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On 2-Resolving Hop Dominating Sets in the Join, Corona and Lexicographic Product of Graphs

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Abstract. Let G be a connected graph. A set S of vertices in G is a 2-resolving hop dominating set of G if S is a 2-resolving set in G and for every vertex $x \in V(G)\backslash S$, there exists $y \in S$ such that $d_G(x, y) = 2$. The minimum cardinality of a set S is called the 2-resolving hop domination number of G and is denoted by $\gamma_{2Rh}(G)$. This study aims to combine the concept of hop domination with the 2-resolving sets of graphs. The main results generated in this study include the characterization of 2-resolving hop dominating sets in the join, corona and lexicographic product of two graphs, as well as their corresponding bounds or exact values.

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1. Introduction

The concept of domination in graphs is one of the most studied problems and one of the fastest growing areas in graph theory. This was formally studied by Claude Berge [1] in 1958 and Oystein Ore in 1962. In 2015, Natarajan and Ayyaswamy introduced and studied the concept of hop domination [13].

On the other hand, in 1975 using the term locating set, the concept of resolving sets for a connected graph was first introduced by Slater [15]. These concepts were studied much earlier in the context of the coin-weighing problem. Later that year, Harary and Melter introduced independently these concepts, but with different terminologies [10]. The term metric dimension was used by Harary and Melter instead of locating number.

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Moreover, in the paper of Brigham et al. [9], the concept of resolving dominating in graphs was studied wherein they defined it as a set that is both resolving and dominating, and determined the resolving domination number $\gamma_R(G)$ of a graph G [9]. In 2021, the concept of resolving hop domination in graphs was studied and published by Mohamad and Rara [11] wherein they characterized the resolving hop dominating sets in some binary operations namely, the join and corona of two graphs and determined the bounds or exact values of the resolving hop domination number of these mentioned graphs [11].

Also, in the same year, Cabaro and Rara published the concept of 2-resolving sets in the join and corona of graphs [4]. Their paper also presents some characterizations involving this concept and investigates the 2-resolving dominating sets in the join, corona and lexicorgraphic product of two graphs [7].

Other variations of 2-resolving sets in graphs were also studied in [4–6, 8].

2. Terminology and Notation

In this study, we consider finite, simple, connected, undirected graphs. For basic graph-theoretic concepts, we then refer readers to [2] and [3]. The following concepts are found in [2], [13], and [14], respectively.

Let G be a connected graph. A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G(u,2) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u,2] = N_G(u,2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X,2) = \bigcup_{u\in X} N_G(u,2)$. The *closed hop neighborhood* of X in G is the set $N_G[X,2] = N_G(X,2) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G[S,2] = V(G)$, that is, for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

For an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, we refer to the k-vector (ordered k-tuple)

$$r_G(v/W) = (d_G(v,w_1), d_G(v,w_2), ..., d_G(v,w_k))$$

as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r_G(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G. An ordered set of vertices $W = \{w_1, ..., w_k\}$ is a *k-resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r_G(u/W)$ and $r_G(v/W)$ of u and v, respectively, differ in at least k positions. If $k = 1$, then the k-resolving set is called a *resolving set* for G. If $k = 2$, then the k-resolving set is called a 2-resolving set for G. If G has a k-resolving set, the minimum cardinality $\dim_k(G)$ of a k-resolving set is called the *k-metric dimension* of G.

A 2-resolving set $S \subseteq V(G)$ which is hop dominating is called a 2-*resolving hop dominating set* or simply 2R-hop dominating set in G. The minimum cardinality of a 2-resolving hop dominating set in G, denoted by $\gamma_{2Rh}(G)$, is called the 2*R-hop domination number* of G. Any 2R-hop dominating set of cardinality $\gamma_{2Rh}(G)$ is then referred to as a γ_{2Rh}-set in G.

3. Preliminary Results

Definition 1. [6] Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subseteq V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $\left|[\big(N_G(x)\backslash N_G(y)\big)\cap S]\cup[\big(N_G(y)\backslash N_G(x)\big)\cap S]\right| \geq 2$, for all $x, y \in V(G)\backslash S$ with $x \neq y$.

(ii) $\big(N_G(v)\backslash N_G(w)\big) \cap S \neq \varnothing$ or $\big(N_G(w)\backslash N_G[v]\big) \cap S \neq \varnothing$, for all $v \in S$ and for all $w \in V(G)\backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Definition 2. [12] A set $D \subseteq V(G)$ is a point-wise non-dominating set of G if for each $v \in V(G)\backslash D$, there exists $u \in D$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G, denoted by $pnd(G)$, is called the point-wise non-domination number of G. Any point-wise non-dominating set D of G with $|D| = pnd(G)$, is called a pnd-set of$G$. A dominating set D which is also a point-wise non-dominating set of G is called a dominating point-wise non-dominating set of G. The smallest cardinality of a dominating point-wise non-dominating set of G will be denoted by $\gamma_{pnd}(G)$. Any dominating point-wise non-dominating set D of G with $|D| = \gamma_{pnd}(G)$, is called a γ_{pnd}-set of G.

Definition 3. A 2-locating set $S \subseteq V(G)$ which is point-wise non-dominating is called a 2-*locating point-wise non-dominating set* in G. The minimum cardinality of a 2-locating point-wise non-dominating set in G, denoted by $ln_2^{pnd}(G)$ is called the 2-*locating point-wise non-domination number* of G. Any 2-locating point-wise non-dominating set of cardinality $ln_2^{pnd}(G)$ is then referred to as a $ln_2^{pnd}(G)$-set in G.

Example 1. For any graph G,

(i) $ln_2^{pnd}(K_n) = n$;

(ii) $ln_2^{pnd}(K_{1,n}) = n + 1$;

(iii) $ln_2^{pnd}(K_{m,n}) = m + n$;

(iv) $ln_2^{pnd}(P_n) = \begin{cases} 3, & n = 3 \\ \lceil \frac{n+1}{2} \rceil, & n \geq 4; \end{cases}$

$$(v)\;\; ln_2^{pnd}(C_n) = \begin{cases} 3, & n = 3 \\ 4, & n = 4 \\ \lceil \frac{n}{2} \rceil, & n \geq 5. \end{cases}$$

Remark 1. [7] Every 2-locating set in G is a 2-resolving set in G. However, a 2-resolving set in G need not be a 2-locating set in G. Thus, $dim_2(G) \leq ln_2(G)$.

Example 2. *For a path graph P_6,*

$$dim_2(P_6) = 2 < 4 = ln_2(P_6).$$

Definition 4. [6] Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

Definition 5. A (2,2)-locating ((2,1)-locating, respectively) set $S \subseteq V(G)$ which is a point-wise non-dominating is called a (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) *set* in G. The minimum cardinality of a (2,2)-locating point-wise non-dominating ((2,1)-locating point-wise non-dominating, respectively) set in G, denoted by $ln_{(2,2)}^{pnd}(G)$ ($ln_{(2,1)}^{pnd}(G)$,respectively) is called the (2,2)-*locating point-wise non-domination* ((2,1)-*locating point-wise non-domination*) *number* of G. Any (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) set of cardinality $ln_{(2,2)}^{pnd}(G)$ ($ln_{(2,1)}^{pnd}(G)$, respectively) is then referred to as a $ln_{(2,2)}^{pnd}$-set ($ln_{(2,1)}^{pnd}$-set) in G.

Example 3. (i) For all $n \geq 5$, $ln_{(2,2)}^{pnd}(P_n) = \begin{cases} 5, & n = 5 \\ \frac{n}{2} + 1, & n \text{ is even} \\ \lceil \frac{n}{2} \rceil, & n \text{ is odd.} \end{cases}$

(ii) For all $n \geq 7$, $ln_{(2,2)}^{pnd}(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \lceil \frac{n}{2} \rceil, & n \text{ is odd.} \end{cases}$

Example 4. (i) For all $n \geq 4$, $ln_{(2,1)}^{pnd}(P_n) = \begin{cases} \frac{n}{2} + 1, & n \text{ is even} \\ \lceil \frac{n}{2} \rceil, & n \text{ is odd.} \end{cases}$

(ii) For all $n \geq 5$, $ln_{(2,1)}^{pnd}(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \lceil \frac{n}{2} \rceil, & n \text{ is odd.} \end{cases}$

Definition 6. A 2-resolving set $S \subseteq V(G)$ which is point-wise non-dominating is called a 2-*resolving point-wise non-dominating set* in G. The minimum cardinality of a 2-resolving point-wise non-dominating set in G, denoted by $dim_{2_{pnd}}(G)$ is called the 2-*resolving point-wise non-domination number* of G. Any 2R-point-wise non-dominating set of cardinality $dim_{2_{pnd}}(G)$ is then referred to as a $dim_{2_{pnd}}(G)$-set in G.

Example 5. For any graph G,

(i) $dim_{2_{pnd}}(K_n) = n$;

(ii) $dim_{2_{pnd}}(K_{m,n}) = m + n \quad \text{where } m, n \geq 1$;

(iii) $dim_{2_{pnd}}(P_n) = \begin{cases} 3, & n = 3 \\ 2, & n \geq 4. \end{cases}$

Remark 2. Every $(2,1)$-locating point-wise non-dominating set in G is a 2-resolving point-wise non-dominating set in G. However, a 2-resolving point-wise non-dominating set in G need not be a $(2,1)$-locating point-wise non-dominating set in G. Thus,

$$dim_{2_{pnd}}(G) \leq ln^{pnd}_{(2,1)}(G).$$

Example 6. For a path graph P_4,

$$\dim_{2_{pnd}}(P_4) = 2 < 3 = ln^{pnd}_{(2,1)}(P_4).$$

Remark 3. Every nontrivial connected graph G admits a 2-resolving hop dominating set. Indeed, the vertex set $V(G)$ of G is a 2-resolving hop dominating set.

Remark 4. [4] Let $S \subseteq G$. Since any pair of vertices u and v where u and v are in the 2-resolving set S of a graph G, $r_G(u/S)$ and $r_G(v/S)$ differ in at least 2 positions. Hence, to show that S is a 2-resolving set of G, we only need to show that every pair u, v of distinct vertices in $V(G)\backslash S$ or $u \in S$ and $v \in V(G)\backslash S$, $r_G(u/S)$ and $r_G(v/S)$ differ in at least 2 positions.

Theorem 1. Let G be any nontrivial connected graph. Then $S \subseteq V(G)$ is a 2-resolving hop dominating set in G if it satisfies the following conditions:

(i) For every pair of vertices $x, y \in V(G)$ where $x \in S$ and $y \in V(G)\backslash S$ or both $x, y \in V(G)\backslash S, r_G(x/S)$ and $r_G(y/S)$ differ in at least 2 positions;

(ii) S is a point-wise non-dominating set in G.

Proof. Suppose $S \subseteq V(G)$ is a 2-resolving hop dominating set in G. Then by Remark 4, (i) holds.

To prove (ii), let $v \in V(G)\backslash S$. Since S is a hop dominating set, there exists $z \in S$ such that $d_G(v, z) = 2$. Hence, $v \notin N_G(z)$. This shows that S is a point-wise non-dominating set of G. Thus, (ii) holds. $\square$

Remark 5. Every 2-resolving hop dominating set in G is a 2-resolving point-wise non-dominating set in G. However, a 2-resolving point-wise non-dominating set in G need not be a 2-resolving hop dominating set in G. Thus, $dim_{2_{pnd}}(G) \leq \gamma_{2Rh}(G)$.

Remark 6. Let G be a connected graph. Then every 2-resolving hop dominating set in G is a 2-resolving set in G. However, a 2-resolving set in G need not be a 2-resolving hop dominating set in G. Thus, $dim_2(G) \leq \gamma_{2Rh}(G)$.

Proposition 1. For a path P_n on n vertices

$$\gamma_{2Rh}(P_n) = \begin{cases} 2, & \text{if } n = 2, 4; \\ 3, & \text{if } n = 3, 5, 6; \\ 2s, & \text{if } n = 6s,\ s \geq 2; \\ 2s+1, & \text{if } n = 6s+1,\ s \geq 1; \\ 2s+2, & \text{if } n = 6s+x;\ 2 \leq x \leq 5, s \geq 1. \end{cases}$$

Proposition 2. For a cycle C_n on n vertices

$$\gamma_{2Rh}(C_n) = \begin{cases} 3, & \text{if } n = 3, 5, 6; \\ 2s & \text{if } n = 6s,\ s \geq 2; \\ 2s+1, & \text{if } n = 6s+1,\ s \geq 1; \\ 2s+2, & \text{if } n = 6s+x;\ 2 \leq x \leq 5, s \geq 1. \end{cases}$$

Proposition 3. Let G be a connected graph of order 4. Then $\gamma_{2Rh}(P_4) = 2$ and $\gamma_{2Rh}(C_4) = 4$, respectively.

Theorem 2. *For a complete graph K_n on n vertices, $\gamma_{2Rh}(K_n) = n$.*

Proof. Suppose that $\gamma_{2Rh}(K_n) < n$. Let S be the minimum 2-resolving hop dominating set of K_n. Let $x \in V(K_n) \backslash S$ and $y \in S$. Then x and y differ at exactly one position, that is a contradiction to the assumption that S is a 2-resolving hop dominating set of K_n. Therefore, $\gamma_{2Rh}(K_n) = n$. □

Theorem 3. A set $S \subseteq V(K_{m,n})$ is a 2-resolving hop dominating set in $K_{m,n}$ if and only if $S = V(K_{m,n})$.

Proof. Let S be a subset of $V(K_{m,n})$. Let U and V be partite sets with $|U| = m$ and $|V| = n$; $m, n \geq 1$. Let $U = \{u_1, u_2, \ldots u_m\}$ and $V = \{v_1, v_2, \ldots v_n\}$. Suppose there exists $u_k \in U \backslash S$. Then $\forall\ i \in \{1, \ldots, m\} \backslash \{k\}$, $r_{K_{m,n}}(u_i/S)$ and $r_{K_{m,n}}(u_k/S)$ differ in at most one position. Similarly, suppose there exists $v_k \in V \backslash S\ \forall\ j \in \{1, \ldots, n\} \backslash \{k\}$, $r_{K_{m,n}}(v_j/S)$ and $r_{K_{m,n}}(v_k/S)$ differ in at most one position. Thus, it follows that $S = V(K_{m,n})$.

The converse is clear. □

Corollary 1. A set $S \subseteq V(K_{1,n})$ is a 2-resolving hop dominating set in $K_{1,n}$ if and only if $S = V(K_{1,n})$.

Corollary 2. *For a complete bipartite graph $K_{m,n}$,*

$$\gamma_{2Rh}(K_{m,n}) = m + n.$$

4. 2-Resolving Hop Dominating Sets in the Join of Graphs

Definition 7. [2] The *join* $G + H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G)\dot{\cup}V(H)$ and edge set

$$E(G+H) = E(G)\dot{\cup}E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Note that the star $K_{1,n}$ can be expressed as the join of the trivial graph K_1 and the empty graph $\overline{K}_n$ of order n, that is, $K_{1,n} = K_1 + \overline{K}_n$. The graphs $F_n = K_1 + P_n$ and $W_n = K_1 + C_n$ of orders $n+1$ are called *fan* and *wheel*, respectively.

Theorem 4. [4] Let G be a connected graph of order greater than 3 and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set in $K_1 + G$ if and only if either $v \notin S$ and S is a $(2,2)$-locating set in G or $S = \{v\} \cup T$ where T is a $(2,1)$-locating set in G.

Theorem 5. Let G be a connected graph and let $K_1 = \{x\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving hop dominating set in $K_1 + G$ if and only if $S = \{x\} \cup T$ where T is a $(2,1)$-locating point-wise non-dominating set in G.

Proof. Let $S \subseteq V(K_1 + G)$ be a 2-resolving hop dominating set in $K_1 + G$. Since S is a hop dominating set, $x \in S$. Hence, $S = \{x\} \cup T$ for $T \subseteq V(G)$. By Theorem 4, T is a $(2,1)$-locating set in G. Now, since S is a hop dominating set $\forall\ v \in V(G)\backslash T$, there exists $z \in T$ such that $d_{K_1+G}(v,z) = 2$. This follows that $v \notin N_G(z)$. Thus, T is a point-wise non-dominating set in G. Therefore, T is a $(2,1)$-locating point-wise non-dominating set in G.

Conversely, suppose $S = \{x\} \cup T$ where T is a $(2,1)$-locating point-wise non-dominating set in G. Then by Theorem 4, S is a 2-resolving set in K_1+G. Since T is a point-wise non-dominating set in G, $\forall\ y \in V(G)\backslash T$, there exists $v \in T$ such that $vy \notin E(G)$. This follows that $d_{K_1+G}(v,y) = 2$. Thus, S is a hop dominating set. Therefore, S is a 2-resolving hop dominating set in $K_1 + G$. □

Corollary 3. Let G be connected nontrivial graph. Then $\gamma_{2Rh}(K_1 + G) = ln^{pnd}_{(2,1)}(G) + 1$.

Proof. Suppose S is a γ_{2Rh}-set in $K_1 + G$. Then by Theorem 5, $S = \{x\} \cup T$ where T is a $(2,1)$- locating point-wise non-dominating set in G. Thus,

$$\gamma_{2Rh}(K_1 + G) = |S| = |T| + 1 \geq ln^{pnd}_{(2,1)}(G) + 1.$$

On the other hand, let T' be a $ln^{pnd}_{(2,1)}$-set of G. Then by Theorem 5, $S = \{x\} \cup T'$ is a 2-resolving hop dominating set of $K_1 + G$. Thus,

$$\gamma_{2Rh}(K_1 + G) \leq |S| = |T'| + 1 = ln^{pnd}_{(2,1)}(G) + 1.$$

Therefore, $\gamma_{2Rh}(K_1 + G) = ln^{pnd}_{(2,1)}(G) + 1$. □

Example 7. For a fan $F_n = P_n + K_1$ on $n+1$ $(n \geq 3)$ vertices

$$\gamma_{2Rh}(F_n) = ln^{pnd}_{(2,1)}(P_n) + 1 = \begin{cases} 4, & \text{if } n = 3 \\ \frac{n}{2} + 2, & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Example 8. For a wheel $W_n = C_n + 1$ on $n+1$ $(n \geq 3)$ vertices

$$\gamma_{2Rh}(W_n) = ln^{pnd}_{(2,1)}(C_n) + 1 = \begin{cases} 4, & \text{if } n = 3 \\ 5, & \text{if } n = 4 \\ \frac{n}{2} + 1, & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 6. [4] Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G+H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H , respectively, where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets.

Theorem 7. [12] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are point-wise non-dominating sets of G and H, respectively.

Theorem 8. Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a 2-resolving hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H $(2,1)$-locating point-wise non-dominating sets.

Proof. Suppose that S is a 2-resolving hop dominating set of $G+H$. Then S is a 2-resolving set of $G+H$. By Theorem 6, $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets. Also, since S is a hop dominating set in $G+H$, it follows by Theorem 7 that S_G and S_H are point-wise non-dominating sets of G and H, respectively. Therefore, S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets of G and H.

Conversely, suppose S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets of G and H, respectively. Then by Theorem 6, S is a 2-resolving set in $G+H$. Similarly, by Theorem 7 it follows that S is a hop dominating set of $G+H$. Therefore, S is a 2-resolving hop dominating set in $G+H$. □

Corollary 4. Let G and H be connected nontrivial graphs. Then,

$$\gamma_{2Rh}(G+H) = \min\{ln^{pnd}_{(2,2)}(G) + ln^{pnd}_{(2)}(H), ln^{pnd}_{(2)}(G) ln^{pnd}_{(2,2)}(H), ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)\}.$$

Proof. Let S be a minimum 2-resolving hop dominating set in $G+H$. Since $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. By Theorem 8, S_G and S_H are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets. If S_G is $(2,2)$-locating point-wise non-dominating set in G, then

$$ln^{pnd}_{(2,2)}(G) + ln^{pnd}_2(H) \leq |S_G| + |S_H| = |S| = \gamma_{2Rh}(G+H).$$

If S_H is a $(2,2)$-locating point-wise non-dominating set in H, then

$$ln^{pnd}_{(2,2)}(H) + ln^{pnd}_2(G) \leq |S_H| + |S_G| = |S| = \gamma_{2Rh}(G+H).$$

If S_G and S_H are both $(2,1)$-locating point-wise non-dominating sets, then

$$ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H) \leq |S_G| + |S_H| = |S| = \gamma_{2Rh}(G+H).$$

Thus,

$$\gamma_{2Rh}(G+H) \geq \min\{ln^{pnd}_{(2,2)}(G) + ln^{pnd}_{(2)}(H), ln^{pnd}_{(2)}(G) ln^{pnd}_{(2,2)}(H), ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)\}.$$

Next, suppose that $ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H) \leq ln^{pnd}_{(2,2)}(G) + ln^{pnd}_2(H)$ and $ln^{pnd}_{2,1}(G) + ln^{pnd}_{(2,1)}(H) \leq ln^{pnd}_2(G) + ln^{pnd}_{(2,2)}(H)$. Let S_G be a minimum $(2,1)$-locating point-wise non-dominating set in G and S_H be a minimum $(2,1)$-locating point-wise non-dominating set in H. Then $S = S_G \cup S_H$ is a 2-resolving hop dominating set in $G+H$, by Theorem 8. Hence, $\gamma_{2Rh}(G+H) \leq |S| = |S_G| + |S_H| = ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)$. Therefore,

$$\gamma_{2Rh}(G+H) \leq ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H).$$

Similarly, if $ln^{pnd}_{(2,2)}(G) + ln^{pnd}_2(H) \leq ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)$ and $ln^{pnd}_{(2,2)}(G) + ln^{pnd}_2(H) \leq ln^{pnd}_2(G) + ln^{pnd}_{(2,2)}(H)$, then

$$\gamma_{2Rh}(G+H) \leq ln^{pnd}_{(2,2)}(G) + ln^{pnd}_2(H).$$

Also, if $ln^{pnd}_2(G) + ln^{pnd}_{(2,2)}(H) \leq ln_{(2,2)}(^{pnd}G) + ln^{pnd}_2(H)$ and
$ln^{pnd}_2(G) + ln^{pnd}_{(2,2)}(H) \leq ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)$, then

$$\gamma_{2Rh}(G+H) \leq ln^{pnd}_2(G) + ln^{pnd}_{(2,2)}(H).$$

Thus,

$$\gamma_{2Rh}(G+H) \leq \min\{ln^{pnd}_{(2,2)}(G) + ln^{pnd}_{(2)}(H), ln^{pnd}_{(2)}(G) ln^{pnd}_{(2,2)}(H), ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)\}.$$

Therefore,

$$\gamma_{2Rh}(G+H) = \min\left\{ln^{pnd}_{(2,2)}(G) + ln^{pnd}_{2}(H), ln^{pnd}_{2}(G) + ln^{pnd}_{(2,2)}(H), ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)\right\}.$$

□

Example 9. For paths P_n and P_m on n and m vertices ($n, m \geq 4$), we have

$$\gamma_{2Rh}(P_n + P_m) = \begin{cases} \left(\frac{n}{2}+1\right) + \left(\frac{m}{2}+1\right), & \text{if } n, m \text{ are even} \\ \left(\frac{n}{2}+1\right) + \lceil\frac{m}{2}\rceil, & \text{if } n \text{ is even}, m \text{ is odd} \\ \lceil\frac{n}{2}\rceil + \left(\frac{m}{2}+1\right), & \text{if } n \text{ is odd}, m \text{ is even} \\ \lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil, & \text{if } n, m \text{ are odd.} \end{cases}$$

In particular, for $n, m = 2, 3$,

$$\gamma_{2Rh}(P_n + P_m) = n + m.$$

4.1. 2-Resolving Hop Dominating Sets in the Corona of Graphs

Let the following notations be described as follows:
For $k \in V(G \circ H)$, $v \in V(G)$ and $S_v \subset V(H^v)$.

$V^k_{S_v}$ is a vector whose components are the distances of k from the elements of S_v.

Definition 8. [2] The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v$, $v \in V(G)$.

Remark 7. [4] Let $v \in V(G)$. For every $x, y \in V(H^v), d_{G\circ H}(x, w) = d_{G\circ H}(y, w)$ and $d_{G\circ H}(v, w) + 1 = d_{G\circ H}(x, w)$ for every $w \in V(G \circ H)\backslash V(H^v)$.

Theorem 9. [12] Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if

$$C = A \cup \left(\bigcup_{v\in V(G)\cap N_G(A)} S_v\right) \cup \left(\bigcup_{w\in V(G)\backslash N_G(A)} E_w\right)$$

where

(i) $A \subseteq V(G)$ such that for each $w \in V(G)\backslash A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H_y) \cap C \neq \varnothing$,

(ii) $S_v \subseteq V(H^v)$ for each $v \in V(G) \cap N_G(A)$, and

(iii) $E_w \subseteq V(H^w)$ is a point-wise non-dominating set of H^w for each $w \in V(G)\backslash N_G(A)$.

Theorem 10. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving hop dominating set of $G \circ H$ if and only if

$$S = A \cup \left(\bigcup_{v \in V(G) \cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in V(G) \backslash N_G(A)} D_w \right)$$

where

- (i) $A \subseteq V(G)$ such that for each $w \in V(G) \backslash A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap S \neq \varnothing$;
- (ii) $S_v \subseteq V(H^v)$ is a 2-locating set of H^v for all $v \in V(G) \cap N_G(A)$; and
- (iii) $D_w \subseteq V(H^w)$ is a 2-locating point-wise non-dominating set of H^w for all $w \in V(G) \backslash N_G(A)$.

Proof. Suppose that $S \subseteq V(G \circ H)$ is a 2-resolving hop dominating set of $G \circ H$. Let $A = V(G) \cap S$, $S_v = S \cap V(H^v)$ for all $v \in V(G) \cap N_G(A)$ and $D_w = S \cap V(H^w)$ for all $w \in V(G) \backslash N_G(A)$. Then

$$S = A \cup \left(\bigcup_{v \in V(G) \cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in V(G) \backslash N_G(A)} D_w \right)$$

where $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ and $D_w \subseteq V(H^w)$. Now, since S is 2-resolving hop dominating set of $G \circ H$. By Theorem 9, (i) holds. Next, suppose that $S_v = \varnothing$ for some $v \in V(G) \cap N_G(A)$. Let $x, y \in V(H^v)$. Then $r_{G \circ H}(x/S) = r_{G \circ H}(y/S)$ which is a contradiction to the assumption of S. Thus, $S_v \neq \varnothing$. Similarly, $D_w \neq \varnothing$ for some $w \in V(G) \backslash N_G(A)$. Next, claim that S_v is a 2-locating set in H^v for each $v \in V(G) \cap N_G(A)$. Let $p, q \in V(H^v) \backslash S_v$ where $p \neq q$. Since S is a 2-resolving set in $G \circ H$, $r_{G \circ H}(p/S)$ and $r_{G \circ H}(q/S)$ differ in at least 2 positions and by Remark 7, $r_{H^v}(p/S_v)$ and $r_{H^v}(q/S_v)$ must differ in at least 2 positions. Thus, it follows that S_v is a 2-locating set of H^v and so (ii) holds. Similarly, D_w is a 2-locating set of H^w. Now, since S is a hop dominating set of $G \circ H$, Theorem 9 follows, showing D_w is a point-wise non-dominating set of H^w that is (iii) holds.

Conversely, let $S = A \cup \left(\bigcup_{v \in V(G) \cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in V(G) \backslash N_G(A)} D_w \right)$ where $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ and $D_w \subseteq V(H^w)$ satisfying the given conditions. Let $x \in V(G \circ H) \backslash S$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Suppose $x = v$. Then $x \notin A$. From the assumption that (i) holds, it follows that there exists $y \in S$ such that $d_{G \circ H}(x, y) = 2$. Next, suppose $x \neq v$. If $v \in N_G(A)$, then there exists a vertex say $z \in A \cap N_G(v)$ such that $z \in S$ and $d_{G \circ H}(x, z) = 2$. Suppose $v \notin N_G(A)$. Then $x \in V(H^v) \backslash D_v$. Since D_v is point-wise non-dominating by Theorem 9, it follows that there exists $y \in D_v$, that is $y \in S$ such that $d_{G \circ H}(x, y) = 2$. This shows that S is a hop dominating set of $G \circ H$. Since S_v or D_v is a

2-locating set of $G \circ H$, by our assumption. By Remark 1, it then follows that S_v or D_v is a 2-resolving set of $G \circ H$.

Accordingly, S is a 2-resolving hop dominating set of $G \circ H$. $\square$

Corollary 5. Let G be a nontrivial graph of order n and H be any graph. Then the following statements hold.

(*i*) $\gamma_{2Rh}(G \circ H) \leq n(1 + ln_2(H))$.

(*ii*) If $ln_2^{pnd}(H) = ln_2(H)$, then $\gamma_{2Rh}(G \circ H) = n(ln_2^{pnd}(H))$.

Proof. (*i*) Let $A = V(G)$ and let S_v be a 2-locating set of H^v for all $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ is a 2-resolving hop dominating set of $G \circ H$ by Theorem 10. Hence,

$$\gamma_{2Rh}(G \circ H) \leq |S| = |V(G)| + |V(G)||S_v| = n(1 + ln_2(H)).$$

(*ii*) Suppose that $ln_2^{pnd}(H) = ln_2(H)$. Now, set $A = \varnothing$ and let D_w be a 2-locating point-wise non-dominating set of H^w for all $w \in V(G)$. Then $S = A \cup \left(\bigcup_{w \in V(G)} D_w \right)$ is a 2-resolving hop dominating set by Theorem 10. Hence,

$$\gamma_{2Rh}(G \circ H) \leq |S| = |A| + |V(G)||D_w| = n(ln_2^{pnd}(H)).$$

Next,let $S_0 = A_0 \cup \left(\bigcup_{v \in V(G) \setminus C_0} S_v \right) \cup \left(\bigcup_{w \in C_0} D_w \right)$ be a 2-resolving hop dominating set of $G \circ H$. By Theorem 10, $A_0 \subseteq V(G)$ where $C_0 = \{x \in V(G) : x \notin N_G(A_0)\}$, S_v is a 2-locating set of H^v for all $v \in V(G) \setminus C_0$ and D_w is a 2-locating point-wise non-dominating set of H^w for all $w \in C_0$. Thus, we can write this as

$$\begin{aligned} \gamma_{2Rh}(G \circ H) &= |S| \qquad (1)\\ &= |A| + |V(G) \setminus C_0||S_v| + |C_0||D_w| \\ &\geq |V(G) \setminus C_0| \ln_2(H) + |C_0| \ln_2^{pnd}(H) \\ &= (|V(G)| - |C_0|) \ln_2(H) + |C_0| \ln_2^{pnd}(H) \\ &= (|V(G)| - |C_0|) \ln_2^{pnd}(H) + |C_0| \ln_2^{pnd}(H) \\ &= n \cdot \ln_2^{pnd}(H) \end{aligned}$$

Therefore, $\gamma_{2Rh}(G \circ H) = n(ln_2^{pnd}(H))$. $\square$

Example 10. (*i*) For $n = 4$ and $m = 3$,

$$\gamma_{2Rh}(P_4 \circ P_3) = 10 < 12 = 4(1 + 2).$$

(ii) For any integer $n \geq 2$ and $m \geq 4$,

$$\gamma_{2Rh}(G \circ P_m) = n \cdot ln_2^{pnd}(P_m) = n\Big(\Big\lceil \frac{m+1}{2} \Big\rceil\Big).$$

Example 11. For any integer $n \geq 2$ and $m \geq 5$,

$$\gamma_{2Rh}(G \circ C_m) = n \cdot ln_2^{pnd}(C_m) = n\Big(\Big\lceil \frac{m}{2} \Big\rceil\Big).$$

4.2. 2-Resolving Hop Dominating Sets in the Lexicographic Product of Graphs

Definition 9. [2] The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$.

Definition 10. [7] A vertex x is said to be 1-*equidistant* to y if $xy \in E(G)$ and $d_G(x,z) = d_G(y,z)$, for all $z \in V(G)\backslash\{x,y\}$ and it is 2-*equidistant* to y if $d_G(x,y) = 2$ and $d_G(x,z) = d_G(y,z)$, for all $z \in V(G)\backslash\{x,y\}$. A vertex is called a *free-vertex* in G if it is neither 1-equidistant nor 2-equidistant to any vertex. The set containing all 1-equidistant, 2-equidistant, and free-vertices in G are denoted by $EQ_1(G), EQ_2(G)$ and $fr(G)$, respectively.

Definition 11. [7] A graph is called *free-equidistant* if all of its vertices are free-vertices.

Remark 8. [16] Let G and H be two nontrivial graphs such that G is connected. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.

(i) $N_{G[H]}(a,b) = (\{a\} \times N_H\{b\}) \cup \{N_G\{a\} \times V(H)\}$

(ii) $d_{G[H]}((a,b),(c,d)) = d_G(a,c)$

(iii) $d_{G[H]}((a,b),(a,d)) = \min\{d_H(b,d), 2\}$.

Theorem 11. [7] Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving hop dominating set in $G[H]$ if and only if

(i) $S = V(G)$;

(ii) T_x is a 2-locating set in H for every $x \in V(G)$;

(iii) T_x or T_y is a $(2,1)$-locating set or one of T_x and T_y is a $(2,2)$-locating set in H whenever $x, y \in EQ_1(G)$; and

(iv) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

Theorem 12. [12] Let G and H be nontrivial connected graphs. A subset $C = \bigcup_{x\in S}[\{x\}\times T_x]$ of $V(G[H])$ is a hop dominating set of $G[H]$ if and only if the following conditions hold:

(i) S is a hop dominating set of G;

(ii) T_x is a point-wise non-dominating set of H for each $x \in S$ with $|N_G(x, 2) \cap S| = 0$.

Theorem 13. Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving hop dominating set in $G[H]$ if and only if

(i) S is a hop dominating set of G, that is $S = V(G)$;

(ii) T_x is a 2-locating set in H for every $x \in V(G)$;

(iii) T_x or T_y is a $(2, 1)$-locating set or one of T_x and T_y is a $(2, 2)$-locating set in H whenever $x, y \in EQ_1(G)$;

(iv) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$; and

(v) T_x is a 2-locating point-wise non-dominating set in H for every $x \in S$ with $|N_G(x, 2)\cap S| = 0$.

Proof. Suppose $W = \bigcup_{x\in S}[\{x\}\times T_x]$ is a 2-resolving hop dominating set of $G[H]$. Then W is a 2-resolving set. By Theorem 11, (i) to (iv) hold. Since W is a hop dominating set then by Theorem 12, it follows that (i) and (v) hold. Now, let $x \in V(G)$ and $p, q \in S\backslash N_G(S, 2)$ where $p \neq q$. Then $(x, p) \neq (x, q)$. If $p, q \notin T_x$ or $[p \in T_x$ and $q \notin T_x]$, then $(x, p), (x, q) \notin W$ or $[(x, p) \in W$ and $(x, q) \notin W]$. Since W is a 2-resolving hop dominating set in $G[H]$, $r_{G[H]}((x, p)/W)$ and $r_{G[H]}((x, q)/W)$ differ in at least 2 positions. Hence, by Remark 8 and Definition 1, T_x is a 2-locating set in H. Thus, (v) holds.

Conversely, suppose (i) to (iv) hold. By Theorem 11, W is a 2-resolving set. Also, since (i) and (v) hold. By Theorem 12, W is a hop dominating set.

Accordingly, W is 2-resolving hop dominating set of $G[H]$. □

The following corollaries are the direct consequences of Theorem 13.

Corollary 6. Let G and H be nontrivial connected graphs. Then,

$$\gamma_{2Rh}(G[H]) \leq n \cdot ln_{(2,1)}(H) + m \cdot \gamma_{2L}(H) + p \cdot ln_2^{pnd}(H),$$

where $n + m + p = |V(G)|$ with $|EQ_1(G)| = n, |EQ_2(G)| = m$ and $|fr(G)| = p$. In particular, if $n = m = 0$, then $\gamma_{2Rh}(G[H]) = |V(G)| \cdot ln_2^{pnd}(H)$.

Corollary 7 follows from Theorem 13.

Corollary 7. Let G and H be nontrivial connected graphs such that G is free-equidistant. Then

$$\gamma_{2Rh}(G[H]) = |V(G)| \cdot ln_2^{pnd}(H).$$

Proof. Since G is free-equidistant, then $|EQ_1(G)| = |EQ_2(G)| = 0$. Therefore, $\gamma_{2Rh}(G[H]) = |V(G)| \cdot ln_2^{pnd}(H)$. $\square$

Example 12. For any integer $n \geq 2$ and $m \geq 4$,

$$\gamma_{2Rh}(G[P_m]) = n \cdot ln_2^{pnd}(P_m) = n\left(\left\lceil \frac{m+1}{2} \right\rceil\right).$$

Example 13. For any integer $n \geq 2$ and $m \geq 5$,

$$\gamma_{2Rh}(G[C_m]) = n \cdot ln_2^{pnd}(C_m) = n\left(\left\lceil \frac{m}{2} \right\rceil\right).$$

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