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On the Transformations Preserving Asymptotic Directions of Hypersurfaces in the Euclidean Space

Yasemin Alagöz* and Ziya Soyuçok

Department of Mathematics, Faculty of Science and Letters, Yildiz Technical University, Istanbul, Turkey

Abstract. We consider the transformations preserving asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we obtain a system of equations which must be satisfied by transformations.

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1. Introduction

In the Euclidean space, the projective transformation preserves the asymptotic lines of a surface [3]. In [4] the inverse of that problem is considered and it is obtained that the most transformation preserving the asymptotic lines of surfaces in 3-dimensional Euclidean space is the projective one. But that paper has very long

*Corresponding author.

Email addresses: ygulluk@yildiz.edu.tr (Y. Alagöz), zsoyucok@yildiz.edu.tr (Z. Soyuçok)

calculations and it seems very difficult to generalize for the n-dimensional space by using given method. Moreover, since it has some errors that transformation is not the general projective transformation [1].

In this paper, we consider the transformations which preserve the asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we obtain a system of equations. The transformations must satisfy these equations system.

2. The Equation of the Asymptotic Directions of a Hypersurface

In the n-dimensional Euclidean space, a hypersurface can be expressed by the equation

$$\mathbf{r}\left(u^1,\ldots,u^{n-1}\right)=(x^1\left(u^1,\ldots,u^{n-1}\right),x^2\left(u^1,\ldots,u^{n-1}\right),\ldots,x^n\left(u^1,\ldots,u^{n-1}\right)) \quad (1)$$

where the metric of the space is given by

$$ds^2=(dx^1)^2+(dx^2)^2+\ldots+(dx^n)^2. \quad (2)$$

We assume that $\mathbf{r}\left(u^1,u^2,\ldots,u^{n-1}\right)$ is a differentiable function of order 3 and the tangent vectors $\mathbf{r}_{,1},\mathbf{r}_{,2},\ldots,\mathbf{r}_{,n-1}$ of the hypersurface are linearly independent where

$$\mathbf{r}_{,i}\equiv\frac{\partial\mathbf{r}}{\partial u^i}, \qquad (i=1,2,\ldots,n-1). \quad (3)$$

The first and second fundamental forms of the hypersurface are

$$I=g_{ij}du^idu^j, \qquad II=L_{ij}du^idu^j, \qquad \left(i,j=1,2,\ldots,n-1\right) \quad (4)$$

where

$$g_{ij}=\mathbf{r}_{,i}.\mathbf{r}_{,j} \quad (5)$$

$$L_{ij}=\mathbf{r}_{,ij}.\mathbf{N}, \qquad \left(\mathbf{r}_{,ij}\equiv\frac{\partial^2\mathbf{r}}{\partial u^i\partial u^j}\right). \quad (6)$$

Here **N** is the unit normal vector of the hypersurface, that is,

$$\mathbf{r}_{,i}.\mathbf{N} = 0 \tag{7}$$

and

$$\mathbf{N}.\mathbf{N} = 1. \tag{8}$$

The differential equation of the asymptotic directions of the hypersurface is given by

$$L_{ij}du^i du^j = 0 \tag{9}$$

[2, p.44] and [5, p.134].

The system (7) can be written as

$$\mathbf{A}\mathbf{N}^T = 0 \tag{10}$$

where

$$\mathbf{A} = \begin{bmatrix} x^1_{,1} & x^2_{,1} & \cdots & x^n_{,1} \\ x^1_{,2} & x^2_{,2} & \cdots & x^n_{,2} \\ \vdots & \vdots & \cdots & \vdots \\ x^1_{,n-1} & x^2_{,n-1} & \cdots & x^n_{,n-1} \end{bmatrix}, \qquad \left(x^k_{,i} = \frac{\partial x^k}{\partial u^i} \right) \tag{11}$$

and

$$\mathbf{N} = \left(N_1, N_2, \ldots, N_n \right). \tag{12}$$

Since the vectors

$$\mathbf{r}_{,i} = \left(x^1_{,i}, x^2_{,i} \ldots, x^n_{,i} \right), \qquad (i = 1, 2, \ldots, n-1) \tag{13}$$

are linearly independent, we can assume that

$$\Delta_n = \det \begin{bmatrix} x^1_{,1} & x^2_{,1} & \cdots & x^{n-1}_{,1} \\ x^1_{,2} & x^2_{,2} & \cdots & x^{n-1}_{,2} \\ \vdots & \vdots & \cdots & \vdots \\ x^1_{,n-1} & x^2_{,n-1} & \cdots & x^{n-1}_{,n-1} \end{bmatrix} \neq 0. \tag{14}$$

Then from (10) and (8) we have

$$\mathbf{N} = \frac{1}{k}\left(\Delta_1, -\Delta_2, \ldots, (-1)^{1+n}\Delta_n\right) \tag{15}$$

where Δ_i is the determinant of the matrix which is obtained by omitting i^{th} column in the coefficients matrix **A** and

$$k = \sqrt{\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2}. \tag{16}$$

Accordingly, from (6) we get

$$L_{ij} = \frac{1}{k}[x^1_{,ij}\Delta_1 - x^2_{,ij}\Delta_2 + \ldots + (-1)^{1+n}\, x^n_{,ij}\Delta_n] \tag{17}$$

and so

$$kL_{ij} = \det \begin{bmatrix} x^1_{,ij} & x^1_{,1} & x^1_{,2} & \cdots & x^1_{,n-1} \\ x^2_{,ij} & x^2_{,1} & x^2_{,2} & \cdots & x^2_{,n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x^n_{,ij} & x^n_{,1} & x^n_{,2} & \cdots & x^n_{,n-1} \end{bmatrix}, \qquad \left(x^k_{,ij} = \frac{\partial^2 x^k}{\partial u^i \partial u^j} \right). \tag{18}$$

Now for a hypersurface S let us choose the parameters as

$$u^1 = x^1, u^2 = x^2, \ldots, u^{n-1} = x^{n-1}. \tag{19}$$

Then, the equation of S becomes

$$\mathbf{r}(x^1, x^2, ..., x^{n-1}) = (x^1, x^2, ..., x^{n-1}, x^n(x^1, ..., x^{n-1})) \tag{20}$$

and from (18) we get

$$kL_{ij} = (-1)^{n+1}\, x^n_{,ij}. \tag{21}$$

See also [2, p.36].

The differential equation of the asymptotic directions of S, from (9), is obtained as

$$x^n_{,ij} dx^i dx^j = 0 \qquad (i, j = 1, 2, \ldots, n-1). \tag{22}$$

3. Conditions for a Transformation Preserving the Asymptotic Directions

Here we determine transformations preserving the asymptotic directions of a hypersurface. In the n-dimensional Euclidean space let us consider the coordinate transformation

$$\mathbf{T}: y^a = y^a\left(x^1, x^2, \ldots, x^n\right), \quad (a = 1, 2, \ldots, n). \tag{23}$$

We assume that $\mathbf{T}$ is differentiable of order 3 and

$$\Delta = \det\begin{bmatrix} \mathbf{T}_{,1} & \mathbf{T}_{,2} & \cdots & \mathbf{T}_{,n} \end{bmatrix} = \begin{vmatrix} \mathbf{T}_{,1} & \mathbf{T}_{,2} & \cdots & \mathbf{T}_{,n} \end{vmatrix} \neq 0 \tag{24}$$

where

$$\mathbf{T}_{,b} = \begin{bmatrix} y^1_{,b} \\ y^2_{,b} \\ \vdots \\ y^n_{,b} \end{bmatrix}, \qquad \left(y^a_{,b} = \frac{\partial y^a}{\partial x^b};\ b = 1, 2, \ldots, n\right). \tag{25}$$

If the transformation $\mathbf{T}$ is applied to the hypersurface S which is defined by the equation (20), then we get

$$\mathbf{T}^{'}: y^a = y^a\left(x^1, x^2, \ldots, x^{n-1}, x^n\left(x^1, x^2, \ldots, x^{n-1}\right)\right). \tag{26}$$

So the transformation $\mathbf{T}$ transforms the hypersurface S to a hypersurface S^* which is given by the equation

$$\mathbf{r}^*\left(x^1, x^2, \ldots, x^{n-1}\right) = \left(y^1, y^2, \ldots, y^n\right) \tag{27}$$

where

$$y^a = y^a(x^1, x^2, \ldots, x^{n-1}, x^n\left(x^1, x^2, \ldots, x^{n-1}\right)), \qquad (a = 1, \ldots, n).$$

For the hypersurface S^*,

$$k^* L^*_{ij} = \begin{vmatrix} \mathbf{T}^{'}_{,1} & \mathbf{T}'_{,2} & \cdots & \mathbf{T}'_{,n-1} & \mathbf{T}'_{,ij} \end{vmatrix} \tag{28}$$

is obtained from (18), where

$$\mathbf{T}'_{,i} = \mathbf{T}_{,i} + \mathbf{T}_{,n} x^n_{,i}, \quad \mathbf{T}'_{,ij} = \mathbf{T}_{,ij} + \mathbf{T}_{,in} x^n_{,j} + \mathbf{T}_{,nj} x^n_{,i} + \mathbf{T}_{,nn} x^n_{,i} x^n_{,j} + \mathbf{T}_{,n} x^n_{,ij}, \tag{29}$$

and

$$\mathbf{T}_{,ij} = \begin{bmatrix} y^1_{,ij} \\ y^2_{,ij} \\ \vdots \\ y^n_{,ij} \end{bmatrix}, \qquad \left(y^a_{,ij} = \frac{\partial^2 y^a}{\partial x^i \partial x^j}; \, i, j = 1, 2, \ldots, n-1 \right). \tag{30}$$

Using (29) and (30), from (28) we can write

$$\begin{aligned} k^* L^*_{ii} = & \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ii} \right| + \left| \mathbf{T}_{,n} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ii} \right| x^n_{,1} \\ & + \left| \mathbf{T}_{,1} \mathbf{T}_{,n} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ii} \right| x^n_{,2} + \ldots + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n} \mathbf{T}_{,n-1} \mathbf{T}_{,ii} \right| x^n_{,n-2} \\ & + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-2} \mathbf{T}_{,n} \mathbf{T}_{,ii} \right| x^n_{,n-1} + 2 \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,in} \right| x^n_{,i} \\ & + 2 \left| \mathbf{T}_{,n} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,in} \right| x^n_{,i} x^n_{,1} + 2 \left| \mathbf{T}_{,1} \mathbf{T}_{,n} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,in} \right| x^n_{,i} x^n_{,2} \\ & + \ldots + 2 \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n} \mathbf{T}_{,n-1} \mathbf{T}_{,in} \right| x^n_{,i} x^n_{,n-2} \\ & + 2 \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-2} \mathbf{T}_{,n} \mathbf{T}_{,in} \right| x^n_{,i} x^n_{,n-1} + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nn} \right| (x^n_{,i})^2 \\ & + \left| \mathbf{T}_{,n} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nn} \right| (x^n_{,i})^2 x^n_{,1} + \left| \mathbf{T}_{,1} \mathbf{T}_{,n} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nn} \right| (x^n_{,i})^2 x^n_{,2} \\ & + \ldots + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n} \mathbf{T}_{,n-1} \mathbf{T}_{,nn} \right| (x^n_{,i})^2 x^n_{,n-2} \\ & + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-2} \mathbf{T}_{,n} \mathbf{T}_{,nn} \right| (x^n_{,i})^2 x^n_{,n-1} + \Delta . x^n_{,ii} \end{aligned} \tag{31}$$

and

$$\begin{aligned} k^* L^*_{ij} = & \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ij} \right| + \left| \mathbf{T}_{,n} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ij} \right| x^n_{,1} \\ & + \left| \mathbf{T}_{,1} \mathbf{T}_{,n} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,ij} \right| x^n_{,2} + \ldots + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n} \mathbf{T}_{,n-1} \mathbf{T}_{,ij} \right| x^n_{,n-2} \\ & + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-2} \mathbf{T}_{,n} \mathbf{T}_{,ij} \right| x^n_{,n-1} + \left| \mathbf{T}_{,1} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nj} \right| x^n_{,i} \\ & + \left| \mathbf{T}_{,n} \mathbf{T}_{,2} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nj} \right| x^n_{,i} x^n_{,1} + \left| \mathbf{T}_{,1} \mathbf{T}_{,n} \ldots \mathbf{T}_{,n-1} \mathbf{T}_{,nj} \right| x^n_{,i} x^n_{,2} \end{aligned}$$

$$
\begin{aligned}
&+\ldots+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n}\mathbf{T}_{,n-1}\mathbf{T}_{,nj}\ \right| x^n_{,i}x^n_{,n-2} \\
&+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-2}\mathbf{T}_{,n}\mathbf{T}_{,nj}\ \right| x^n_{,i}x^n_{,n-1}+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,in}\ \right| x^n_{,j} \\
&+\left|\ \mathbf{T}_{,n}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,in}\ \right| x^n_{,j}x^n_{,1}+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,n}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,in}\ \right| x^n_{,j}x^n_{,2} \\
&+\ldots+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n}\mathbf{T}_{,n-1}\mathbf{T}_{,in}\ \right| x^n_{,j}x^n_{,n-2} \\
&+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-2}\mathbf{T}_{,n}\mathbf{T}_{,in}\ \right| x^n_{,j}x^n_{,n-1}+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,nn}\ \right| x^n_{,i}x^n_{,j} \\
&+\left|\ \mathbf{T}_{,n}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,nn}\ \right| x^n_{,i}x^n_{,j}x^n_{,1}+\left|\ \mathbf{T}_{,1}\mathbf{T}_{,n}\ldots\mathbf{T}_{,n-1}\mathbf{T}_{,nn}\ \right| x^n_{,i}x^n_{,j}x^n_{,2} \\
&+\ldots+\left|\mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n}\mathbf{T}_{,n-1}\mathbf{T}_{,nn}\right| x^n_{,i}x^n_{,j}x^n_{,n-2} \\
&+\left|\mathbf{T}_{,1}\mathbf{T}_{,2}\ldots\mathbf{T}_{,n-2}\mathbf{T}_{,n}\mathbf{T}_{,nn}\right| x^n_{,i}x^n_{,j}x^n_{,n-1}+\Delta.x^n_{,ij}. \qquad (32)
\end{aligned}
$$

The differential equation of the asymptotic directions of S^*, according to (9), is

$$L^*_{ij}dx^i dx^j = 0. \qquad (33)$$

In order that the transformation **T** transforms the asymptotic directions of the hypersurface S to the asymptotic directions of the hypersurface S^* it must transform the equation (22) to the equation (33). Accordingly, our conditions are

$$L^*_{ij} = t x^n_{,ij} \qquad (34)$$

where t is an arbitrary function of the variables $x^1, x^2, \ldots, x^{n-1}$.

The equations (31) and (32) can be written as follows:

$$
\begin{aligned}
k^*L^*_{ii} \ = \ & \Delta_0(ii)+\Delta_1(ii)x^n_{,1}+\Delta_2(ii)x^n_{,2}+\ldots+\Delta_{n-1}(ii)x^n_{,n-1} \\
&+2[\Delta_0(in)+\Delta_1(in)x^n_{,1}+\Delta_2(in)x^n_{,2}+\ldots+\Delta_{n-1}(in)x^n_{,n-1}]x^n_{,i} \\
&+[\Delta_0(nn)+\Delta_1(nn)x^n_{,1}+\Delta_2(nn)x^n_{,2}+\ldots+\Delta_{n-1}(nn)x^n_{,n-1}](x^n_{,i})^2 \\
&+\Delta.x^n_{,ii} \qquad (35)
\end{aligned}
$$

and

$$k^*L^*_{ij} \ = \ \Delta_0(ij)+\Delta_1(ij)x^n_{,1}+\Delta_2(ij)x^n_{,2}+\ldots+\Delta_{n-1}(ij)x^n_{,n-1}$$

$$+[\Delta_0(in)+\Delta_1(in)x^n_{,1}+\Delta_2(in)x^n_{,2}+...+\Delta_{n-1}(in)x^n_{,n-1}]x^n_{,j}$$
$$+[\Delta_0(jn)+\Delta_1(jn)x^n_{,1}+\Delta_2(jn)x^n_{,2}+...+\Delta_{n-1}(jn)x^n_{,n-1}]x^n_{,i}$$
$$+[\Delta_0(nn)+\Delta_1(nn)x^n_{,1}+\Delta_2(nn)x^n_{,2}+...+\Delta_{n-1}(nn)x^n_{,n-1}]x^n_{,i}x^n_{,j}$$
$$+\Delta.x^n_{,ij} \tag{36}$$

where $\Delta_0(ab)$ denotes the determinant which is obtained by replacing the nth column with $\mathbf{T}_{,ab}$ in the determinant Δ which is defined by (24), and $\Delta_k(ab)$ denotes the determinant which is obtained by replacing the nth column with $\mathbf{T}_{,ab}$ and kth column with $\mathbf{T}_{,n}$ in the determinant Δ. For example,

$$\Delta_2(44)=\left|\mathbf{T}_1\mathbf{T}_n\mathbf{T}_3\ldots\mathbf{T}_{n-1}\mathbf{T}_{44}\right|.$$

The equations (34) must be satisfied by any hypersurface. So, using the quantities given by (35) in (34) we have the following conditions:

$$\Delta_1(nn)=0,\Delta_2(nn)=0,\ldots,\Delta_{n-1}(nn)=0, \tag{37}$$

$$\Delta_0(ii)=0,\Delta_1(ii)=0,\ldots,\Delta_{i-1}(ii)=0,\Delta_{i+1}(ii)=0,\ldots,\Delta_{n-1}(ii)=0, \tag{38}$$

$$\Delta_1(in)=0,\Delta_2(in)=0,\ldots,\Delta_{i-1}(in)=0,\Delta_{i+1}(in)=0,\ldots,\Delta_{n-1}(in)=0, \tag{39}$$

$$\Delta_i(ii)+2\Delta_0(in)=0, \quad \Delta_0(nn)+2\Delta_i(in)=0. \tag{40}$$

From (37) and (38) we respectively get

$$\mathbf{T}_{,nn}=2A_n\mathbf{T}_{,n} \tag{41}$$

and

$$\mathbf{T}_{,ii}=2A_i\mathbf{T}_{,i} \tag{42}$$

and so

$$\mathbf{T}_{,bb}=2A_b\mathbf{T}_{,b} \tag{43}$$

where $A_1, A_2, \ldots, A_n$ are arbitrary functions of variables $x^1, x^2, \ldots, x^n$.
Using (43) in (39) and (40), we have

$$\mathbf{T}_{,in} = A_n \mathbf{T}_{,i} + A_i \mathbf{T}_{,n}. \tag{44}$$

Now let us use the quantities given by (36) in (34) which must be satisfied by any hypersurface. Then we have the following conditions:

$$\Delta_0(ij) = 0, \Delta_1(ij) = 0, ..., \Delta_{i-1}(ij) = 0, \Delta_{i+1}(ij) = 0, \tag{45}$$

$$..., \Delta_{j-1}(ij) = 0, \Delta_{j+1}(ij) = 0, ..., \Delta_{n-1}(ij) = 0$$

$$\Delta_i(ij) + \Delta_0(jn) = 0, \quad \Delta_j(ij) + \Delta_0(in) = 0, \tag{46}$$

$$\Delta_i(in) + \Delta_j(jn) + \Delta_0(nn) = 0 \tag{47}$$

and (37) and (39) again.
From (44) and (45), using (46) we get

$$\mathbf{T}_{,ij} = A_j \mathbf{T}_{,i} + A_i \mathbf{T}_{,j}. \tag{48}$$

The results (43), (44) and (48) can be expressed by a single equation as

$$\mathbf{T}_{,ab} = A_b \mathbf{T}_{,a} + A_a \mathbf{T}_{,b}, \qquad (a, b = 1, 2, \ldots, n). \tag{49}$$

(47) is automatically satisfied by these results.
From (37) to (40) and from (45) to (47) all equations are satisfied by (49). Thus we have the following theorem.

Theorem 1. *A transformation* **T** *which preserves the asymptotic directions of a hypersurface must satisfy the equations*

$$\mathbf{T}_{,ab} = A_b \mathbf{T}_{,a} + A_a \mathbf{T}_{,b}, \qquad (a, b = 1, 2, \ldots, n) \tag{50}$$

where $A_1, A_2, \ldots, A_n$ *are arbitrary functions of variables* $x^1, x^2, \ldots, x^n$.

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