V_s*-sets) is a* Λ_s*-set (resp.* V_s*-set).*

(3) Every intersection of Λ_s*-sets (resp.* V_s*-sets) is a* Λ_s*-set (resp.* V_s*-set).*

(4) A subset A *is a* Λ_s*-set if and only* $X - A$ *is a* V_s*-set.*

3. On (Λ, s)-closed sets

In this section, we introduce the notion of (Λ, s)-closed sets. Moreover, some properties of (Λ, s)-closed sets are discussed.

Definition 2. *A subset* A *of a topological space* (X,τ) *is called* (Λ, s)*-closed if* $A = T \cap C$, *where* T *is a* Λ_s*-set and* C *is a semi-closed set. The family of all* (Λ, s)*-closed sets in a topological space* (X,τ) *is denoted by* $(\Lambda, s)C(X)$.

Lemma 3. [1] *For a subset* A *of a topological space* (X,τ), *the following properties hold:*

(1) $sCl(A) = A \cup Int(Cl(A))$;

(2) $sInt(A) = A \cap Cl(Int(A))$.

Theorem 1. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is (Λ, s)-closed;

(2) $A = T \cap sCl(A)$, where T is a Λ_s-set;

(3) $A = A^{\Lambda_s} \cap sCl(A)$;

(4) $Int(Cl(A)) \cap A^{\Lambda_s} \subseteq A$.

Proof. (1) $\Rightarrow$ (2): Suppose that $A = T \cap C$, where T is a Λ_s-set and C is a semi-closed set. Since $A \subseteq C$, we have $s\mathrm{Cl}(A) \subseteq C$ and hence $A = T \cap C \supseteq T \cap s\mathrm{Cl}(A) \supseteq A$. Consequently, we obtain $A = T \cap s\mathrm{Cl}(A)$.

(2) $\Rightarrow$ (3): Suppose that $A = T \cap sCl(A)$, where T is a Λ_s-set. Since $A \subseteq T$, we have $A^{\Lambda_s} \subseteq T^{\Lambda_s} = T$ and hence $A \subseteq A^{\Lambda_s} \cap s\mathrm{Cl}(A) \subseteq T \cap s\mathrm{Cl}(A) = A$. Thus, $A = A^{\Lambda_s} \cap s\mathrm{Cl}(A)$.

(3) $\Rightarrow$ (4): Let $A = A^{\Lambda_s} \cap s\mathrm{Cl}(A)$. Thus, by Lemma 3,

$$\begin{aligned} A &= A^{\Lambda_s} \cap [A \cup \mathrm{Int}(\mathrm{Cl}(A))] \\ &= (A^{\Lambda_s} \cap A) \cup [A^{\Lambda_s} \cap \mathrm{Int}(\mathrm{Cl}(A))] \\ &= A \cup [A^{\Lambda_s} \cap \mathrm{Int}(\mathrm{Cl}(A))] \end{aligned}$$

and hence $\mathrm{Int}(\mathrm{Cl}(A)) \cap A^{\Lambda_s} \subseteq A$.

(4) $\Rightarrow$ (1): Let $\mathrm{Int}(\mathrm{Cl}(A)) \cap A^{\Lambda_s} \subseteq A$. Then, we have $A \cup [A^{\Lambda_s} \cap \mathrm{Int}(\mathrm{Cl}(A))] = A$ and by Lemma 3, $A = (A \cup A^{\Lambda_s}) \cap [A \cup \mathrm{Int}(\mathrm{Cl}(A))] = A^{\Lambda_s} \cap s\mathrm{Cl}(A)$. This shows that A is (Λ, s)-closed.

Definition 3. *A subset A of a topological space (X, τ) is said to be (Λ, s)-open if the complement of A is (Λ, s)-closed. The family of all (Λ, s)-open sets in a topological space (X, τ) is denoted by $(\Lambda, s)O(X)$.*

Proposition 1. *Let $A_\gamma(\gamma \in \nabla)$ be a subset of a topological space (X, τ). Then, the following properties hold:*

(1) If A_γ is (Λ, s)-closed for each $\gamma \in \nabla$, then $\cap\{A_\gamma \mid \gamma \in \nabla\}$ is (Λ, s)-closed.

(2) If A_γ is (Λ, s)-open for each $\gamma \in \nabla$, then $\cup\{A_\gamma \mid \gamma \in \nabla\}$ is (Λ, s)-open.

Proof. (1) Suppose that A_γ is (Λ, s)-closed for each $\gamma \in \nabla$. Then, for each γ, there exist a Λ_s-set T_γ and a semi-closed set C_γ such that $A_\gamma = T_\gamma \cap C_\gamma$. We have

$$\cap_{\gamma \in \nabla} A_\gamma = \cap_{\gamma \in \nabla}(T_\gamma \cap C_\gamma) = (\cap_{\gamma \in \nabla} T_\gamma) \cap (\cap_{\gamma \in \nabla} C_\gamma).$$

By Lemma 2, $\cap_{\gamma \in \nabla} T_\gamma$ is a Λ_s-set and $\cap_{\gamma \in \nabla} C_\gamma$ is a semi-closed set. Thus, $\cap_{\gamma \in \nabla} A_\gamma$ is (Λ, s)-closed.

(2) Let A_γ is (Λ, s)-open for each $\gamma \in \nabla$. Then, $X - A_\gamma$ is (Λ, s)-closed for each $\gamma \in \nabla$. Thus, by (1), we have $X - \cup_{\gamma \in \nabla} A_\gamma = \cap_{\gamma \in \nabla}(X - A_\gamma)$ is (Λ, s)-closed and hence $\cup_{\gamma \in \nabla} A_\gamma$ is (Λ, s)-open.

Theorem 2. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is (Λ, s)-open;

(2) $A = T \cup G$, where T is a V^s-set and U is a semi-open set;

(3) $A = T \cup s\mathit{Int}(A)$, where T is a V^s-set;

(4) $A = A^{V^s} \cup s\mathit{Int}(A)$;

(5) $A \subseteq \mathit{Cl}(\mathit{Int}(A)) \cup A^{V_s}$.

Proof. (1) $\Rightarrow$ (2): Suppose that A is (Λ, s)-open. Then, $X - A$ is (Λ, s)-closed and hence $X - A = T \cap F$, where T is a Λ_s-set and F is a semi-closed set. Thus, we have $A = (X - A) \cup (X - F)$, where $X - T$ is a V_s-set and $X - F$ is a semi-open set.

(2) $\Rightarrow$ (3): Suppose that $A = T \cup U$, where T is a V^s-set and U is a semi-open set. Since $U \subseteq A$ and U is semi-open, we have $U \subseteq s\mathrm{Int}(A)$ and hence $A = T \cup U \subseteq T \cup s\mathrm{Int}(A) \subseteq A$. Thus, $A = T \cup s\mathrm{Int}(A)$.

(3) $\Rightarrow$ (4): Suppose that $A = T \cup s\mathrm{Int}(A)$, where T is a V^s-set. Since $T \subseteq A$, we have $A^{V^s} \supseteq T^{V^s}$ and hence $A \supseteq A^{V^s} \cup s\mathrm{Int}(A) \supseteq T^{V^s} \cup s\mathrm{Int}(A) = T \cup s\mathrm{Int}(A) = A$. This shows that $A = A^{V^s} \cup s\mathrm{Int}(A)$.

(4) $\Rightarrow$ (5): Let $A = A^{V^s} \cup s\mathrm{Int}(A)$. Thus, by Lemma 3,

$$\begin{aligned} A &= A^{V^s} \cup s\mathrm{Int}(A) \\ &= A^{V^s} \cup [A \cap \mathrm{Cl}(\mathrm{Int}(A))] \\ &= [A^{V_s} \cup A] \cap [\mathrm{Cl}(\mathrm{Int}(A)) \cup A^{V_s}] \\ &= A \cap [\mathrm{Cl}(\mathrm{Int}(A)) \cup A^{V_s}] \end{aligned}$$

and hence $A \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cup A^{V_s}$.

(5) $\Rightarrow$ (1): Let $A \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cup A^{V_s}$. Then, we have

$$\begin{aligned} \mathrm{Int}(\mathrm{Cl}(X - A)) \cap [X - A]^{\Lambda_s} &= [X - \mathrm{Cl}(\mathrm{Int}(A))] \cap [X - A^{V_s}] \\ &= X - [\mathrm{Cl}(\mathrm{Int}(A)) \cup A^{V_s}] \\ &\subseteq X - A \end{aligned}$$

and by Theorem 1, $X - A$ is (Λ, s)-closed. Thus, A is (Λ, s)-open.

Definition 4. *Let A be a subsets of a topological space (X, τ). A point $x \in X$ is called a (Λ, s)-cluster point of A if for every (Λ, s)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ, s)-cluster points of A is called the (Λ, s)-closure of A and is denoted by $A^{(\Lambda, s)}$.*

Lemma 4. *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq A^{(\Lambda,s)}$ *and* $[A^{(\Lambda,s)}]^{(\Lambda,s)} = A^{(\Lambda,s)}$.

(2) If $A \subseteq B$*, then* $A^{(\Lambda,s)} \subseteq B^{(\Lambda,s)}$.

(3) $A^{(\Lambda,s)} = \cap\{F | A \subseteq F$ *and* F *is* (Λ, s)*-closed*$\}$.

(4) $A^{(\Lambda,s)}$ *is* (Λ, s)*-closed.*

(5) A *is* (Λ, s)*-closed if and only if* $A = A^{(\Lambda,s)}$.

Proposition 2. *For a subset* A *of a topological space* (X, τ)*, the following properties hold:*

(1) If A *is* (Λ, s)*-closed, then* $A = A^{\Lambda_s} \cap A^{(\Lambda,s)}$.

(2) If A *is semi-closed, then* A *is* (Λ, s)*-closed.*

Proof. (1) Let A be a (Λ, s)-closed set. Then, there exist a Λ_s-set T and a semi-closed set C such that $A = T \cap C$. By $A \subseteq T$, we have $A \subseteq A^{\Lambda_s} \subseteq T^{\Lambda_s} = T$, and also by $A \subseteq C$, $A \subset A^{(\Lambda,s)} \subseteq C^{(\Lambda,s)} = C$. Now $A \subseteq A^{\Lambda_s} \cap A^{(\Lambda,s)} \subseteq T \cap C = A$. Thus, $A = A^{\Lambda_s} \cap A^{(\Lambda,s)}$.

(2) It is sufficient to observe that $A = X \cap A$, where the whole set X is a Λ_s-set.

Definition 5. *Let* A *be a subset of a topological space* (X, τ)*. Then,* $\Gamma_{(\Lambda,s)}(A)$ *is defined as follows:* $\Gamma_{(\Lambda,s)}(A) = \cap\{U \in (\Lambda, s)O(X) \mid A \subseteq U\}$.

Proposition 3. *For subsets* A *and* B *of a topological space* (X, τ)*, the following properties hold:*

(1) If $A \subseteq B$*, then* $\Gamma_{(\Lambda,s)}(A) \subseteq \Gamma_{(\Lambda,s)}(B)$.

(2) If $A \in (\Lambda, s)O(X)$*, then* $\Gamma_{(\Lambda,s)}(A) = A$.

(3) $\Gamma_{(\Lambda,s)}[\Gamma_{(\Lambda,s)}(A)] = \Gamma_{(\Lambda,s)}(A)$.

Proposition 4. *Let* (X, τ) *be a topological space and* $x, y \in X$*. Then,* $y \in \Gamma_{(\Lambda,s)}(\{x\})$ *if and only if* $x \in \{y\}^{(\Lambda,s)}$.

Proof. Let $y \notin \Gamma_{(\Lambda,s)}(\{x\})$. Then, there exists a (Λ, s)-open set V containing x such that $y \notin V$. Thus, $x \notin \{y\}^{(\Lambda,s)}$. The converse is similarly shown.

Definition 6. *Let* (X, τ) *be a topological space and* $x \in X$*. Then,* $\langle x\rangle_{(\Lambda,s)}$ *is defined as follows:* $\langle x\rangle_{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\}) \cap \{x\}^{(\Lambda,s)}$.

Proposition 5. *For a topological space* (X, τ)*, the following properties hold:*

(1) for each $x \in X$*,* $\Gamma_{(\Lambda,s)}[\langle x\rangle_{(\Lambda,s)}] = \Gamma_{(\Lambda,s)}(\{x\})$*;*

(2) for each $x \in X$*,* $[\langle x\rangle_{(\Lambda,s)}]^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$.

Proof. (1) Let $x \in X$. Then, we have $\{x\} \subseteq \{x\}^{(\Lambda,s)} \cap \Gamma_{(\Lambda,s)}(\{x\}) = \langle x \rangle_s$. By Proposition 3, $\Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}[\langle x \rangle_{(\Lambda,s)}]$. Next, we show the opposite implication. Suppose that $y \notin \Gamma_{(\Lambda,s)}(\{x\})$. There exists a (Λ, s)-open set V such that $x \in V$ and $y \notin V$. Since $\langle x \rangle_s \subseteq \Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}(V) = V$, we have $\Gamma_{(\Lambda,s)}[\langle x \rangle_{(\Lambda,s)}] \subseteq V$. Since $y \notin V$, $y \notin \Gamma_{(\Lambda,s)}[\langle x \rangle_{(\Lambda,s)}]$. Thus, $\Gamma_{(\Lambda,s)}[\langle x \rangle_{(\Lambda,s)}] \subseteq \Gamma_{(\Lambda,s)}(\{x\})$ and hence

$$\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}[\langle x \rangle_{(\Lambda,s)}].$$

(2) By the definition of $\langle x \rangle_{(\Lambda,s)}$, we have $\{x\} \subseteq \langle x \rangle_{(\Lambda,s)}$ and $\{x\}^{(\Lambda,s)} \subseteq [\langle x \rangle_{(\Lambda,s)}]^{(\Lambda,s)}$ by Lemma 4. On the other hand, we have $\langle x \rangle_{(\Lambda,s)} \subseteq \{x\}^{(\Lambda,s)}$ and

$$[\langle x \rangle_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq [\{x\}^{(\Lambda,s)}]^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}.$$

This shows that $[\langle x \rangle_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq \{x\}^{(\Lambda,s)}$.

Theorem 3. *For any points x and y in a topological space (X, τ), the following properties are equivalent:*

(1) $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$.

(2) $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$.

Proof. (1) ⇒ (2): Suppose that $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$. There exists a point $z \in X$ such that $z \in \Gamma_{(\Lambda,s)}(\{x\})$ and $z \notin \Gamma_{(\Lambda,s)}(\{y\})$ or $z \in \Gamma_{(\Lambda,s)}(\{y\})$ and $z \notin \Gamma_{(\Lambda,s)}(\{x\})$. We prove only the first case being the second analogous. From $z \in \Gamma_{(\Lambda,s)}(\{x\})$ it follows that $\{x\} \cap \{z\}^{(\Lambda,s)} \neq \emptyset$ which implies $x \in \{z\}^{(\Lambda,s)}$. By $z \notin \Gamma_{(\Lambda,s)}(\{y\})$, we have $\{y\} \cap \{z\}^{(\Lambda,s)} = \emptyset$. Since $x \in \{z\}^{(\Lambda,s)}$, $\{x\}^{(\Lambda,s)} \subseteq \{z\}^{(\Lambda,s)}$ and $\{y\} \cap \{x\}^{(\Lambda,s)} = \emptyset$. Therefore, it follows that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. Thus, $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$ implies that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$.

(2) ⇒ (1): Suppose that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. Then, there exists a point $z \in X$ such that $z \in \{x\}^{(\Lambda,s)}$ and $z \notin \{y\}^{(\Lambda,s)}$ or $z \in \{y\}^{(\Lambda,s)}$ and $z \notin \{x\}^{(\Lambda,s)}$. We prove only the first case being the second analogous. It follows that there exists a (Λ, s)-open set containing z and therefore x but not y, namely, $y \notin \Gamma_{(\Lambda,s)}(\{x\})$ and thus $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$.

Theorem 4. *For any points x and y in a topological space (X, τ), the following properties hold:*

(1) $y \in \Gamma_{(\Lambda,s)}(\{x\})$ *if and only if* $x \in \{y\}^{(\Lambda,s)}$.

(2) $\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}(\{y\})$ *if and only if* $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$.

Proof. (1) Let $x \notin \{y\}^{(\Lambda,s)}$. Then, there exists $U \in (\Lambda, s)O(X)$ such that $x \in U$ and $y \notin U$. Thus, $y \notin \Gamma_{(\Lambda,s)}(\{x\})$. The converse is similarly shown.

(2) Suppose that $\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}(\{y\})$ for any $x, y \in X$. Since $x \in \Gamma_{(\Lambda,s)}(\{x\})$, $x \in \Gamma_{(\Lambda,s)}(\{y\})$ and by (1), $y \in \{x\}^{(\Lambda,s)}$. By Lemma 4, we have $\{y\}^{(\Lambda,s)} \subseteq \{x\}^{(\Lambda,s)}$. Similarly, we have $\{x\}^{(\Lambda,s)} \subseteq \{y\}^{(\Lambda,s)}$ and hence $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$.

Conversely, suppose that $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$. Since $x \in \{x\}^{(\Lambda,s)}$, we have $x \in \{y\}^{(\Lambda,s)}$ and by (1), $y \in \Gamma_{(\Lambda,s)}(\{x\})$. By Proposition 3,

$$\Gamma_{(\Lambda,s)}(\{y\}) \subseteq \Gamma_{(\Lambda,s)}(\Gamma_{(\Lambda,s)}(\{x\})) = \Gamma_{(\Lambda,s)}(\{x\}).$$

Similarly, we have $\Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}(\{y\})$. Thus, $\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}(\{y\})$.

4. On some low separation axioms

In this section, we introduce the notion of (Λ, s)-R_0 spaces and investigate some characterizations of (Λ, s)-R_0 spaces.

Definition 7. *A topological space (X, τ) is called (Λ, s)-R_0 if, for each (Λ, s)-open set U and each $x \in U$, $\{x\}^{(\Lambda,s)} \subseteq U$.*

Theorem 5. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is (Λ, s)-R_0;

(2) for any $F \in (\Lambda, s)C(X)$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in (\Lambda, s)O(X)$;

(3) for any $F \in (\Lambda, s)C(X)$, $x \notin F$ implies $F \cap \{x\}^{(\Lambda,s)} = \emptyset$;

(4) for any distinct points x and y of X, $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$ or $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $F \in (\Lambda, s)C(X)$ and $x \notin F$. Then by (1), $\{x\}^{(\Lambda,s)} \subseteq X - F$. Put $U = X - \{x\}^{(\Lambda,s)}$, then $U \in (\Lambda, s)O(X)$, $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let $F \in (\Lambda, s)C(X)$ and $x \notin F$. There exists $U \in (\Lambda, s)O(X)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in (\Lambda, s)O(X)$, $U \cap \{x\}^{(\Lambda,s)} = \emptyset$ and $F \cap \{x\}^{(\Lambda,s)} = \emptyset$.

(3) $\Rightarrow$ (4): Let x, y be distinct points of X. Suppose that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. By (3), $x \in \{y\}^{(\Lambda,s)}$ and $y \in \{x\}^{(\Lambda,s)}$. Thus, $\{x\}^{(\Lambda,s)} \subseteq \{y\}^{(\Lambda,s)} \subseteq \{x\}^{(\Lambda,s)}$. Consequently, we obtain $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$.

(4) $\Rightarrow$ (1): Let $V \in (\Lambda, s)O(X)$ and $x \in V$. For each $y \notin V$, $V \cap \{y\}^{(\Lambda,s)} = \emptyset$ and hence $x \notin \{y\}^{(\Lambda,s)}$. Thus, $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. By (4), for each $y \notin V$, $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$. Since $X - V$ is (Λ, s)-closed, $\{y\}^{(\Lambda,s)} \subseteq X - V$ and $X - V = \cup_{y \in X-V}\{y\}^{(\Lambda,s)}$. Thus, $(X - V) \cap \{x\}^{(\Lambda,s)} = [\cup_{y\in X-V}\{y\}^{(\Lambda,s)}] \cap \{x\}^{(\Lambda,s)} = \cup_{y\in X-V}[\{y\}^{(\Lambda,s)} \cap \{x\}^{(\Lambda,s)}] = \emptyset$ and hence $\{x\}^{(\Lambda,s)} \subseteq V$. This shows that (X, τ) is a (Λ, s)-R_0 space.

Corollary 1. *A topological space (X, τ) is (Λ, s)-R_0 if and only if for any points x, y in X, $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$ implies $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$.*

Proof. This is obvious by Theorem 5.

Conversely, let $U \in (\Lambda, s)O(X)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,s)} = \emptyset$. Thus, $x \notin \{y\}^{(\Lambda,s)}$ and $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. By the hypothesis, $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$ and hence $y \notin \{x\}^{(\Lambda,s)}$. This shows that $\{x\}^{(\Lambda,s)} \subseteq U$. Consequently, we obtain (X, τ) is (Λ, s)-R_0.

Theorem 6. *A topological space (X, τ) is (Λ, s)-R_0 if and only if for any $x, y \in X$, $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$ implies $\Gamma_{(\Lambda,s)}(\{x\}) \cap \Gamma_{(\Lambda,s)}(\{y\}) = \emptyset$.*

Proof. Let (X, τ) be a (Λ, s)-R_0 space. Suppose that $\Gamma_{(\Lambda,s)}(\{x\}) \cap \Gamma_{(\Lambda,s)}(\{y\}) \neq \emptyset$. Let $z \in \Gamma_{(\Lambda,s)}(\{x\}) \cap \Gamma_{(\Lambda,s)}(\{y\})$. Then, we have $z \in \Gamma_{(\Lambda,s)}(\{x\})$ and Theorem 4, $x \in \{z\}^{(\Lambda,s)}$. Thus, $x \in \{z\}^{(\Lambda,s)} \cap \{x\}^{(\Lambda,s)}$ and by Corollary 1, $\{z\}^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$. Similarly, we have $\{z\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$ and hence $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$. By Theorem 4, $\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}(\{y\})$.

Conversely, suppose that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. By Theorem 4, $\Gamma_{(\Lambda,s)}(\{x\}) \neq \Gamma_{(\Lambda,s)}(\{y\})$ and hence $\Gamma_{(\Lambda,s)}(\{x\}) \cap \Gamma_{(\Lambda,s)}(\{y\}) = \emptyset$. Thus, $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$. In fact, assume $z \in \{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)}$. Then, we have $z \in \{x\}^{(\Lambda,s)}$ implies $x \in \Gamma_{(\Lambda,s)}(\{z\})$ and hence $x \in \Gamma_{(\Lambda,s)}(\{z\}) \cap \Gamma_{(\Lambda,s)}(\{x\})$. By the hypothesis, $\Gamma_{(\Lambda,s)}(\{z\}) = \Gamma_{(\Lambda,s)}(\{x\})$ and by Theorem 4, $\{z\}^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$. Similarly, we have $\{z\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$ and hence $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$. This contradicts that $\{x\}^{(\Lambda,s)} \neq \{y\}^{(\Lambda,s)}$. Thus, $\{x\}^{(\Lambda,s)} \cap \{y\}^{(\Lambda,s)} = \emptyset$ and by Theorem 4, (X, τ) is a (Λ, s)-R_0 space.

Theorem 7. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)-R_0;

(2) $\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\})$ *for each* $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and $x \notin \Gamma_{(\Lambda,s)}(\{x\})$. There exists $V \in (\Lambda, s)O(X)$ such that $x \in V$ and $y \notin V$; hence $\{y\}^{(\Lambda,s)} \cap V = \emptyset$. Since (X, τ) is (Λ, s)-R_0, we have $\{x\}^{(\Lambda,s)} \subseteq V$. Therefore, $\{y\}^{(\Lambda,s)} \cap \{x\}^{(\Lambda,s)} = \emptyset$. Thus, $y \notin \{x\}^{(\Lambda,s)}$ and hence

$$\{x\}^{(\Lambda,s)} \subseteq \Gamma_{(\Lambda,s)}(\{x\}).$$

By Corollary 1, $\{x\}^{(\Lambda,s)} = \{y\}^{(\Lambda,s)}$. Thus, $y \in \{x\}^{(\Lambda,s)}$ and so $\Gamma_{(\Lambda,s)}(\{x\}) \subseteq \{x\}^{(\Lambda,s)}$. This shows that $\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\})$.

(2) $\Rightarrow$ (1): Let $U \in (\Lambda, s)O(X)$ and $x \in U$. By (2) and Proposition 3,

$$\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}(U) = U.$$

Consequently, we obtain (X, τ) is a (Λ, s)-R_0 space.

Corollary 2. *Let (X, τ) be a (Λ, s)-R_0 topological space and $x \in X$. If $\langle x\rangle_{(\Lambda,s)} = \{x\}$, then $\{x\}^{(\Lambda,s)} = \{x\}$.*

Proof. This is a consequence of Theorem 7.

Theorem 8. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)-R_0;

(2) $x \in \{y\}^{(\Lambda,s)}$ *if and only if* $y \in \{x\}^{(\Lambda,s)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $x \in \{y\}^{(\Lambda,s)}$. By Theorem 4, $y \in \Gamma_{(\Lambda,s)}(\{x\})$ and hence $\Gamma_{(\Lambda,s)}(\{x\}) \cap \Gamma_{(\Lambda,s)}(\{y\}) \neq \emptyset$. By Theorem 6, $\Gamma_{(\Lambda,s)}(\{x\}) = \Gamma_{(\Lambda,s)}(\{y\})$ and so $x \in \Gamma_{(\Lambda,s)}(\{y\})$. Thus, by Theorem 4, $y \in \{x\}^{(\Lambda,s)}$. The converse is similarly shown.

(2) $\Rightarrow$ (1): Let $U \in (\Lambda, s)O(X)$ and $x \in U$. If $y \notin U$, then $x \notin \{y\}^{(\Lambda,s)}$ and hence $y \notin \{x\}^{(\Lambda,s)}$. This implies that $\{x\}^{(\Lambda,s)} \subseteq U$. Hence, (X, τ) is a (Λ, s)-R_0 space.

Lemma 5. *Let (X, τ) be a topological space. Then, $[\langle x\rangle_{(\Lambda,s)}]^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$ for each $x \in X$.*

Proof. Since $\{x\} \subseteq \langle x\rangle_{(\Lambda,s)}$, $\{x\}^{(\Lambda,s)} \subseteq [\langle x\rangle_{(\Lambda,s)}]^{(\Lambda,s)}$ by Lemma 4. On the other hand, we have $\langle x\rangle_{(\Lambda,s)} \subseteq \{x\}^{(\Lambda,s)}$ and $[\langle x\rangle_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq [\{x\}^{(\Lambda,s)}]^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$. Thus, $[\langle x\rangle_{(\Lambda,s)}]^{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$.

Theorem 9. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is (Λ, s)-R_0;

(2) $\langle x\rangle_{(\Lambda,s)} = \{x\}^{(\Lambda,s)}$ for each $x \in X$;

(3) $\langle x\rangle_{(\Lambda,s)}$ is (Λ, s)-closed for each $x \in X$.

Proof. (1) $\Rightarrow$ (2): By Theorem 7, $\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\})$ for each $x \in X$. Thus, $\{x\}^{(\Lambda,s)} = \{x\}^{(\Lambda,s)} \cap \Gamma_{(\Lambda,s)}(\{x\}) = \langle x\rangle_{(\Lambda,s)}$.

(2) $\Rightarrow$ (1): Let $U \in (\Lambda, s)O(X)$ and $x \in U$. By (2), we have

$$\{x\}^{(\Lambda,s)} = \langle x\rangle_{(\Lambda,s)} = \{x\}^{(\Lambda,s)} \cap \Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}(U) = U$$

and hence (X, τ) is (Λ, s)-R_0.

(2) $\Leftrightarrow$ (3): This is a consequence of Lemma 5.

Theorem 10. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is (Λ, s)-R_0;

(2) for each nonempty set A of X and each $U \in (\Lambda, s)O(X)$ such that $A \cap U \neq \emptyset$, there exists a (Λ, s)-closed set F such that $A \cap F \neq \emptyset$ and $F \subseteq U$;

(3) $F = \Gamma_{(\Lambda,s)}(F)$ for every (Λ, s)-closed set F;

(4) $\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\})$ for each $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let A be any nonempty set of X and $U \in (\Lambda, s)O(X)$ such that $A \cap U \neq \emptyset$. Then, there exists $x \in U \cap A$ and hence $\{x\}^{(\Lambda,s)} \subseteq U$. Put $F = \{x\}^{(\Lambda,s)}$, then F is (Λ, s)-closed, $A \cap F \neq \emptyset$ and $F \subseteq U$.

(2) $\Rightarrow$ (3): Let F be any (Λ, s)-closed set of X. On the other hand, we have $F \subseteq \Gamma_{(\Lambda,s)}(F)$. Next, we show that $F \supseteq \Gamma_{(\Lambda,s)}(F)$. Suppose that $x \notin F$. Then, $x \in X - F$

and $X - F \in (\Lambda, s)O(X)$. By (2), there exists a (Λ, s)-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, we have $F \subseteq U \in (\Lambda, s)O(X)$ and $x \notin U$. Thus, $x \notin \Gamma_{(\Lambda,s)}(F)$ and hence $F \subseteq \Gamma_{(\Lambda,s)}(F)$. This shows that $F = \Gamma_{(\Lambda,s)}(F)$.

(3) $\Rightarrow$ (4): Let $x \in X$ and $y \notin \Gamma_{(\Lambda,s)}(\{x\})$. There exists a (Λ, s)-open set U such that $x \in U$ and $y \notin U$. Thus, $\{y\}^{(\Lambda,s)} \cap U = \emptyset$ and by (3), $\Gamma_{(\Lambda,s)}(\{y\}^{(\Lambda,s)}) \cap U = \emptyset$. Since $x \notin \Gamma_{(\Lambda,s)}(\{y\}^{(\Lambda,s)})$, there exists a (Λ, s)-open set G such that $\{y\}^{(\Lambda,s)} \subseteq G$ and $x \notin G$. Hence, $\{x\}^{(\Lambda,s)} \cap G = \emptyset$. Since $y \in G$, $y \notin \{x\}^{(\Lambda,s)}$. Therefore, $\{x\}^{(\Lambda,s)} \subseteq \Gamma_{(\Lambda,s)}(\{x\})$. Moreover, $\{x\}^{(\Lambda,s)} \subseteq \Gamma_{(\Lambda,s)}(\{x\}) \subseteq \Gamma_{(\Lambda,s)}[\{x\}^{(\Lambda,s)}] = \{x\}^{(\Lambda,s)}$. This shows that $\{x\}^{(\Lambda,s)} = \Gamma_{(\Lambda,s)}(\{x\})$.

(4) $\Rightarrow$ (1): This is obvious by Theorem 7.

Definition 8. *A topological space (X, τ) is called (Λ, s)-symmetric if, for each $x, y \in X$, $x \in \{y\}^{(\Lambda,s)}$ implies $y \in \{x\}^{(\Lambda,s)}$.*

Theorem 11. *A topological space (X, τ) is (Λ, s)-R_0 if and only if (X, τ) is (Λ, s)-symmetric.*

Proof. Let $x \in \{y\}^{(\Lambda,s)}$ and U be a (Λ, s)-open set such that $y \in U$. Since (X, τ) is (Λ, s)-R_0, we have $x \in \{x\}^{(\Lambda,s)} \subseteq U$. Thus, every (Λ, s)-open set which contains y contains x.

Conversely, let $U \in (\Lambda, s)O(X)$ and $x \in U$. If $y \notin U$, then $x \notin \{y\}^{(\Lambda,s)}$ and hence $y \notin \{x\}^{(\Lambda,s)}$. This implies that $\{y\}^{(\Lambda,s)} \subseteq U$. Thus, (X, τ) is a (Λ, s)-R_0 space.

5. On generalized (Λ, s)-closed sets

In this section, we introduce the notion of generalized (Λ, s)-closed sets and investigate several fundamental properties of generalized (Λ, s)-closed sets. Furthermore, some characterizations of (Λ, s)-normal spaces are discussed.

Definition 9. *A subset A of a topological space (X, τ) is said to be generalized (Λ, s)-closed (briefly g-(Λ, s)-closed) if $A^{(\Lambda,s)} \subseteq U$ whenever $A \subseteq U$ and $U \in (\Lambda, s)O(X)$.*

Remark 1. *Every (Λ, s)-closed set is g-(Λ, s)-closed.*

The converse of Remark 1 need not be true as shown in the following example.

Example 1. *Let $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1, 2\}, X\}$. Then, $A = \{1\}$ is a g-(Λ, s)-closed set, which is not (Λ, s)-closed.*

Theorem 12. *A topological space (X, τ) is (Λ, s)-symmetric if and only if $\{x\}$ is g-(Λ, s)-closed for each $x \in X$.*

Proof. Let $V \in (\Lambda, s)O(X)$ and $x \in V$. Suppose that $\{x\}^{(\Lambda,s)} \nsubseteq V$. Therefore, $(X - V) \cap \{x\}^{(\Lambda,s)} \neq \emptyset$. Let $y \in (X - V) \cap \{x\}^{(\Lambda,s)}$. Thus, $y \in \{x\}^{(\Lambda,s)}$ and hence $x \in \{y\}^{(\Lambda,s)}$. Now, we have $x \in \{y\}^{(\Lambda,s)}$ which is a subset of the complement of V and $x \notin V$. This is a contradiction.

Conversely, let $x \in \{y\}^{(\Lambda,s)}$. Suppose that $y \notin \{x\}^{(\Lambda,s)}$. Thus, $y \in X - \{x\}^{(\Lambda,s)}$ and hence $\{y\}^{(\Lambda,s)} \subseteq X - \{x\}^{(\Lambda,s)}$. Now, the complement of $\{x\}^{(\Lambda,s)}$ contains x which is a contradiction.

Theorem 13. *A subset A of a topological space (X, τ) is g-(Λ, s)-closed if and only if $A^{(\Lambda,s)} - A$ contains no nonempty (Λ, s)-closed set.*

Proof. Let F be a (Λ, s)-closed subset of $A^{(\Lambda,s)} - A$. Now, $A \subseteq X - F$ and since A is g-(Λ, s)-closed, we have $A^{(\Lambda,s)} \subseteq X - F$ and $F \subseteq X - A^{(\Lambda,s)}$. Thus,

$$F \subseteq A^{(\Lambda,s)} \cap (X - A^{(\Lambda,s)}) = \emptyset$$

and F is empty.

Conversely, let $A \subseteq U$ and U be (Λ, s)-open. If $A^{(\Lambda,s)} \not\subseteq U$, then $A^{(\Lambda,s)} \cap (X - U)$ is a nonempty (Λ, s)-closed subset of $A^{(\Lambda,s)} - A$.

Proposition 6. *Let A, B subsets of a topological space (X, τ). If A is g-(Λ, s)-closed and $A \subseteq B \subseteq A^{(\Lambda,s)}$, then B is g-(Λ, s)-closed.*

Proof. Let $B \subseteq U$ and $U \in (\Lambda, s)O(X)$. Then, we have $A \subseteq U$. Since A is g-(Λ, s)-closed, $A^{(\Lambda,s)} \subseteq U$. Since $A \subseteq B \subseteq A^{(\Lambda,s)}$, $B^{(\Lambda,s)} = A^{(\Lambda,s)}$ and hence $B^{(\Lambda,s)} \subseteq U$. Thus, B is g-(Λ, s)-closed.

Definition 10. *Let A be a subset of a topological space (X, τ). The union of all (Λ, s)-open sets contained in A is called the (Λ, s)-interior of A and is denoted by $A_{(\Lambda,s)}$.*

Lemma 6. *Let A and B be subsets of a topological space (X, τ). For the (Λ, s)-interior, the following properties hold:*

(1) $A_{(\Lambda,s)} \subseteq A$ *and* $[A_{(\Lambda,s)}]_{(\Lambda,s)} = A_{(\Lambda,s)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,s)} \subseteq B_{(\Lambda,s)}$.

(3) $A_{(\Lambda,s)}$ *is* (Λ, s)*-open.*

(4) A *is* (Λ, s)*-open if and only if* $A_{(\Lambda,s)} = A$.

Theorem 14. *A subset A of a topological space (X, τ) is g-(Λ, s)-open if and only if $F \subseteq A_{(\Lambda,s)}$ whenever $F \subseteq A$ and F is (Λ, s)-closed.*

Proof. Suppose that A is g-(Λ, s)-open. Let $F \subseteq A$ and F be (Λ, s)-closed. Then, we have $X - A \subseteq X - F$. Since $X - F$ is (Λ, s)-open and $X - A$ is g-(Λ, s)-closed, $X - A_{(\Lambda,s)} = [X - A]^{(\Lambda,s)} \subseteq X - F$ and hence $F \subseteq A_{(\Lambda,s)}$.

Conversely, let $X - A \subseteq U$ and $U \in (\Lambda, s)O(X)$. Then, we have $X - U \subseteq A$ and $X - U$ is (Λ, s)-closed. By the hypothesis, $X - U \subseteq A_{(\Lambda,s)}$ and hence $[X - A]^{(\Lambda,s)} = X - A_{(\Lambda,s)} \subseteq U$. Thus, $X - A$ is g-(Λ, s)-closed and so A is g-(Λ, s)-open.

Corollary 3. *Let A, B subsets of a topological space (X, τ). If A is g-(Λ, s)-open and $A_{(\Lambda,s)} \subseteq B \subseteq A$, then B is g-(Λ, s)-open.*

Proof. This follows from Proposition 6.

Lemma 7. *Let A be a subset of a topological space (X, τ) and $G \in (\Lambda, s)O(X)$. If $A \cap G = \emptyset$, then $A^{(\Lambda,s)} \cap G = \emptyset$.*

Theorem 15. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-(Λ, s)-closed;

(2) $A^{(\Lambda,s)} - A$ contains no nonempty (Λ, s)-closed set;

(3) $A^{(\Lambda,s)} - A$ is g-(Λ, s)-open.

Proof. (1) ⇒ (2): This follows from Theorem 13.

(2) ⇒ (3): Let $F \subseteq A^{(\Lambda,s)} - A$ and F be (Λ, s)-closed. By (2), we have $F = \emptyset$ and $F \subseteq [A^{(\Lambda,s)} - A]_{(\Lambda,s)}$. It follows from Theorem 14 that $A^{(\Lambda,s)} - A$ is g-(Λ, s)-open.

(3) ⇒ (1): Let $A \subseteq U$ and $U \in (\Lambda, s)O(X)$. Thus, $A^{(\Lambda,s)} - U \subseteq A^{(\Lambda,s)} - A$. Since $A^{(\Lambda,s)} - A$ is g-(Λ, s)-open and $A^{(\Lambda,s)} - U$ is (Λ, s)-closed. By Theorem 14,

$$A^{(\Lambda,s)} - U \subseteq [A^{(\Lambda,s)} - A]_{(\Lambda,s)} = \emptyset.$$

Thus, $A^{(\Lambda,s)} \subseteq U$ and hence A is g-(Λ, s)-closed. Now, the proof of $[A^{(\Lambda,s)} - A]_{(\Lambda,s)} = \emptyset$ is given as follows. Suppose that $[A^{(\Lambda,s)} - A]_{(\Lambda,s)} \neq \emptyset$. Let $x \in [A^{(\Lambda,s)} - A]_{(\Lambda,s)}$. Then, there exists $G \in (\Lambda, s)O(X)$ such that $x \in G \subseteq A^{(\Lambda,s)} - A$. Since $G \subseteq X - A$, we have $G \cap A = \emptyset$ and by Lemma 7, $G \cap A^{(\Lambda,s)} = \emptyset$. This implies that $G \subseteq X - A^{(\Lambda,s)}$. Thus, $G \subseteq [X - A^{(\Lambda,s)}] \cap A^{(\Lambda,s)} = \emptyset$. This is a contradiction.

Theorem 16. *A subset A of a topological space (X, τ) is g-(Λ, s)-closed if and only if $F \cap A^{(\Lambda,s)} = \emptyset$ whenever $A \cap F = \emptyset$ and F is (Λ, s)-closed.*

Proof. Let F be a (Λ, s)-closed set such that $A \cap F = \emptyset$. Then, we have $A \subseteq X - F$. Since A is g-(Λ, s)-closed and $X - F$ is (Λ, s)-open, $A^{(\Lambda,s)} \subseteq X - F$. Thus, $F \cap A^{(\Lambda,s)} = \emptyset$.

Conversely, let $A \subseteq U$ and $U \in (\Lambda, s)O(X)$. Then, we have $A \cap (X - U) = \emptyset$ and $X - U$ is (Λ, s)-closed. By the hypothesis, $(X - U) \cap A^{(\Lambda,s)} = \emptyset$ and hence $A^{(\Lambda,s)} \subseteq U$. This shows that A is (Λ, s)-closed.

Theorem 17. *A subset A of a topological space (X, τ) is g-(Λ, s)-closed if and only if $A \cap \{x\}^{(\Lambda,s)} \neq \emptyset$ for every $x \in A^{(\Lambda,s)}$.*

Proof. Suppose that $A \cap \{x\}^{(\Lambda,s)} = \emptyset$ for some $x \in A^{(\Lambda,s)}$. Then, $A \subseteq X - \{x\}^{(\Lambda,s)}$. Since A is g-(Λ, s)-closed and $X - \{x\}^{(\Lambda,s)}$ is (Λ, s)-open, $A^{(\Lambda,s)} \subseteq X - \{x\}^{(\Lambda,s)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda,s)}$.

Conversely, suppose that A is not g-(Λ, s)-closed. Thus, $\emptyset \neq A^{(\Lambda,s)} - U$ for some $U \in (\Lambda, s)O(X)$ containing A. There exists $x \in A^{(\Lambda,s)} - U$. Since $x \notin U$, by Lemma 7, $U \cap \{x\}^{(\Lambda,s)} = \emptyset$ and hence $A \cap \{x\}^{(\Lambda,s)} \subseteq U \cap \{x\}^{(\Lambda,s)} = \emptyset$. This shows that $A \cap \{x\}^{(\Lambda,s)} = \emptyset$ for some $x \in A^{(\Lambda,s)}$.

Corollary 4. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-(Λ, s)-open;

(2) $A - A_{(\Lambda,s)}$ contains no nonempty (Λ, s)-closed set;

(3) $A - A_{(\Lambda,s)}$ is g-(Λ, s)-open;

(4) $(X - A) \cap \{x\}^{(\Lambda,s)} \neq \emptyset$ for every $x \in X - A_{(\Lambda,s)}$.

Proof. This follows from Theorem 15 and Theorem 16.

Theorem 18. *A subset A of a topological space (X, τ) is g-(Λ, s)-open if and only if $U = X$ whenever U is (Λ, s)-open and $(X - A) \cup A_{(\Lambda,s)} \subseteq U$.*

Proof. Let U be a (Λ, s)-open set and $(X - A) \cup A_{(\Lambda,s)} \subseteq U$. Then, we have

$$X - U \subseteq (X - A)^{(\Lambda,s)} - (X - A).$$

Since $X - A$ is g-(Λ, s)-closed and $X - U$ is (Λ, s)-closed. By Theorem 13, $X - U = \emptyset$ and hence $X = U$.

Conversely, suppose that $F \subseteq A$ and F is (Λ, s)-closed. Then,

$$(X - A) \cup A_{(\Lambda,s)} \subseteq (X - F) \cup A_{(\Lambda,s)} \in (\Lambda, s)O(X).$$

By the hypothesis, we have $X = (X - F) \cup A_{(\Lambda,s)}$ and hence

$$F = F \cap [(X - F) \cup A_{(\Lambda,s)}] = F \cap A_{(\Lambda,s)} \subseteq A_{(\Lambda,s)}.$$

It follows from Theorem 14 that A is g-(Λ, s)-open.

Definition 11. *A subset A of a topological space (X, τ) is said to be locally (Λ, s)-closed if $A = U \cap F$, where $U \in (\Lambda, s)O(X)$ and F is a (Λ, s)-closed set.*

Theorem 19. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is locally (Λ, s)-closed;

(2) $A = U \cap A^{(\Lambda,s)}$ for some $U \in (\Lambda, s)O(X)$;

(3) $A^{(\Lambda,s)} - A$ is (Λ, s)-closed;

(4) $A \cup [X - A_{(\Lambda,s)}]$ *is* (Λ, s)*-open;*

(5) $A \subseteq [A \cup [X - A^{(\Lambda,s)}]]_{(\Lambda,s)}$.

Proof. (1) ⇒ (2): Suppose that $A = U \cap F$, where $U \in (\Lambda, s)O(X)$ and F is (Λ, s)-closed. Since $A \subseteq F$, we have $A^{(\Lambda,s)} \subseteq F^{(\Lambda,s)} = F$. Since $A \subseteq U$,

$$A \subseteq U \cap A^{(\Lambda,s)} \subseteq U \cap F = A.$$

Thus, $A = U \cap A^{(\Lambda,s)}$.

(2) ⇒ (3): Suppose that $A = U \cap A^{(\Lambda,s)}$ for some $U \in (\Lambda, s)O(X)$. Then, we have $A^{(\Lambda,s)} - A = (X - [U \cap A^{(\Lambda,s)}]) \cap A^{(\Lambda,s)} = (X - U) \cap A^{(\Lambda,s)}$. This shows that $A^{(\Lambda,s)} - A$ is (Λ, s)-closed.

(3) ⇒ (4): Since $X - [A^{(\Lambda,s)} - A] = [X - A^{(\Lambda,s)}] \cup A$ and by (3), we obtain $A \cup [X - A^{(\Lambda,s)}]$ is (Λ, s)-open.

(4) ⇒ (5): By (4), $A \subseteq A \cup [X - A^{(\Lambda,s)}] = [A \cup (X - A^{(\Lambda,s)})]_{(\Lambda,s)}$.

(5) ⇒ (1): We put $U = [A \cup [X - A^{(\Lambda,s)}]]_{(\Lambda,s)}$. Then, we have U is (Λ, s)-open and $A = A \cap U \subseteq U \cap A^{(\Lambda,s)} \subseteq [A \cup [X - A^{(\Lambda,s)}]] \cap A^{(\Lambda,s)} = A \cap A^{(\Lambda,s)} = A$. Therefore, we obtain $A = U \cap A^{(\Lambda,s)}$, where $U \in (\Lambda, s)O(X)$ and $A^{(\Lambda,s)}$ is (Λ, s)-closed. Thus, A is locally (Λ, s)-closed.

Theorem 20. *A subset* A *of a topological space* (X, τ) *is* (Λ, s)*-closed if and only if* A *is locally* (Λ, s)*-closed and g-*(Λ, s)*-closed.*

Proof. Let A be (Λ, s)-closed. By Remark 1, A is g-(Λ, s)-closed. Since X is (Λ, s)-open and $A = X \cap A$, we have A is locally (Λ, s)-closed.

Conversely, suppose that A is locally (Λ, s)-closed and g-(Λ, s)-closed. Since A is locally (Λ, s)-closed and by Theorem 19, $A \subseteq [A \cup (X - A^{(\Lambda,s)})]_{(\Lambda,s)}$. Since $[A \cup (X - A^{(\Lambda,s)})]_{(\Lambda,s)}$ is (Λ, s)-open and A is g-(Λ, s)-closed,

$$A^{(\Lambda,s)} \subseteq [A \cup [X - A^{(\Lambda,s)}]]_{(\Lambda,s)} \subseteq A \cup [X - A^{(\Lambda,s)}]$$

and hence $A^{(\Lambda,s)} \subseteq A$. Thus, $A^{(\Lambda,s)} = A$ and by Lemma 4, A is (Λ, s)-closed.

Definition 12. *A subset* A *of a topological space* (X, τ) *is said to be:*

(i) (Λ, s)*-dense if* $A^{(\Lambda,s)} = X$*;*

(ii) (Λ, s)*-codense if its complement is* (Λ, s)*-dense.*

Definition 13. *A topological space* (X, τ) *is said to be* (Λ, s)*-submaximal if, for each* (Λ, s)*-dense subset of* X *is* (Λ, s)*-open.*

Theorem 21. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)*-submaximal;*

(2) every subset of X is a locally (Λ, s)-closed set;

(3) every subset of X is the union of a (Λ, s)-open set and a (Λ, s)-closed set;

(4) every (Λ, s)-dense set of X is the intersection of a (Λ, s)-closed set and a (Λ, s)-open set;

(5) every (Λ, s)-codense set of X is the union of a (Λ, s)-open set and a (Λ, s)-closed set.

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ) is (Λ, s)-submaximal. Let A be any subset of X. Then, $[X - [A^{(\Lambda,s)} - A]]^{(\Lambda,s)} = [A \cup [X - A^{(\Lambda,s)}]]^{(\Lambda,s)} = X$. Therefore, $X - [A^{(\Lambda,s)} - A]$ is (Λ, s)-dense and so $X - [A^{(\Lambda,s)} - A]$ is (Λ, s)-open. Thus, $X - [A^{(\Lambda,s)} - A] = A \cup [X - A^{(\Lambda,s)}]$ is (Λ, s)-open. This shows that $A = [A \cup [X - A^{(\Lambda,s)}]] \cap A^{(\Lambda,s)}$ is locally (Λ, s)-closed.

(2) $\Leftrightarrow$ (3): Suppose that every subset of X is a locally (Λ, s)-closed set. Let A be any subset of X. By (2), we have $X - A = U \cap F$, where U is a (Λ, s)-open set and F is a (Λ, s)-closed set. This implies that $A = (X - U) \cup (X - K)$, where $X - U$ is a (Λ, s)-closed set and $X - F$ is a (Λ, s)-open set. The converse is similar.

(2) $\Rightarrow$ (4) and (4) $\Leftrightarrow$ (5) are obvious.

(4) $\Rightarrow$ (1): Let A be a (Λ, s)-dense set. By (4), there exist a (Λ, s)-open set U and a (Λ, s)-closed set F such that $A = U \cap F$. Since $A \subseteq F$ and A is a (Λ, s)-dense set, $X \subseteq F$. Thus, $F = X$ and hence $A = U$ is (Λ, s)-open. This shows that (X, τ) is (Λ, s)-submaximal.

Definition 14. *A subset A of a topological space (X, τ) is said to be:*

(i) a $\Gamma_{(\Lambda,s)}$-set if $A = \Gamma_{(\Lambda,s)}(A)$;

(ii) a $\circledast\Gamma_{(\Lambda,s)}$-set if $\Gamma_{(\Lambda,s)}(A) \subseteq F$ whenever $A \subseteq F$ and F is a (Λ, s)-closed set.

Definition 15. *A topological space (X, τ) is called (Λ, s)-$T_{\frac{1}{2}}$ if every g-(Λ, s)-closed set of X is (Λ, s)-closed.*

Lemma 8. *For a topological space (X, τ), the following properties hold:*

(1) for each $x \in X$, the singleton $\{x\}$ is (Λ, s)-closed or $X - \{x\}$ is g-(Λ, s)-closed;

(2) for each $x \in X$, the singleton $\{x\}$ is (Λ, s)-open or $X - \{x\}$ is a $\circledast\Gamma_{(\Lambda,s)}$-set.

Proof. (1) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, s)-closed. Then, we have $X - \{x\}$ is not (Λ, s)-open and X is the only (Λ, s)-open set which contains $X - \{x\}$ and hence $X - \{x\}$ is g-(Λ, s)-closed.

(2) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, s)-open. Then, we have $X - \{x\}$ is not (Λ, s)-closed and the only (Λ, s)-closed set which contains $X - \{x\}$ is X and hence $X - \{x\}$ is a $\circledast\Gamma_{(\Lambda,s)}$-set.

Theorem 22. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)*-*$T_{\frac{1}{2}}$*;*

(2) for each $x \in X$*, the singleton* $\{x\}$ *is* (Λ, s)*-open or* (Λ, s)*-closed;*

(3) every $\circledast\Gamma_{(\Lambda,s)}$*-set is a* $\Gamma_{(\Lambda,s)}$*-set.*

Proof. (1) $\Rightarrow$ (2): By Lemma 8, for each $x \in X$, the singleton $\{x\}$ is (Λ, s)-closed or $X - \{x\}$ is g-(Λ, s)-closed. Since (X, τ) is a (Λ, s)-$T_{\frac{1}{2}}$-space, $X - \{x\}$ is (Λ, s)-closed and hence $\{x\}$ is (Λ, s)-open in the latter case. Therefore, the singleton $\{x\}$ is (Λ, s)-open or (Λ, s)-closed.

(2) $\Rightarrow$ (3): Suppose that there exists a $\circledast\Gamma_{(\Lambda,s)}$-set A which is not a $\Gamma_{(\Lambda,s)}$-set. There exists $x \in \Gamma_{(\Lambda,s)}(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is (Λ, s)-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, s)-closed. Since A is a $\circledast\Gamma_{(\Lambda,s)}$-set, $\Gamma_{(\Lambda,s)}(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is (Λ, s)-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, s)-open. By Proposition 3, $\Gamma_{(\Lambda,s)}(A) \subseteq \Gamma_{(\Lambda,s)}(X - \{x\}) = X - \{x\}$. This is a contradiction. Thus, every $\circledast\Gamma_{(\Lambda,s)}$-set is a $\Gamma_{(\Lambda,s)}$-set.

(3) $\Rightarrow$ (1): Suppose that (X, τ) is not a (Λ, s)-$T_{\frac{1}{2}}$-space. Then, there exists a g-(Λ, s)-closed set A which is not (Λ, s)-closed. Since A is not (Λ, s)-closed, there exists $x \in A^{(\Lambda,s)}$ such that $x \notin A$. By Lemma 8, the singleton $\{x\}$ is (Λ, s)-open or $X - \{x\}$ is a $\Gamma_{(\Lambda,s)}$-set. (a) In case $\{x\}$ is (Λ, s)-open, since $x \in A^{(\Lambda,s)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Gamma_{(\Lambda,s)}$-set, if $\{x\}$ is not (Λ, s)-closed, $X - \{x\}$ is not (Λ, s)-open and $\Gamma_{(\Lambda,s)}(X - \{x\}) = X$. Hence, $X - \{x\}$ is not a $\Gamma_{(\Lambda,s)}$-set. This contradicts (3). If $\{x\}$ is (Λ, s)-closed, $A \subseteq X - \{x\} \in (\Lambda, s)O(X)$ and A is g-(Λ, s)-closed. Thus, we have $A^{(\Lambda,s)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda,s)}$. Therefore, (X, τ) is (Λ, s)-$T_{\frac{1}{2}}$.

Now, as an application of g-(Λ, s)-closed sets, we introduce the concept of (Λ, s)-normality in a topological space (X, τ). This concept enables us to unify several modifications of normal spaces.

Definition 16. *A topological space* (X, τ) *is said to be* (Λ, s)*-normal if, for any disjoint* (Λ, s)*-closed sets* F_1 *and* F_2*, there exist disjoint* (Λ, s)*-open sets* U_1 *and* U_2 *such that* $F_1 \subseteq U_1$ *and* $F_2 \subseteq U_2$*.*

Theorem 23. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)*-normal;*

(2) for every disjoint (Λ, s)*-closed sets* F_1 *and* F_2*, there exist disjoint g-*(Λ, s)*-open sets* U_1 *and* U_2 *such that* $F_1 \subseteq U_1$ *and* $F_2 \subseteq U_2$*;*

(3) for each (Λ, s)*-closed set* F *and each* (Λ, s)*-open set* G *containing* F*, there exists a g-*(Λ, s)*-open set* U *such that* $F \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G$*;*

(4) for each (Λ, s)*-closed set* F *and each g-*(Λ, s)*-open set* G *containing* F*, there exists a* (Λ, s)*-open set* U *such that* $F \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G_{(\Lambda,s)}$*;*

(5) for each (Λ, s)-closed set F and each g-(Λ, s)-open set G containing F, there exists a g-(Λ, s)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G_{(\Lambda,s)}$;

(6) for each g-(Λ, s)-closed set F and each (Λ, s)-open set G containing F, there exists a (Λ, s)-open set U such that $F^{(\Lambda,s)} \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G$;

(7) for each g-(Λ, s)-closed set F and each (Λ, s)-open set G containing F, there exists a g-(Λ, s)-open set U such that $F^{(\Lambda,s)} \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let F be a (Λ, s)-closed set and G be a (Λ, s)-open set containing F. Then, F and $X - G$ are two disjoint (Λ, s)-closed sets. Hence by (2), there exist disjoint g-(Λ, s)-open sets U and V such that $F \subseteq U$ and $X - G \subseteq V$. Since V is g-(Λ, s)-open and $X - G$ is (Λ, s)-closed, by Theorem 14, $X - G \subseteq V_{(\Lambda,s)}$. Since $U \cap V = \emptyset$, we have $U^{(\Lambda,s)} \subseteq (X - V)^{(\Lambda,s)} = X - V_{(\Lambda,s)} \subseteq G$. Thus, $F \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G$.

(3) $\Rightarrow$ (1): Let F_1 and F_2 be any disjoint (Λ, s)-closed sets. Then, we have $X - F_2$ is a (Λ, s)-open set containing F_1. Thus by (3), there exists a g-(Λ, s)-open set U such that $F_1 \subseteq U \subseteq U^{(\Lambda,s)} \subseteq X - F_2$ and hence $F_2 \subseteq X - U^{(\Lambda,s)}$. Since F_1 is (Λ, s)-closed and U is g-(Λ, s)-open, by Theorem 14, we have $F_1 \subseteq U_{(\Lambda,s)}$. This shows that (X, τ) is (Λ, s)-normal.

(6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (5): Let F be a (Λ, s)-closed set and G be a g-(Λ, s)-open set containing F. Since G is g-(Λ, s)-open and F is (Λ, s)-closed, by Theorem 14, $F \subseteq G_{(\Lambda,s)}$. Thus by (3), there exists a g-(Λ, s)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G_{(\Lambda,s)}$.

(5) $\Rightarrow$ (6): Let F be a g-(Λ, s)-closed set and G be a (Λ, s)-open set containing F. Then, we have $F^{(\Lambda,s)} \subseteq G$. Since G is g-(Λ, s)-open and by (5), there exists a g-(Λ, s)-open set U such that $F^{(\Lambda,s)} \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G$. Since U is g-(Λ, s)-open and $F^{(\Lambda,s)}$ is (Λ, s)-closed, by Theorem 14, $F^{(\Lambda,s)} \subseteq U_{(\Lambda,s)}$. Put $V = U_{(\Lambda,s)}$. Then, V is (Λ, s)-open and $F^{(\Lambda,s)} \subseteq V \subseteq V^{(\Lambda,s)} = [U_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq U^{(\Lambda,s)} \subseteq G$.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2): The proofs are obvious.

(6) $\Rightarrow$ (4): Let F be a (Λ, s)-closed set and G be a g-(Λ, s)-open set containing F. By Theorem 14, $F \subseteq G_{(\Lambda,s)}$. Since F is g-(Λ, s)-closed and $G_{(\Lambda,s)}$ is (Λ, s)-open, by (6), there exists a (Λ, s)-open set U such that $F = F^{(\Lambda,s)} \subseteq U \subseteq U^{(\Lambda,s)} \subseteq G_{(\Lambda,s)}$.

6. On (Λ, s)-extremally disconnected spaces

In this section, we introduce the notion of (Λ, s)-extremally disconnected spaces and investigate several characterizations of such spaces.

Definition 17. *A subset A of a topological space (X, τ) is said to be:*

(i) $s(\Lambda, s)$-open if $A \subseteq [A_{(\Lambda,s)}]^{(\Lambda,s)}$;

(ii) $p(\Lambda, s)$-open if $A \subseteq [A^{(\Lambda,s)}]_{(\Lambda,s)}$;

(iii) $\alpha(\Lambda, s)$*-open if* $A \subseteq [[A_{(\Lambda,s)}]^{(\Lambda,s)}]_{(\Lambda,s)}$*;*

(iv) $\beta(\Lambda, s)$*-open if* $A \subseteq [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)}$*;*

(v) $b(\Lambda, s)$*-open set if* $A \subseteq [A_{(\Lambda,s)}]^{(\Lambda,s)} \cup [A^{(\Lambda,s)}]_{(\Lambda,s)}$.

The family of all $s(\Lambda, s)$-open (resp. $p(\Lambda, s)$-open, $\alpha(\Lambda, s)$-open, $\beta(\Lambda, s)$-open, $b(\Lambda, s)$-open) sets in a topological space (X, τ) is denoted by $s(\Lambda, s)O(X)$ (resp. $p(\Lambda, s)O(X)$, $\alpha(\Lambda, s)O(X)$, $\beta(\Lambda, s)O(X)$, $b(\Lambda, s)O(X)$).

The complement of a $s(\Lambda, s)$-open (resp. $p(\Lambda, s)$-open, $\alpha(\Lambda, s)$-open, $\beta(\Lambda, s)$-open, $b(\Lambda, s)$-open) set is called $s(\Lambda, s)$*-closed* (resp. $p(\Lambda, s)$*-closed*, $\alpha(\Lambda, s)$*-closed*, $\beta(\Lambda, s)$*-closed*, $b(\Lambda, s)$*-closed*). The family of all $s(\Lambda, s)$-closed (resp. $p(\Lambda, s)$-closed, $\alpha(\Lambda, s)$-closed, $\beta(\Lambda, s)$-closed, $b(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $s(\Lambda, s)C(X)$ (resp. $p(\Lambda, s)C(X)$, $\alpha(\Lambda, s)C(X)$, $\beta(\Lambda, s)C(X)$, $b(\Lambda, s)C(X)$).

Definition 18. *A subset A of a topological space (X, τ) is said to be* $r(\Lambda, s)$*-open (resp.* $r(\Lambda, s)$*-closed) if* $A = [A^{(\Lambda,s)}]_{(\Lambda,s)}$ *(resp.* $A = [A_{(\Lambda,s)}]^{(\Lambda,s)}$*).*

The family of all $r(\Lambda, s)$-open (resp. $r(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $r(\Lambda, s)O(X)$ (resp. $r(\Lambda, s)C(X)$).

Proposition 7. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \beta(\Lambda, s)O(X)$*;*

(2) $A^{(\Lambda,s)} \in r(\Lambda, s)C(X)$*;*

(3) $A^{(\Lambda,s)} \in \beta(\Lambda, s)O(X)$*;*

(4) $A^{(\Lambda,s)} \in s(\Lambda, s)O(X)$*;*

(5) $A^{(\Lambda,s)} \in b(\Lambda, s)O(X)$.

Proof. (1) $\Rightarrow$ (2): Let $A \in \beta(\Lambda, s)O(X)$. Then, we have $A \subseteq [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)}$ and hence $A^{(\Lambda,s)} \subseteq [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq A^{(\Lambda,s)}$. Thus, $A^{(\Lambda,s)} = [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)}$. Therefore, $A^{(\Lambda,s)} \in r(\Lambda, s)C(X)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (1): Let $A^{(\Lambda,s)} \in b(\Lambda, s)O(X)$. Then, we have

$$\begin{aligned} A^{(\Lambda,s)} &\subseteq [[A^{(\Lambda,s)}]^{(\Lambda,s)}]_{(\Lambda,s)} \cup [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \\ &= [A^{(\Lambda,s)}]_{(\Lambda,s)} \cup [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \\ &= [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \end{aligned}$$

and hence $A \subseteq [[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)}$. Thus, $A \in \beta(\Lambda, s)O(X)$.

Corollary 5. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \beta(\Lambda, s)C(X)$;

(2) $A_{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(3) $A_{(\Lambda,s)} \in \beta(\Lambda, s)C(X)$;

(4) $A_{(\Lambda,s)} \in s(\Lambda, s)C(X)$;

(5) $A_{(\Lambda,s)} \in b(\Lambda, s)C(X)$.

Definition 19. *A topological space (X, τ) is called (Λ, s)-extremally disconnected if $U^{(\Lambda,s)}$ is (Λ, s)-open in X for every (Λ, s)-open set U of X.*

Theorem 24. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is (Λ, s)-extremally disconnected;*

(2) for each $V \in \beta(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(3) for each $V \in b(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(4) for each $V \in s(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(5) for each $V \in \alpha(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(6) for each $V \in (\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(7) for each $V \in r(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$;

(8) for each $V \in p(\Lambda, s)O(X)$, $V^{(\Lambda,s)} \in r(\Lambda, s)O(X)$.

Proof. The proof follows from Theorem 2 of [3].

Theorem 25. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is (Λ, s)-extremally disconnected;*

(2) $r(\Lambda, s)C(X) \subseteq (\Lambda, s)O(X)$;

(3) $r(\Lambda, s)C(X) \subseteq \alpha(\Lambda, s)O(X)$;

(4) $r(\Lambda, s)C(X) \subseteq p(\Lambda, s)O(X)$;

(5) $s(\Lambda, s)O(X) \subseteq \alpha(\Lambda, s)O(X)$;

(6) $s(\Lambda, s)C(X) \subseteq \alpha(\Lambda, s)C(X)$;

(7) $s(\Lambda, s)C(X) \subseteq p(\Lambda, s)C(X)$;

(8) $s(\Lambda, s)O(X) \subseteq p(\Lambda, s)O(X)$*;*

(9) $\beta(\Lambda, s)O(X) \subseteq p(\Lambda, s)O(X)$*;*

(10) $\beta(\Lambda, s)C(X) \subseteq p(\Lambda, s)C(X)$*;*

(11) $b(\Lambda, s)C(X) \subseteq p(\Lambda, s)C(X)$*;*

(12) $b(\Lambda, s)O(X) \subseteq p(\Lambda, s)O(X)$*;*

(13) $r(\Lambda, s)O(X) \subseteq p(\Lambda, s)C(X)$*;*

(14) $r(\Lambda, s)O(X) \subseteq (\Lambda, s)C(X)$*;*

(15) $r(\Lambda, s)O(X) \subseteq \alpha(\Lambda, s)C(X)$*.*

Proof. The proof follows from Theorem 3 of [3].

Theorem 26. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)*-extremally disconnected;*

(2) $F_{(\Lambda,s)}$ *is* (Λ, s)*-closed for every* (Λ, s)*-closed set* F *of* X*;*

(3) $[A_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq [A^{(\Lambda,s)}]_{(\Lambda,s)}$ *for every subset* A *of* X*.*

Proof. (1) $\Rightarrow$ (2): Let F be any (Λ, s)-closed set. Then, we have $X - F$ is (Λ, s)-open. Since (X, τ) is (Λ, s)-extremally disconnected, $[X - F]^{(\Lambda,s)} = X - F_{(\Lambda,s)}$ is (Λ, s)-open and hence $F_{(\Lambda,s)}$ is (Λ, s)-closed.

(2) $\Rightarrow$ (3): Let A be any subset of X. Then, $X - A_{(\Lambda,s)}$ is (Λ, s)-closed and by (2), $[X - A_{(\Lambda,s)}]_{(\Lambda,s)}$ is (Λ, s)-closed. Thus, $[A_{(\Lambda,s)}]^{(\Lambda,s)}$ is (Λ, s)-open and hence

$$[A_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq [A^{(\Lambda,s)}]_{(\Lambda,s)}.$$

(3) $\Rightarrow$ (1): Let U be any $s(\Lambda, s)$-open set. Thus, by (3),

$$U^{(\Lambda,s)} = [U_{(\Lambda,s)}]^{(\Lambda,s)} \subseteq [U^{(\Lambda,s)}]_{(\Lambda,s)}$$

and so $U^{(\Lambda,s)}$ is (Λ, s)-open. This shows that (X, τ) is (Λ, s)-extremally disconnected.

Theorem 27. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* (Λ, s)*-extremally disconnected;*

(2) for every (Λ, s)*-open sets* U_1 *and* U_2 *such that* $U_1 \cap U_2 = \emptyset$*, there exist disjoint* (Λ, s)*-closed sets* F_1 *and* F_2 *such that* $U_1 \subseteq F_1$ *and* $U_2 \subseteq F_2$*;*

(3) $U_1^{(\Lambda,s)} \cap U_2^{(\Lambda,s)} = \emptyset$ *for every* (Λ, s)*-open sets* U_1 *and* U_2 *such that* $U_1 \cap U_2 = \emptyset$*;*

(4) $[[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \cap U^{(\Lambda,s)} = \emptyset$ *for every subset* A *of* X *and every* (Λ,s)*-open set* U *such that* $A \cap U = \emptyset$.

Proof. (1) ⇒ (2): Suppose that (X,τ) is (Λ,s)-extremally disconnected. Let U_1 and U_2 be (Λ,s)-open sets such that $U_1 \cap U_2 = \emptyset$. Then, we have $U_1^{(\Lambda,s)}$ and $X - U_1^{(\Lambda,s)}$ are disjoint (Λ,s)-closed sets containing U_1 and U_2, respectively.

(2) ⇒ (3): Let U_1 and U_2 be (Λ,s)-open sets such that $U_1 \cap U_2 = \emptyset$. By (2), there exist disjoint (Λ,s)-closed sets F_1 and F_2 such that $U_1 \subseteq F_1$ and $U_2 \subseteq F_2$. Thus,

$$U_1^{(\Lambda,s)} \cap U_2^{(\Lambda,s)} \subseteq F_1 \cap F_2 = \emptyset$$

and hence $U_1^{(\Lambda,s)} \cap U_2^{(\Lambda,s)} = \emptyset$.

(3) ⇒ (4): Let A be any subset of X and U be any (Λ,s)-open set such that $A \cap U = \emptyset$. Since $[A^{(\Lambda,s)}]_{(\Lambda,s)}$ is (Λ,s)-open and $[A^{(\Lambda,s)}]_{(\Lambda,s)} \cap U = \emptyset$. By (3),

$$[[A^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \cap U^{(\Lambda,s)} = \emptyset.$$

(4) ⇒ (1): Let U be any (Λ,s)-open set. Then, we have $[X - U^{(\Lambda,s)}] \cap U = \emptyset$. Since $X - U^{(\Lambda,s)}$ is (Λ,s)-open and by (4), $[[U^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} \cap [X - U^{(\Lambda,s)}]^{(\Lambda,s)} = \emptyset$. Since U is (Λ,s)-open, we have $U^{(\Lambda,s)} \cap [X - [U^{(\Lambda,s)}]_{(\Lambda,s)}] = \emptyset$ and hence $U^{(\Lambda,s)} \subseteq [U^{(\Lambda,s)}]_{(\Lambda,s)}$. This implies that $U^{(\Lambda,s)}$ is (Λ,s)-open. Thus, (X,τ) is (Λ,s)-extremally disconnected.

Theorem 28. *For a topological space* (X,τ)*, the following properties are equivalent:*

(1) (X,τ) *is* (Λ,s)*-extremally disconnected;*

(2) for every $r(\Lambda,s)$*-open set of* X *is* (Λ,s)*-closed;*

(3) for every $r(\Lambda,s)$*-closed set of* X *is* (Λ,s)*-open.*

Proof. (1) ⇒ (2): Suppose that (X,τ) is (Λ,s)-extremally disconnected. Let U be any $r(\Lambda,s)$-open set of X. Then, we have $U = [U^{(\Lambda,s)}]_{(\Lambda,s)}$. Since U is (Λ,s)-open, $U^{(\Lambda,s)}$ is (Λ,s)-open. Thus, $U = [U^{(\Lambda,s)}]_{(\Lambda,s)} = U^{(\Lambda,s)}$ and hence U is (Λ,s)-closed.

(2) ⇒ (1): Suppose that for every $r(\Lambda,s)$-open set of X is (Λ,s)-closed. Let U be any (Λ,s)-open set. Since $[U^{(\Lambda,s)}]_{(\Lambda,s)}$ is $r(\Lambda,s)$-open, we have $[U^{(\Lambda,s)}]_{(\Lambda,s)}$ is (Λ,s)-closed and hence $U^{(\Lambda,s)} \subseteq [[U^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)} = [U^{(\Lambda,s)}]_{(\Lambda,s)}$. Thus, $U^{(\Lambda,s)}$ is (Λ,s)-open. This shows that (X,τ) is (Λ,s)-extremally disconnected.

(2) ⇔ (3): The proof is obvious.

7. Characterizations of almost (Λ,s)-continuous functions

In this section, we introduce the notion of almost (Λ,s)-continuous functions. Moreover, some characterizations of almost (Λ,s)-continuous functions are discussed.

Definition 20. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost (Λ, s)-continuous at a point $x \in X$ if, for each (Λ, s)-open set V of Y containing $f(x)$, there exists a (Λ, s)-open set U of X containing x such that $f(U) \subseteq [V^{(\Lambda,s)}]_{(\Lambda,s)}$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost (Λ, s)-continuous if f has this property at each point $x \in X$.*

Theorem 29. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost (Λ, s)-continuous at $x \in X$;

(2) $x \in [f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$ for every (Λ, s)-open set V of Y containing $f(x)$;

(3) $x \in [f^{-1}(V)]_{(\Lambda,s)}$ for every $r(\Lambda, s)$-open set V of Y containing $f(x)$;

(4) for every $r(\Lambda, s)$-open set V of Y containing $f(x)$, there exists a (Λ, s)-open set U of X containing x such that $f(U) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any (Λ, s)-open set of Y containing $f(x)$. Then, there exists a (Λ, s)-open set U of X containing x such that $f(U) \subseteq [V^{(\Lambda,s)}]_{(\Lambda,s)}$. Thus,

$$x \in U \subseteq f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)}).$$

Since $U \in (\Lambda, s)O(X)$, we have $x \in [f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$.

(2) ⇒ (3): Let V be any $r(\Lambda, s)$-open set of Y containing $f(x)$. Since $V = [V^{(\Lambda,s)}]_{(\Lambda,s)}$ and by (2), we have $x \in [f^{-1}(V)]_{(\Lambda,s)}$.

(3) ⇒ (4): Let V be any $r(\Lambda, s)$-open set of Y containing $f(x)$. Thus, by (3), we have $x \in [f^{-1}(V)]_{(\Lambda,s)}$. Then, there exists a (Λ, s)-open set U of X containing x such that $U \subseteq f^{-1}(V)$ and hence $f(U) \subseteq V$.

(4) ⇒ (1): Let V be any (Λ, s)-open set of Y containing $f(x)$. Then,

$$f(x) \in V \subseteq [V^{(\Lambda,s)}]_{(\Lambda,s)}.$$

Since $[V^{(\Lambda,s)}]_{(\Lambda,s)}$ is $r(\Lambda, s)$-open and by (4), there exists a (Λ, s)-open set U of X containing x such that $f(U) \subseteq [V^{(\Lambda,s)}]_{(\Lambda,s)}$. This shows that f is almost (Λ, s)-continuous.

Theorem 30. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost (Λ, s)-continuous;

(2) $f^{-1}(V) \subseteq [f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$ for every (Λ, s)-open set V of Y;

(3) $[f^{-1}([F_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)} \subseteq f^{-1}(F)$ for every (Λ, s)-closed set F of Y;

(4) $[f^{-1}([[B^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)} \subseteq f^{-1}(B^{(\Lambda,s)})$ for every subset B of Y;

(5) $f^{-1}(B_{(\Lambda,s)}) \subseteq [f^{-1}([[B_{(\Lambda,s)}]^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$ for every subset B of Y;

(6) $f^{-1}(V)$ is (Λ, s)-open in X for every $r(\Lambda, s)$-open set V of Y;

(7) $f^{-1}(F)$ *is* (Λ,s)*-closed in* X *for every* $r(\Lambda,s)$*-closed set* F *of* Y*.*

Proof. (1) ⇒ (2): Let V be any (Λ,s)-open set of Y and $x\in f^{-1}(V)$. By (1), there exists a (Λ,s)-open set U of X containing x such that $f(U)\subseteq[V^{(\Lambda,s)}]_{(\Lambda,s)}$. This implies that $x\in[f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$. Thus, $f^{-1}(V)\subseteq[f^{-1}([V^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$.

(2) ⇒ (3): Let F be any (Λ,s)-closed set of Y. Thus, by (2), we have

$$\begin{aligned}X-f^{-1}(V)&=f^{-1}(Y-F)\\&\subseteq[f^{-1}([[Y-F]^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}\\&=[f^{-1}(Y-[F_{(\Lambda,s)}]^{(\Lambda,s)})]_{(\Lambda,s)}\\&=X-[f^{-1}([F_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)}\end{aligned}$$

and hence $[f^{-1}([F_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)}\subseteq f^{-1}(F)$.

(3) ⇒ (4): Let B be any subset of Y. Since $B^{(\Lambda,s)}$ is (Λ,s)-closed and by (3), we have $[f^{-1}([B^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)}\subseteq f^{-1}(B^{(\Lambda,s)})$.

(4) ⇒ (5): Let B be any subset of Y. By (4),

$$\begin{aligned}f^{-1}(B_{(\Lambda,s)})&=X-f^{-1}([Y-B]^{(\Lambda,s)})\\&\subseteq X-[f^{-1}([[[Y-B]^{(\Lambda,s)}]_{(\Lambda,s)}]^{(\Lambda,s)})]^{(\Lambda,s)}\\&=[f^{-1}([[B_{(\Lambda,s)}]^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}.\end{aligned}$$

(5) ⇒ (6): Let V be any $r(\Lambda,s)$-open set of Y. Since $[[V_{(\Lambda,s)}]^{(\Lambda,s)}]_{(\Lambda,s)}=V$ and by (5), $f^{-1}(V)\subseteq[f^{-1}(V)]_{(\Lambda,s)}$. Thus, $f^{-1}(V)=[f^{-1}(V)]_{(\Lambda,s)}$ and hence $f^{-1}(V)$ is (Λ,s)-open.

(6) ⇒ (7): The proof is obvious.

(7) ⇒ (1): Let V be any $r(\Lambda,s)$-open set of Y containing $f(x)$. Thus, by (7), we have $X-f^{-1}(V)=f^{-1}(Y-V)=[f^{-1}(Y-V)]^{(\Lambda,s)}=X-[f^{-1}(V)]_{(\Lambda,s)}$ and hence $f^{-1}(V)=[f^{-1}(V)]_{(\Lambda,s)}$. Since $x\in[f^{-1}(V)]_{(\Lambda,s)}$, there exists a (Λ,s)-open set U of X containing x such that $U\subseteq f^{-1}(V)$. Thus, $f(U)\subseteq V$ and by Theorem 29, f is almost (Λ,s)-continuous.

Theorem 31. *For a function* $f:(X,\tau)\to(Y,\sigma)$*, the following properties are equivalent:*

(1) f *is almost* (Λ,s)*-continuous;*

(2) $[f^{-1}(U)]^{(\Lambda,s)}\subseteq f^{-1}(U^{(\Lambda,s)})$ *for every* $\beta(\Lambda,s)$*-open set* U *of* Y*;*

(3) $[f^{-1}(U)]^{(\Lambda,s)}\subseteq f^{-1}(U^{(\Lambda,s)})$ *for every* $s(\Lambda,s)$*-open set* U *of* Y*;*

(4) $f^{-1}(U)\subseteq[f^{-1}([U^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$ *for every* $p(\Lambda,s)$*-open set* U *of* Y*.*

Proof. (1) ⇒ (2): Let U be any $\beta(\Lambda,s)$-open set of Y. Since $U^{(\Lambda,s)}$ is $r(\Lambda,s)$-closed, by Theorem 30, $[f^{-1}(U)]^{(\Lambda,s)}\subseteq[f^{-1}(U^{(\Lambda,s)})]^{(\Lambda,s)}=f^{-1}(U^{(\Lambda,s)})$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let F be any $r(\Lambda, s)$-closed set of Y. Then, we have F is $s(\Lambda, s)$-open. By (3), $[f^{-1}(F)]^{(\Lambda,s)} \subseteq f^{-1}[F^{(\Lambda,s)}] = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is (Λ, s)-closed and by Theorem 30, f is almost (Λ, s)-continuous.

(1) ⇒ (4): Let U be any $p(\Lambda, s)$-open set of Y. Then, $U \subseteq [U^{(\Lambda,s)}]_{(\Lambda,s)}$ and $[U^{(\Lambda,s)}]_{(\Lambda,s)}$ is $r(\Lambda, s)$-open. By Theorem 30, $f^{-1}(U) \subseteq f^{-1}([U^{(\Lambda,s)}]_{(\Lambda,s)}) = [f^{-1}([U^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)}$.

(4) ⇒ (1): Let U be any $r(\Lambda, s)$-open set of Y. Then, we have U is $p(\Lambda, s)$-open and by (4), $f^{-1}(U) \subseteq [f^{-1}([U^{(\Lambda,s)}]_{(\Lambda,s)})]_{(\Lambda,s)} = [f^{-1}(U)]_{(\Lambda,s)}$. Thus, $f^{-1}(U)$ is (Λ, s)-open and by Theorem 30, f is almost (Λ, s)-continuous.

8. Conclusion

The notions of closed sets and low separation axioms are fundamental with respect to the investigation of topological spaces. Various types of generalizations of closed sets and some new separation axioms have been researched by many mathematicians. Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity in topological spaces. Using different forms of open sets, several authors have introduced and studied various types of weak forms of continuity. This work is concerned with the concepts of (Λ, s)-closed sets. Moreover, some properties of generalized (Λ, s)-closed sets are obtained. Several characterizations of some low separation axioms are established. Characterizations of (Λ, s)-extremally disconnected are obtained. Furthermore, some characterizations of almost (Λ, s)-continuous functions are explored. The ideas and results of this work may motivate further research.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] D. Andrijević. Semi-preopen sets. *Matematički Vesnik*, 38:24–32, 1986.

[2] A. V. Arhangel'skiĭ and P. J. Collins. On submaximal spaces. *Topology and its Applicaions*, 64:219–241, 1995.

[3] C. Boonpok and J. Khampakdee. (Λ, sp)-open sets in topological spaces. *European Journal of Pure and Applied Mathematics*, 15(2):572–588, 2022.

[4] S. Buadong, C. Viriyapong, and C. Boonpok. On generalized topology and minimal structure spaces. *International Journal of Mathematical Analysis*, 5(31):1507–1516, 2011.

[5] M. Caldas and J. Dontchev. G.Λ_s-sets and g.V_s-sets. *arXiv:math/9810080v1 [math:GN]*, 1998.

[6] D. Cameron. Properties of S-closed spaces. *Proceedings of the University of Oklahoma topology conference, Oklahoma*, 1978.

[7] F. Cammaroto and T. Noiri. On Λ_m-sets and related topological spaces. *Acta Mathematica Hungarica*, 109:261–279, 2005.

[8] A. S. Davis. Indexed systems of neighborhoods for general topological spaces. *The American Mathematical Monthly*, 68:886–893, 1961.

[9] C. Dorsett. R_0 and R_1 topological spaces. *Matematički Vesnik*, 2(15)(30):117–122, 1978.

[10] K. K. Dube. A note on R_0 topological spaces. *Matematički Vesnik*, 11:203–208, 1974.

[11] W. Dungthaisong, C. Boonpok, and C. Viriyapong. Generalized closed sets in bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 5(24):1175–1184, 2011.

[12] W. Dunham. $T_{\frac{1}{2}}$-spaces. *Kyungpook Mathematical Journal*, 17:161–169, 1977.

[13] W. Dunham and N. Levine. Further results on generalized closed sets in topology. *Kyungpook Mathematical Journal*, 20:169–175, 1980.

[14] L. Gillman and M. Jerison. *Rings of continuous functions, The University Series in Higher Mathematics*. Van Nostrand, Princeton, New York, 1960.

[15] R. A. Herrman. RC-convergence. *Proceedings of the American Mathematical Soceity*, 75:311–317, 1979.

[16] E. Hewitt. A problem of set-theoretic topology. *Duke Mathematical Journal*, 10:309–333, 1943.

[17] N. Levine. Semi-open sets and semi-continuity in topological spaces. *The American Mathematical Monthly*, 70:36–41, 1963.

[18] N. Levine. Generalized closed sets in topology. *Rendiconti del Circolo Matematico de Palermo (2)*, 19:89–96, 1970.

[19] S. Lugojan. Generalized topology. *Studii şi Cercetări de Matematică*, 34:348–360, 1982.

[20] S. A. Naimpally. On R_0-topological spaces. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, 10:53–54, 1967.

[21] T. Noiri. On S-closed subspaces. *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, 64:273–285, 1978.

[22] T. Noiri. A note on extremally disconnected spaces. *Proceedings of the American Mathematical Society*, 79(2):327–330, 1980.

[23] T. Noiri. Characterizations of extremally disconnected spaces. *Indian Journal of Pure and Applied Mathematics*, 19:325–329, 1988.

[24] N. A. Shanin. On separation in topological spaces. *Doklady Akademii Nauk SSSR*, 38:110–113, 1943.

[25] D. Sivaraj. A note on extremally disconnected spaces. *Indian Journal of Pure and Applied Mathematics*, 17(12):1373–1375, 1986.

[26] T. Thompson. S-closed spaces. *Proceedings of the American Mathematical Society*, 60:335–338, 1976.

[27] P. Torton, C. Viriyapong, and C. Boonpok. Some separation axioms in bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 6(56):2789–2796, 2012.

[28] E. K. van Douwen. Applications of maximal topologies. *Topology and its Applications*, 51:125–139, 1993.