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Regularity on variants of transformation semigroups that preserve an equivalence relation

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Abstract. The variant of a semigroup S with respect to an element $a \in S$, is the semigroup with underlying set S and a new binary operation $*$ defined by $x * y = xay$ for $x, y \in S$. Let $T(X)$ be the full transformation semigroup on a nonempty set X. For an arbitrary equivalence E on X, let

$$T_E(X) = \{\alpha \in T(X) : \forall a, b \in X, (a, b) \in E \Rightarrow (a\alpha, b\alpha) \in E\}.$$

Then $T_E(X)$ is a subsemigroup of $T(X)$. In this paper, we investigate regular, left regular and right regular elements for the variant of some subsemigroups of the semigroup $T_E(X)$.

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1. Introduction and preliminaries

Let S be a semigroup and a belong to S. We define a new binary operation $*$ on S by putting $x * y = xay$ for all $x, y \in S$. The operation $*$ is clearly associative. Hence $(S, *)$ is a semigroup and it is called a *variant* of S. We usually write (S, a) rather than $(S, *)$ to make the element explicit.

Variants of abstract semigroups were first studied by Hickey [5]. Although concrete semigroups of relations had earlier been considered by Magill [10]. The study of semigroup variants goes back to the 1960 monograph of Lyapin [9] and a 1967 paper by Magill and Subbiah [6] that considers semigroups of functions $X \to Y$ under an operation defined by $f \cdot g = f \circ \theta \circ g$, where θ is some fixed function $Y \to X$. In the case that $X = Y$, this provides an alternative product on the full transformation semigroup $T(X)$ (consisting of all functions $X \to X$).

For an element a of a semigroup S, a is called *regular* if there exists $x \in S$ such that $a = axa$. A semigroup S is *regular semigroup* if every element of S is regular. Regular

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semigroups were introduced by Green [4] in his influential 1951 paper "On the structure of semigroups". The concept of regularity in a semigroup was adapted from an analogous condition for rings, already considered by Neumann [12]. It was Green's study of regular semigroups that led him to define his celebrated relations. According to a footnote in Green 1951, the suggestion that the notion of regularity be applied to semigroups was first made by Rees [15, 16]. This property of regular elements was first observed by Thierrin [17] in 1952.

Another important kind of the regularity was introduced by Clifford [1] in 1941, who studied elements a of a semigroup S having the property that there exists $x \in S$ such that $a = axa$ and $ax = xa$, which we now call a *completely regular element*, and semigroups whose any element is completely regular, are called *completely regular semigroups.* The complete regularity was also investigated by Croisot [2] in 1953, who also studied elements a of a semigroup S for which $a \in Sa^2$ (resp. $a \in a^2S$), called *left regular* (resp. *right regular*) *elements*, and semigroups whose every element is left regular (resp. right regular), called *left regular* (resp. *right regular*) *semigroups.*

In [13], Pei has introduced a family of subsemigroup of $T(X)$ defined by

$$T_E(X) = \{\alpha \in T(X) : \forall a, b \in X, (a, b) \in E \Rightarrow (a\alpha, b\alpha) \in E\}$$

where E is an arbitrary equivalence relation on X. In [13], the author investigated regularity and Green's relations for $T_E(X)$. In [11], Namnak and Laysirikul investigated a necessary and sufficient condition when elements of $T_E(X)$ to be left regular, right regular and completely regular.

For a fixed element θ in $T_E(X)$, the variant semigroup of $T_E(X)$ with the sandwich function θ will be denoted simply by $T_E(X, \theta)$. Green's equivalences for elements in the sandwich semigroup $T_E(X, \theta)$ were characterized by Pei alone [13].

Deng, Zeng and Xu [7] introduced a subsemigroup of $T(X)$ defined by

$$T_{E^*}(X) =: \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ if and only if } (x\alpha, y\alpha) \in \sigma\},$$

the so-called *semigroups of transformations that preserve double direction equivalence on* X. They investigated the regularity and Green's relations on $T_{E^*}(X)$. Later, Laysilikul and Namnak [8] investigated a necessary and sufficient condition for the left regularity, the right regularity and the completely regularity of elements in $T_{E^*}(X)$. Deng [3] discussed the Green's $*$-relations, certain $*$-ideal and certain Rees quotient semigroup for the semigroup $T_{E^*}(X)$ and proved that regular and abundant in the semigroup $T_{E^*}(X)$ coincided. The variant semigroup of $T_{E^*}(X)$ with the sandwich function θ, and denoted by $(T_{E^*}(X), \theta)$. Yonthanthum [18] investigated a necessary and sufficient condition for an element of $(T_{E^*}(X), \theta)$ to be regular and determined when $(T_{E^*}(X), \theta)$ is a regular semigroup.

However, it easy to see that the notations $T_E(X)$ $(T_{E^*}(X))$ and $T_E(X, \theta)$ $(T_{E^*}(X, \theta))$ have the same elements. Hence, these are the same set but need not be the same semigroup. If θ is the identity transformation, then these semigroups are coincided. Therefore semigroups $T_E(X, \theta)$ $(T_{E^*}(X, \theta))$ is a generalization of $T_E(X)$ $(T_{E^*}(X))$ and $T_E(X, \theta)$.

In this paper, for a fixed element $\theta \in T_E(X)$ ($\theta \in T_{E^*}(X)$), the variant semigroup of $T_E(X)$ ($T_{E^*}(X)$) with the sandwich function θ will be denoted by $T_E(X,\theta)$ ($T_{E^*}(X,\theta)$). This paper aims to characterize the regular, the left regular, the right regular and the completely regular for elements of $T_E(X,\theta)$ and $T_{E^*}(X,\theta)$. Moreover, we give a necessary and sufficient condition for the identity transformation id_X of the semigroup $T_E(X)$ to be regular, left regular, right regular and completely regular elements in $T_E(X,\theta)$ and $T_{E^*}(X,\theta)$.

In this introductory section, we present many notations and lemma most of which will be indispensable for our research. For arbitrary semigroup S, let $Reg(S), LReg(S), RReg(S)$ and $CReg(S)$, denote the set of all regular elements, the set of all left regular elements, the set of all right regular elements and the set of all completely regular elements of S, respectively.

For a nonempty set X and $\alpha \in T(X)$, we denote by $\pi(\alpha)$ the partition of X induced by α, namely,

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}.$$

Then $\pi(\alpha) = X/\ker(\alpha)$ where $\ker(\alpha) = \{(x,y) \in X \times X : x\alpha = y\alpha\}$. For $A \subseteq X$, we define

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

In the remainder, let E be an equivalence relation on a nonempty set X. Denote by X/E the quotient set. For $x \in X$, we write E_x as for the set of all elements of X that are equivalent to x, that is, $E_x = \{y \in X : (x,y) \in E\}$. The following lemma is needed.

Lemma 1. *[13] Let $\alpha \in T(X)$. Then $\alpha \in T_E(X)$ if and only if for each $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq B$.*

2. Regularity of variants of transformation semigroups that preserve an equivalence relation

In this section, we characterize the regular, left regular, right regular and completely regular elements of the variant semigroup $T_E(X,\theta)$. The identity transformation on X, namely, id_X is the identity of semigroup $T_E(X)$ but need not to be the identity of the semigroup $T_E(X,\theta)$. In addition, we give a necessary and sufficient condition for id_X of the semigroup $T_E(X)$ to be regular, left regular, right regular and completely regular elements in $T_E(X,\theta)$.

Theorem 1. *Let $\alpha \in T_E(X,\theta)$. Then $\alpha \in Reg(T_E(X,\theta))$ if and only if*

(i) $\ker(\alpha) = \ker(\alpha\theta)$ *and*

(ii) for every $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha\theta \subseteq B\theta\alpha\theta$.

Proof. Assume that α is regular of $T_E(X,\theta)$. Then $\alpha = \alpha*\beta*\alpha$ for some $\beta \in T_E(X,\theta)$. Thus $\alpha = \alpha\theta\beta\theta\alpha$. Clearly, $\ker(\alpha) \subseteq \ker(\alpha\theta)$. For the converse conclusion, let $x, y \in X$ be such that $(x,y) \in \ker(\alpha\theta)$. Then $x\alpha\theta = y\alpha\theta$ and so

$$x\alpha = x\alpha\theta(\beta\theta\alpha) = y\alpha\theta(\beta\theta\alpha) = y\alpha.$$

This means that $(x, y) \in \ker(\alpha)$. Hence (1) holds. For each $A \in X/E$, by Lemma 1 there exists $B \in X/E$ such that $A\beta \subseteq B$. Thus $A\beta\theta \subseteq B\theta$. If $a \in A \cap X\alpha\theta$, then $a \in A$ and $a = x\alpha\theta$ for some $x \in X$. Thus $a\beta\theta \in B\theta$. This implies that

$$a = x\alpha\theta = x\alpha\theta\beta\theta\alpha\theta = a\beta\theta\alpha\theta \in B\theta\alpha\theta.$$

Hence $A \cap X\alpha\theta \subseteq B\theta\alpha\theta$.

Conversely, suppose that the conditions (1) and (2) hold. For each $A \in X/E$, we choose $A' \in X/E$ such that $A \cap X\alpha\theta \subseteq A'\theta\alpha\theta$. Let $x \in X$. If $x \in X\alpha\theta$, then by (2), we choose and fix an element $x' \in (E_x)'$ such that $x = x'\theta\alpha\theta$. Otherwise, if $x \notin X\alpha\theta$, we choose and fix an element $x' \in (E_x)'$. Define $\beta : X \to X$ by

$$x\beta = x' \text{ for all } x \in X.$$

Then β is a well-defined mapping. Let $x, y \in X$ be such that $(x, y) \in E$. Then $E_x = E_y$ and so $E_{x'} = E_{y'}$. This implies that $\beta \in T_E(X, \theta)$. It remains to be verified that $\alpha * \beta * \alpha = \alpha$. If $x \in X$, then $x\alpha\theta\beta\theta\alpha = (x\alpha\theta)'\theta\alpha$ with $(x\alpha\theta)'\theta\alpha\theta = x\alpha\theta$. Thus $((x\alpha\theta)'\theta, x) \in \ker(\alpha\theta) = \ker(\alpha)$. This implies that $x\alpha\theta\beta\theta\alpha = (x\alpha\theta)'\theta\alpha = x\alpha$ and therefore $\alpha = \alpha * \beta * \alpha$. Hence α is regular in $T_E(X, \theta)$.

Let E be an equivalence relation on X and Y be a subset of X. A mapping $\alpha : Y \to X$ is called *E-preserving* if for all $x, y \in Y, (x, y) \in E$ implies $(x\alpha, y\alpha) \in E$. If α satisfies the condition that $(x\alpha, y\alpha) \in E$ if and only if $(x, y) \in E$, then α is called E^*-*preserving* . It is easy to see that every $\alpha \in T_E(X)$ is E-preserving but need not be E^*-preserving .

The following result is immediate from Theorem 1.

Corollary 1. *$id_X \in Reg(T_E(X, \theta))$ if and only if θ is an E^*-preserving bijection.*

Proof. Suppose that $id_X \in Reg(T_E(X, \theta))$. Let $x, y \in X$ be such that $x\theta = y\theta$. Then $(x, y) \in \ker(\theta)$. By Theorem 1, $\ker(id_X) = \ker(id_X\theta) = \ker(\theta)$, which implies that $x = xid_X = yid_X = y$. Hence θ is an injection. For each $x \in X$, there exists $B \in X/E$ such that $x\theta \in E_{x\theta} \cap Xid_X\theta \subseteq B\theta id_X\theta$. Thus $x\theta = x'\theta^2$ for some $x' \in B$. Since θ is injective, we deduce that $x = x'\theta$. Consequently, θ is a bijection. Finally, for any $x, y \in X$, if $(x\theta, y\theta) \in E$, then there exists $B \in X/E$ such that $x\theta, y\theta \in E_{x\theta} \cap Xid_X\theta \subseteq B\theta id_X\theta$. Therefore $x\theta = x'\theta^2$ and $y\theta = y'\theta^2$ where $x', y' \in B$. It follows from θ is injective that $x = x'\theta$ and $y = y'\theta$. Since $(x', y') \in E$ and $\theta \in T_E(X), (x, y) = (x'\theta, y'\theta) \in E$. We conclude that θ is an E^*-preserving bijection.

Conversely, if θ is an E^*-preserving bijection, then $\theta^{-1} \in T_E(X, \theta)$ and

$$id_X = id_X\theta(\theta^{-1}\theta^{-1})\theta id_X = id_X * (\theta^{-1}\theta^{-1}) * id_X.$$

Hence $id_X \in Reg(T_E(X, \theta))$.

In what follows we investigate when an element in $T_E(X, \theta)$ is left regular.

Theorem 2. *Let $\alpha \in T_E(X, \theta)$. Then $\alpha \in LReg(T_E(X, \theta))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha), x\theta\alpha\theta \in P$ for some $x \in B$.*

Proof. Assume that $\alpha \in LReg(T_E(X,\theta))$. Then $\alpha = \beta * \alpha * \alpha$ for some $\beta \in T_E(X,\theta)$ and so $\alpha = \beta\theta\alpha\theta\alpha$. Let $A \in X/E$. By Lemma 1, there exists $B \in X/E$ such that $A\beta \subseteq B$. Suppose that $P \in \pi_A(\alpha)$ and let $x \in P \cap A$. Hence $x\beta \in B$ and $x\alpha = x\beta\theta\alpha\theta\alpha$ which means $x\beta\theta\alpha\theta \in (x\alpha)\alpha^{-1} = P$.

Conversely, for each $A \in X/E$, we choose $A' \in X/E$ such that for every $P \in \pi_A(\alpha), x\theta\alpha\theta \in P$ for some $x \in A'$. Let $x \in X$. Since X/E and $\pi(\alpha)$ are partitions of X, there exist $A \in X/E$ and $P \in \pi(\alpha)$ such that $x \in A$ and $x \in P$. Hence $P \in \pi_A(\alpha)$. By assumption, we choose and fix an element $x' \in A'$ such that $x'\theta\alpha\theta \in P$ and $A' \in X/E$. We also have that $x'\theta\alpha\theta\alpha = x\alpha$. Define $\beta : X \to X$ by

$$x\beta = x' \text{ for all } x \in X.$$

Let $x, y \in X$ be such that $(x,y) \in E$. Then $E_x = E_y$ and thus $E_{x'} = E_{y'}$. This implies that $\beta \in T_E(X,\theta)$. If $x \in X$, then $x\beta\theta\alpha\theta\alpha = x'\theta\alpha\theta\alpha = x\alpha$ which implies that $\alpha = \beta\theta\alpha\theta\alpha$. Therefore $\alpha = \beta * \alpha * \alpha$ and hence α is left regular, as required.

Corollary 2. *$id_X \in LReg(T_E(X,\theta))$ if and only if θ is a surjection.*

Proof. Suppose that $id_X \in LReg(T_E(X,\theta))$. Let $x \in X$. Since $\{x\} \in \pi_{E_x}(id_X)$ and by Theorem 2, there exists $B \in X/E$ such that $b\theta id_X\theta \in \{x\}$ for some $b \in B$. Therefore $x = b\theta id_X\theta = b\theta\theta$. Hence θ is a surjection on X.

Conversely, assume that θ is a surjection. Thus θ^2 is also surjective. Let $A \in X/E$. Note that $\pi_A(id_X) = \{\{a\} : a \in A\}$. Let $P \in \pi_A(id_X)$, then $P = \{a\}$ where $a \in A$ and so $a = x\theta\theta$ for some $x \in X$. Choose $B = E_x$ and so $x\theta id_X\theta = x\theta\theta = a \in \{a\} = P$. Hence by Theorem 2, we conclude that $id_X \in LReg(T_E(X,\theta))$.

Theorem 3. *Let $\alpha \in T_E(X,\theta)$. Then $\alpha \in RReg(T_E(X,\theta))$ if and only if $(\theta\alpha\theta)|_{X\alpha}$ is an E^*-preserving injection.*

Proof. Suppose that $\alpha \in RReg(T_E(X,\theta))$. Then there is $\beta \in T_E(X,\theta)$ such that $\alpha = \alpha * \alpha * \beta$ and so $\alpha = \alpha\theta\alpha\theta\beta$. Since $\theta\alpha\theta \in T_E(X), (\theta\alpha\theta)|_{X\alpha}$ is E-preserving. Let $x, y \in X\alpha$ be such that $x = x'\alpha$ and $y = y'\alpha$ where $x', y' \in X$. If $x\theta\alpha\theta = y\theta\alpha\theta$, then $x = x'\alpha = x'\alpha\theta\alpha\theta\beta = x\theta\alpha\theta\beta = y\theta\alpha\theta\beta = y'\alpha\theta\alpha\theta\beta = y'\alpha = y$. It follows that $(\theta\alpha\theta)|_{X\alpha}$ is an injection. If $(x\theta\alpha\theta, y\theta\alpha\theta) \in E$, then since $\beta \in T_E(X,\theta)$, we have that $(x\theta\alpha\theta\beta, y\theta\alpha\theta\beta) \in E$. Moreover,

$$(x,y) = (x'\alpha, y'\alpha) = (x'\alpha\theta\alpha\theta\beta, y'\alpha\theta\alpha\theta\beta) = (x\theta\alpha\theta\beta, y\theta\alpha\theta\beta) \in E$$

which implies that $(\theta\alpha\theta)|_{X\alpha}$ is an E^*-preserving injection.

Conversely, assume that $(\theta\alpha\theta)|_{X\alpha}$ is an E^*-preserving injection. Let $A \in X/E$ be such that $A \cap X\alpha\theta\alpha\theta \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\alpha\theta\alpha\theta$. For each $x \in A \cap X\alpha\theta\alpha\theta$, there exists a unique element $x' \in X\alpha$ such that $x = x'\theta\alpha\theta$ by the condition $(\theta\alpha\theta)|_{X\alpha}$ is injective. We observe that $(x'\theta\alpha\theta, x'_A\theta\alpha\theta) = (x, x_A) \in E$. It follows from $(\theta\alpha\theta)|_{X\alpha}$ is an E^*-preserving mapping that $(x', x'_A) \in E$. Define $\beta_A : A \to E_{x'_A}$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha\theta\alpha\theta, \\ x'_A & \text{otherwise.} \end{cases}$$

Then we define the map $\beta : X \to X$ by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha\theta\alpha\theta \neq \emptyset, \\ id_A & \text{otherwise,} \end{cases}$$

for all $A \in X/E$. Since X/E is a partition of X, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. By the definition of β, we have $(x\beta, y\beta) = (x\beta|_A, y\beta|_A)$. If $A \cap \alpha\beta\alpha\beta = \emptyset$, then $(x\beta, y\beta) = (xid_A, yid_A) = (x, y) \in E$. If $A \cap \alpha\beta\alpha\beta \neq \emptyset$, then $x\beta, y\beta \in A\beta = A\beta_A \subseteq E_{x'_A}$. Consequently, $\beta \in T_E(X, \theta)$.

Finally, to show that $\alpha = \alpha\theta\alpha\theta\beta$, let $x \in X$. Then $x\alpha\theta\alpha\theta \in X\alpha\theta\alpha\theta$. Then there exists $A \in X/E$ such that $x\alpha\theta\alpha\theta \in A$. By the definition of $\beta_A, x\alpha\theta\alpha\theta\beta_A = (x\alpha\theta\alpha\theta)'$ where $(x\alpha\theta\alpha\theta)'\theta\alpha\theta = x\alpha\theta\alpha\theta = (x\alpha)\theta\alpha\theta$. Since $(x\alpha\theta\alpha\theta)'$ is unique, we get that $(x\alpha\theta\alpha\theta)' = x\alpha$. Thus $x\alpha\theta\alpha\theta\beta = x\alpha\theta\alpha\theta\beta_A = x\alpha$. Therefore $\alpha = \alpha * \alpha * \beta$. Hence $\alpha \in RReg(T_E(X, \theta))$, as asserted.

Corollary 3. *$id_X \in RReg(T_E(X, \theta))$ if and only if θ is an E^*-preserving injection.*

Proof. Assume that $id_X \in RReg(T_E(X, \theta))$. By Theorem 3, we get that $(\theta id_X\theta)|_{Xid_X}$ is an E^*-preserving injection. This implies that $\theta\theta$ is an E^*-preserving injection. Hence θ is an E^*-preserving injection.

This converse of corollary is clear.

Final of this section, we give a characterization of completely regular elements in $T_E(X, \theta)$. Recall that, an element a of a semigroup S is completely regular if and only if a is both left and right regular [14]. Hence, as an immediate consequence of Theorems 2 and 3, we have the following.

Theorem 4. *Let $\alpha \in T_E(X, \theta)$. Then $\alpha \in CReg(T_E(X, \theta))$ if and only if*

(i) for every $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha), x\theta\alpha\theta \in P$ for some $x \in B$ and

(ii) $(\theta\alpha\theta)|_{X\alpha}$ is an E^*-preserving injection.

As an immediate consequence of Corollaries 2 and 3.

Corollary 4. *$id_X \in CReg(T_E(X, \theta))$ if and only if θ is an E^*-preserving bijection.*

Theorem 5. *Let $\alpha \in T_E(X, \theta)$. If $\alpha \in CReg(T_E(X, \theta))$, then every $A \in X/E$, there exists $B \in X/E$ such that $|P \cap B\theta\alpha\theta| = |P \cap X\theta\alpha\theta| = 1$ for all $P \in \pi_A(\alpha)$.*

Proof. Assume that $\alpha \in CReg(T_E(X, \theta))$. Then α is regular, left regular and right regular. Let $A \in X/E$. By Theorem 2, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha), x\theta\alpha\theta \in P$ for some $x \in B$. For each $P \in \pi_A(\alpha)$, we have $x\theta\alpha\theta \in P$ for some $x \in B$. Therefore $P \cap B\theta\alpha\theta \neq \emptyset$ and $P \cap X\theta\alpha\theta \neq \emptyset$. Let $y \in P \cap X\theta\alpha\theta$. Then $y\alpha = x\theta\alpha\theta\alpha$. By Theorem 3, we obtain that $(\theta\alpha\theta)|_{X\alpha}$ is injective. Claim that $\alpha|_{X\theta\alpha\theta}$ is also injective, let $x_1, x_2 \in X\theta\alpha\theta$ be such that $x_1\alpha = x_2\alpha$. Then $x_1 = x'_1\theta\alpha\theta$ and

$x_2 = x_2'\theta\alpha\theta$ for some $x_1', x_2' \in X$. Thus $x_1'\theta\alpha\theta\alpha = x_2'\theta\alpha\theta\alpha$ and so $x_1'\theta\alpha\theta\alpha\theta = x_2'\theta\alpha\theta\alpha\theta$. Since $\theta\alpha\theta|_{X\alpha}$ is injective, $x_1'\theta\alpha = x_2'\theta\alpha$ which implies that $x_1 = x_1'\theta\alpha\theta = x_2'\theta\alpha\theta = x_2$. So, we have the claim. This implies that $y = x\theta\alpha\theta$ and hence $|P \cap X\theta\alpha\theta| = 1$. It follows from $P \cap B\theta\alpha\theta \subseteq P \cap X\theta\alpha\theta$ that $|P \cap B\theta\alpha\theta| = |P \cap X\theta\alpha\theta| = 1$.

3. Regularity of variants of transformation semigroups that preserve double direction equivalence

In this section, we characterize the regular, left regular, right regular and completely regular elements of the variant semigroup $T_{E^*}(X,\theta)$. In addition, we give a necessary and sufficient condition for the identity transformation id_X of the semigroup $T_{E^*}(X)$ to be regular, left regular, right regular and completely regular elements in $T_{E^*}(X,\theta)$.

Theorem 6. *Let $\alpha \in T_{E^*}(X,\theta)$. Then $\alpha \in Reg(T_{E^*}(X,\theta))$ if and only if*

(i) $\ker(\alpha) = \ker(\alpha\theta)$ *and*

(ii) for every $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha\theta = B\theta\alpha\theta$.

Proof. Suppose that $\alpha \in Reg(T_{E^*}(X,\theta))$. Then $\alpha \in Reg(T_E(X,\theta))$. By Theorem 1$(i)$, we have (i) holds. Let $A \in X/E$. Then by Theorem 1(ii), there exists $B \in X/E$ such that $A \cap X\alpha\theta \subseteq B\theta\alpha\theta$. Since $\theta, \alpha\theta \in T_E(X)$ and by Lemma 1, there are $C, D \in X/E$ such that $B\theta \subseteq C$ and $C\alpha\theta \subseteq D$. Therefore $A \cap X\alpha\theta \subseteq D$. Since A and D are equivalence classes of X, $A = D$. This implies that $A \cap X\alpha\theta \subseteq B\theta\alpha\theta \subseteq C\alpha\theta = C\alpha\theta \cap X\alpha\theta \subseteq A \cap X\alpha\theta$. Hence $A \cap X\alpha\theta = B\theta\alpha\theta$.

Conversely, assume that conditions (i) and (ii) hold. Define $\beta : X \to X$ as in the proof of Theorem 1. Then $\beta \in T_E(X)$ and $\alpha = \alpha * \beta * \alpha$. It remains to show that $\beta \in T_{E^*}(X)$. Let $x, y \in X$ be such that $(x\beta, y\beta) \in E$. Then $(x', y') \in E$ where $E_x \cap X\alpha\theta = E_{x'}\theta\alpha\theta$ and $E_y \cap X\alpha\theta = E_{y'}\theta\alpha\theta$. Thus $E_{x'} = E_{y'}$ and so $E_x \cap X\alpha\theta = E_{x'}\theta\alpha\theta = E_{y'}\theta\alpha\theta = E_y \cap X\alpha\theta$, which implies that $E_x = E_y$. Therefore $(x, y) \in E$ and hence $\beta \in T_{E^*}(X)$.

Corollary 5. *$id_X \in Reg(T_{E^*}(X,\theta))$ if and only if θ is a bijection.*

Now, we discuss a characterization of an element in the semigroup $T_{E^*}(X,\theta)$ to be left regular.

Theorem 7. *Let $\alpha \in T_{E^*}(X,\theta)$. Then $\alpha \in LReg(T_{E^*}(X,\theta))$ if and only if for every $P \in \pi(\alpha), P \cap X\theta\alpha\theta \neq \emptyset$.*

Proof. Suppose that $\alpha \in LReg(T_{E^*}(X,\theta))$. Then $\alpha \in LReg(T_E(X,\theta))$. Let $P \in \pi(\alpha)$ and $p \in P$. Then $P \in \pi_{E_p}(\alpha)$. By Theorem 2, there exists $B \in X/E$ such that $x\theta\alpha\theta \in P$ for some $x \in B$. Hence $P \cap X\theta\alpha\theta \neq \emptyset$.

Conversely, for every $P \in \pi(\alpha), P \cap X\theta\alpha\theta \neq \emptyset$. We choose and fix an element $x_P\theta\alpha\theta \in P$. For each $x \in X$, we let $P_x \in \pi(\alpha)$ be such that $x \in P_x$. Define $\beta : X \to X$ by $x\beta = x_{P_x}$ for all $x \in X$. Next, we will show that $\beta \in T_{E^*}(X,\theta)$, let $x, y \in X$. If $(x, y) \in E$, then

$(x\alpha, y\alpha) \in E$ and $(x\beta, y\beta) = (x_{P_x}, x_{P_y})$ where $x_{P_x}\theta\alpha\theta \in P_x$ and $x_{P_y}\theta\alpha\theta \in P_y$. We then have $(x_{P_x}\theta\alpha\theta\alpha, x_{P_y}\theta\alpha\theta\alpha) = (x\alpha, y\alpha) \in E$. Since $\theta\alpha\theta\alpha \in T_{E^*}(X), (x\beta, y\beta) = (x_{P_x}, x_{P_y}) \in E$. On the other hand, if $(x\beta, y\beta) \in E$, then $(x_{P_x}, x_{P_y}) \in E$ where $x_{P_x}\theta\alpha\theta \in P_x$ and $x_{P_y}\theta\alpha\theta \in P_y$ and so $(x\alpha, y\alpha) = (x_{P_x}\theta\alpha\theta\alpha, x_{P_y}\theta\alpha\theta\alpha) \in E$. By $\alpha \in T_{E^*}(X)$, it follows that $(x, y) \in E$. Hence $\beta \in T_{E^*}(X)$. Finally, to show that $\alpha = \beta * \alpha * \alpha$. Let $x \in X$. Then $x_{P_x}\theta\alpha\theta \in P_x$ and hence $x\beta\theta\alpha\theta\alpha = x_{P_x}\theta\alpha\theta\alpha = x\alpha$, as required.

Corollary 6. *$id_X \in LReg(T_{E^*}(X, \theta))$ if and only if θ is a surjection.*

Proof. The necessity is clear from Corollary 2. To prove the sufficiency, we suppose that θ is a surjection. Then $\theta\theta$ is also a surjection. Since $\pi(id_X) = \{\{x\} : x \in X\}$, $P \cap X\theta id_X\theta = P \cap X\theta\theta \neq \emptyset$ for all $P \in \pi(id_X)$. Hence $id_X \in LReg(T_{E^*}(X, \theta))$, by Theorem 7.

Next, we characterize a right regular element of $T_{E^*}(X, \theta)$. The following lemma is needed.

Lemma 2. *[8] Let $\alpha \in T_{E^*}(X, \theta)$ and $A, B \in X/E$. If $A\alpha \subseteq B$, then $B\alpha^{-1} = A$.*

Theorem 8. *Let $\alpha \in T_{E^*}(X, \theta)$. Then $\alpha \in RReg(T_{E^*}(X, \theta))$ if and only if*

(i) $(\theta\alpha\theta)|_{X\alpha}$ is an injection and

(ii) if there exists $A \in X/E$ such that $A \cap X(\alpha\theta)^2 = \emptyset$, then there exists an injection $\varphi : \{A \in X/E : A \cap X(\alpha\theta)^2 = \emptyset\} \to \{A \in X/E : A \cap X\alpha = \emptyset\}$.

Proof. Assume that $\alpha \in RReg(T_{E^*}(X, \theta))$. Then $\alpha \in RReg(T_E(X, \theta))$. By Theorem 3, we then have (i) hold. Next, we prove that (ii) holds in the following. Suppose that $\{A \in X/E : A \cap X(\alpha\theta)^2 = \emptyset\} \neq \emptyset$. Let $A \in X/E$ be such that $A \cap X(\alpha\theta)^2 = \emptyset$. Since $\alpha \in RReg(T_{E^*}(X, \theta))$, there exists $\beta \in T_{E^*}(X, \theta)$ such that $\alpha = \alpha * \alpha * \beta$ and so $\alpha = \alpha\theta\alpha\theta\beta$. By Lemma 1, we let $A' \in X/E$ such that $A\beta \subseteq A'$. Claim that $A' \cap X\alpha = \emptyset$, suppose not. Let $x \in X$ be such that $x\alpha \in A'$ and choose $a \in A$. Then $a\beta \in A'$ and so $(x\alpha\theta\alpha\theta\beta, a\beta) = (x\alpha, a\beta) \in E$. Since $\beta \in T_{E^*}(X)$, we get $(x\alpha\theta\alpha\theta, a) \in E$. Hence $x\alpha\theta\alpha\theta \in A$ which is a contradiction. Thus $A' \cap X\alpha = \emptyset$. Define $\varphi : \{A \in X/E : A \cap X(\alpha\theta)^2 = \emptyset\} \to \{A \in X/E : A \cap X\alpha = \emptyset\}$ by

$$A\varphi = A' \text{ for all } A \in X/E \text{ and } A \cap X(\alpha\theta)^2 = \emptyset.$$

To show that φ is injective, let $A, B \in \{A \in X/E : A \cap X(\alpha\theta)^2 = \emptyset\}$ be such that $A\varphi = B\varphi$. By the definition of $\varphi, A\varphi = A'$ and $B\varphi = B'$ where $A\beta \subseteq A'$ and $B\beta \subseteq B'$ for some $A', B' \in X/E$. It follows from Lemma 2 that $A = A'\beta^{-1}$ and $B = B'\beta^{-1}$. Since $A' = B'$, we deduce that $A = B$. Therefore φ is an injection. Hence (ii) holds.

Conversely, suppose that the conditions (i) and (ii) hold. For each $x \in X(\alpha\theta)^2$, we choose and fix an element $x' \in X\alpha$ such that $x = x'\theta\alpha\theta$. Let $A \in X/E$ be such that $A \cap X(\alpha\theta)^2 \neq \emptyset$. Then we fix $x_A \in A \cap X\alpha$ and define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X(\alpha\theta)^2, \\ x'_A & \text{otherwise.} \end{cases}$$

Let $A \in X/E$ be such that $A \cap X(\alpha\theta)^2 = \emptyset$ and $x \in A$. By (ii), we fix $\tilde{x} \in A\varphi$ and define $\beta_A : A \to X$ by

$$x\beta_A = \tilde{x} \text{ for all } x \in A.$$

Let $\beta : X \to X$ by $\beta|_A = \beta_A$ for all $A \in X/E$. Since X/E is a partition of X, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. There are two cases to consider.

Case 1. $A \cap X(\alpha\theta)^2 = \emptyset$. Then $(x\beta, y\beta) = (\tilde{x}, \tilde{y}) \in E$.

Case 2. $A \cap X(\alpha\theta)^2 \neq \emptyset$. Without loss of generality, we assume that $x, y \in X(\alpha\theta)^2$. Hence $x\beta = x'$ and $y\beta = y'$ where $x = x'\theta\alpha\theta$ and $y = y'\theta\alpha\theta$, respectively. Since $\theta\alpha\theta \in T_{E^*}(X, \theta)$ and $(x'\theta\alpha\theta, y'\theta\alpha\theta) \in E$, we conclude that $(x\beta, y\beta) = (x', y') \in E$.

On the other hand, let $x, y \in X$ be such that $(x\beta, y\beta) \in E$. Thus $x\beta, y\beta \in B$ for some $B \in X/E$. If $B \cap X\alpha = \emptyset$, then by the definition of β, $x\beta, y\beta \in B = E_x\varphi = E_y\varphi$. Since φ is an injection, $E_x = E_y$ and hence $(x, y) \in E$. If $B \cap X\alpha \neq \emptyset$, then by the definition of β, we may assume that $x\beta = x', y\beta = y'$ for some $x', y' \in X\alpha$ with $x = x'\theta\alpha\theta$ and $y = y'\theta\alpha\theta$. Since $(x', y') = (x\beta, y\beta) \in E$ and $\theta\alpha\theta \in T_{E^*}(X)$, we deduce that $(x, y) = (x'\theta\alpha\theta, y'\theta\alpha\theta) \in E$. It follows that $\beta \in T_{E^*}(X)$. Let $x \in X$, then $x(\alpha\theta)^2 \in X(\alpha\theta)^2$ and there exists $(x(\alpha\theta)^2)' \in X\alpha$ such that $(x(\alpha\theta)^2)'\theta\alpha\theta = x(\alpha\theta)^2 = (x\alpha)\theta\alpha\theta$. We note by (1) that $(x(\alpha\theta)^2)' = x\alpha$. Therefore

$$x(\alpha * \alpha * \beta) = x\alpha\theta\alpha\theta\beta = x(\alpha\theta)^2\beta = (x(\alpha\theta)^2)' = x\alpha.$$

Hence α is right regular, as required.

Corollary 7. *$id_X \in RReg(T_{E^*}(X, \theta))$ if and only if*

(i) θ is an injection and

(ii) $A \cap X\theta \neq \emptyset$ for all $A \in X/E$.

Proof. Assume that $id_X \in RReg(T_{E^*}(X, \theta))$. By Corollary 3, we get that θ is injective. Suppose that $A \cap X\theta^2 = \emptyset$ for some $A \in X/E$. By (2) of Theorem 8, there exists an injection $\varphi : \{A \in X/E : A \cap X(id_X\theta)^2 = \emptyset\} \to \{A \in X/E : A \cap Xid_X = \emptyset\}$. This is a contradiction with $\{A \in X/E : A \cap Xid_X = \emptyset\} = \{A \in X/E : A \cap X = \emptyset\} = \emptyset$. Thus $A \cap X\theta^2 \neq \emptyset$ for all $A \in X/E$. It follows from $X\theta^2 \subseteq X\theta$ that $A \cap X\theta \neq \emptyset$ for all $A \in X/E$.

The converse of corollary follows from Theorem 8.

The following result is obtained directly from Theorem 7 and Theorem 8.

Theorem 9. *Let $\alpha \in T_{E^*}(X, \theta)$. Then $\alpha \in CReg(T_{E^*}(X, \theta))$ if and only if*

(i) for every $P \in \pi(\alpha), P \cap X\theta\alpha\theta \neq \emptyset$,

(ii) $(\theta\alpha\theta)|_{X\alpha}$ is an injection and

(iii) if there exists $A \in X/E$ such that $A \cap X(\alpha\theta)^2 = \emptyset$, then there exists an injection $\varphi : \{A \in X/E : A \cap X(\alpha\theta)^2 = \emptyset\} \to \{A \in X/E : A \cap X\alpha = \emptyset\}$.

Corollary 8. *$id_X \in CReg(T_{E^*}(X, \theta))$ if and only if θ is a bijection.*

4. Conclusion and discussion

In this work, we presented necessary and sufficient conditions when elements of the semigroups $T_E(X,\theta)$ and $T_{E^*}(X,\theta)$ to be regular, left regular, right regular and completely regular. The identity transformation is a regular, a left regular, a right regular and a completely regular element for the semigroups $T_E(X)$ and $T_{E^*}(X)$ but need not for $T_E(X,\theta)$ and $T_{E^*}(X,\theta)$. Hence, we presented properties for θ that the identity transformation is regular, left regular, right regular and completely regular.

The important theory in algebraic semigroups theory is Cayley's Theorem for semigroups that every semigroup is isomorphic to a subsemigroup of some full transformation semigroup. Hence in order to study structure of semigroups, it suffices to consider subsemigroups of the full transformation semigroup. Therefore, several researchers are interested in the characterizations of subsemigroups of the full transformation semigroup.

In future work, we intend to expand other algebraic structures for the variants of transformation semigroups that preserve an equivalence relation and other transformation semigroups.

References

[1] AH Clifford. Semigroups admitting relative inverses. *Annals of Mathematics*, 42:1037–1049, 1941.

[2] R Croisot. Demi-groupes inversifs et demi-groupes reunions de demi-groupes simples. *Annales scientifiques de l'École Normale Supérieure*, 70(4):361–379, 1953.

[3] L Deng. On certain semigroups of transformations that preserve double direction equivalence. *Bulletin of the Iranian Mathematical Society*, 42:1015–1024, 2016.

[4] JA Green. On the structure of semigroups. *Annals of Mathematics*, 54:163–172, 1951.

[5] JB Hickey. Semigroups under a sandwich operation. *Proceeding of the Edinburgh Mathematical Society*, 26:371–382, 1983.

[6] Jr KD Magill and S Subbiah. Green's relations for regular elements of sandwich semigroups. ii. semigroups of continuous functions. *Journal of the Australian Mathematical Society. Series A*, 25:45–65, 1978.

[7] J Zeng L Deng and B Xu. Green's relations and regularity for semigroups of transformations that preserve double direction equivalence. *Semigroup Forum*, 80:416–425, 2010.

[8] E Laysirikul and C Namnak. Regularity conditions for semigroups of transformations that preserving double direction equivalence. *Proceedings of the* 17^{th} *Annual Meeting in Mathematics.*, pages 155–160, 2012.

[9] ES Lyapin. Semigroups (in russian). *Gosudarstv. Izdat. Fiz-Mat. Lit. Moscow*, 1960.

[10] KD Magill. Semigroup structures for families of functions i. some homomorphism theorems. *Journal of the Australian Mathematical Society*, 7:81–94, 1967.

[11] C Namnak and E Laysirikul. Regularity for semigroups of transformations that preserve equivalence. *JP Journal of Algebra, Number Theory and Applications*, 28:97–105, 2013.

[12] J Neumann. On regular rings. *Proceedings of the National Academy of Sciences of the United States of America*, 22:707–713, 1936.

[13] H Pei. Regularity and green's relations for semigroup of transformations that preserve an equivalence. *Communications in Algebra*, 14:497–503, 2005.

[14] M Petrich and NR Reilly. *Completely Regular Semigroups.* Wiley, New York, 1999.

[15] D Rees. On semigroups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 36:387–400, 1940.

[16] D Rees. Note on semigroups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(4), 1941.

[17] G Thierrin. Sur les elements inversifs et les elements unitaires dun demi-groupe inversif. *Proceedings of the Academy of sciences*, 234:33–34, 1952.

[18] W Yonthanthum. Regular elements of the variant semigroups of transformations preserving double direction equivalences. *Thai Journal of Mathematics*, 16:165–171, 2018.