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Reverse Derivations on δ-prime rings

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Abstract. In this paper, we generalized Posner's theorem, then, Mayne's theorem has been extended to get a main result, and presented by the following theorem, if δ is a nonzero centralizing reverse derivations on a nonzero δ-ideal U of δ –prime ring R, then R is commutative.

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Key Words and Phrases: Reverse derivation, δ-prime ring, δ-ideal

1. Introduction

Throughout this paper, R assumed to be an associative ring with unity and the center $Z(R)$, the commutator is defined as $[u, v] = uv - vu$, is also called a Lie commutator for elements $u, v \in R$, the symbol $C(R)$ stand for the set of all commutator ideals generated by $[u, v]$, the set $annU = \{r \in R : rU = 0\}$ is the annihilator of U of R. The smallest positive integer n such that $n \cdot u = 0$ for all $u \in R$ is the characteristic of the ring R.

We will employ some commutator properties like $[uv, w] = u[v, w]+[u, w]v$ and $[u, vw] = v[u, w] + [u, v]w \ \forall u, v, w \in R$. Furthermore, we recall that R is called a prime ring if $uRv = \{0\}$, then either $u = 0$ or $v = 0$, and by analogy, R is called a semiprime ring, for $u \in R$, if $uRu = \{0\}$, then $u = 0$.

A map F from R to R is said to be a centralizing on U if $[u, F(u)] \in Z(R)$ for all $u \in U$. An additive map $\delta : R \to R$ is called a derivation on R, if the condition $\delta(uv) = \delta(u)v + u\delta(v), \ \forall u, v \in R$ holds, while an additive map $\delta : R \to R$ is said to be a reverse derivation on R if satisfies the rule $\delta(uv) = \delta(v)u + v\delta(u), \ \forall u, v \in R$.

Furthermore, for a fixed element $u \in R$ the additive map $\partial_u : R \to R$ defined by, $\partial_u(v) = uv - vu$, where $v \in R$ is called a partial derivation generated by $u \in R$ ($\partial_u(U) = [u, U] = \{[u, t] : t \in U\}$).

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Let Δ be a subset of the set of all derivations $\mathfrak{D}$ on R. An ideal U of R such that $\delta(U) \subseteq U$ is used to be called a δ- ideal of R. A ring R is called δ-prime if, for any two δ- ideals U, V of R, the condition $UV = 0$ infers that either $U = 0$ or $V = 0$, equivalently, we call a ring R that δ- semiprime, for $U \subseteq \mathfrak{D}$, if $[U, U] \neq 0$, implies $U \neq 0$.

Moreover, other terminologies are standard and they were considered as in [10],[11] and [12].

Posner's First Theorem demonstrated that the composition of two nonzero derivations on a prime ring R, with $charR \neq 2$, is not a derivation. The Second Theorem of Posner proved that if the nonzero derivation d on a prime ring R is a centralizing on R, then R is commutative [26].

Mayne generalized Posner's theorem when a ring R has an automorphism or a nonzero centralizing derivation on some ideal $U \neq 0$, concluding that R is commutative [22].

Likewise, some generalizations of these results with different ways for a prime and semiprime rings are condidered in [6, 7, 13, 14, 21–24, 26, 28, 29]. In general, they have showed commutativity of prime and semiprime rings admit centralizing derivations on specific subsets of R.

Overall, Brešar and Vukman who started the researching on reverse derivation concept (see [8]), recently, Samman and Alyamani [27] came, with many properties of reverse derivations in prime (resp. semiprime) rings.

2. Preliminaries

Many researchers studied the properties of Lie rings with derivations $\mathfrak{D}$ of differentially simple, prime and semiprime rings (see for example [1–4], [14, 15], [16, 17] and [18, 28], where further references can be found for the widening in this field.

Passman in [25], has investigated the commutative rings with semiprime Lie ring $\mathfrak{D}$. There are many papers in this line, as [9, 19, 20].

The following main lemmas that will be used to prove our new results, to which we shall refer, stated as in the following:

Lemma 1. *[26, Lemma 3] Let R be a prime ring, and d a derivation of R such that $ad(a) - d(a)a = 0$ for all $a \in R$. Then R is commutative.*

Lemma 2. *[21, Theorem] Let R be a prime ring with a nontrivial centralizing automorphism. Then, R is a commutative integral domain.*

Lemma 3. *[22, Theorem]*
Let R be a prime ring and $A \neq \{0\}$ be an ideal of R. If R has a nontrivial automorphisim or derivation T such that $uu^T - u^T u$ is in the center of R and u^T is in U for every u in U, then R is commutative.

Lemma 4. *[4, Lemma 13] Let $A \neq 0$ be a Lie δ-ideal of a δ-semiprime and subring of a ring R of $charR \neq 2$. Then $A \subseteq Z(R)$ or A contains a non-central δ-ideal of R.*

Therefore the purpose of our research is to study the structure of reverse derivation on $\delta-$ prime ring and some properties of centralizing reverse derivations on nonzero $\delta-$ ideal of $\delta-$ prime ring.

3. Some properties and examples of Reverse Derivation

In fact, it is not necessary that every derivation is a reverse derivation on a ring R or vice versa. Taking into consideration when the ring R is commutative, then the derivation and the reverse derivation are coincides. Therefore it is possible to define an example about this case.

Example 1. *Let* $R = \left\{ \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} : u, v \in T \right\}$, *such that* T *is a ring with* $T^2 \neq \{0\}$. *Let* $\delta : R \to R$ *be an additive mapping defined as*

$$\delta\left(\begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} : \ \forall u, v \in T.$$

and $\grave{\delta} : R \to R$ *be an additive mapping defined as*

$$\grave{\delta}\left(\begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} : \ \forall u, v \in T.$$

It is easy to see that δ is a derivation, but not a reverse derivation, while $\grave{\delta}$ is both a derivation and a reverse derivation.

Now, if the ring R is commutative, then

$$\delta(uv) = \delta(vu).$$

if δ is a derivation on a ring R, so

$$\delta(uv) = \delta(u)v + u\delta(v)$$

and

$$\delta(vu) = \delta(v)u + v\delta(u).$$

This means

$$\delta(u)v + u\delta(v) = \delta(v)u + v\delta(u).$$

Thus, δ is a reverse derivation too.

Lemma 5. *Let* R *be a* δ*-prime ring,* $U \neq \{0\}$ *a* δ*-ideal of* R*, which* $\delta \neq 0$ *a reverse derivation on a* R*. If* $\delta(U) = 0$*, then* $\delta(R) = 0$*.*

proof. Since U is δ-ideal of R, then

$$RU \subseteq U \text{ and } UR \subseteq U.$$

So

$$\delta(RU) \subseteq \delta(U) = 0 \text{ and } \delta(UR) \subseteq \delta(U) = 0,$$

then

$$\delta(RU) = \delta(UR) = 0.$$

Since δ is a reverse derivation then,

$$\delta(RU) = \delta(U)R + U\delta(R) = 0 \quad \text{and} \quad \delta(UR) = \delta(R)U + R\delta(U) = 0,$$

this means

$$U\delta(R) = \delta(R)U = 0.$$

Thus, we deduce that $\delta(R) \subseteq annU$, but U is a δ-ideal, hence $\delta(R) = 0$.

Lemma 6. *Let $\delta \neq 0$ be a reverse derivation on a δ-ring R and let $U \neq \{0\}$ be δ-ideal of R. If*

$$[\delta(u), u] = 0 \ \forall u \in U, \tag{1}$$

then R is commutative.

proof. Now, let linearize the identity (1) on U, then we have for all $u, v \in U$

$$0 = [\delta(u+v), u+v] = [\delta(u), u] + [\delta(u), v] +$$

$$[\delta(v), u] + [\delta(v), v] = [\delta(u), v] + [\delta(v), u].$$

$$[\delta(u), v] = [u, \delta(v)]. \tag{2}$$

Write $v\delta(u)$ instead of $\delta(u)$ in (2), then we get

$$[u, \delta(v)] = [v\delta(u), v] =$$

$$v[\delta(u), v] + [v, v]\delta(u) =$$

$$v[\delta(u), v] = v[u, \delta(v)] =$$

$$vu\delta(v) - v\delta(v)u = vu\delta(v) - \delta(v)vu = [vu, \delta(v)] =$$

$$[\delta(vu), v] = [\delta(u)v + u\delta(v), v] =$$

$$[\delta(u)v, v] + [u\delta(v), v] = [\delta(u), v]v + [u, v]\delta(v)$$

$$[v, u]\delta(v) = 0. \tag{3}$$

Now, replace u by uv in (3), then,

$$[v, uv]\delta(v) = 0,$$

$$[v, u]v\delta(v) = 0,$$

thus

$$[v, u]U\delta(v) = 0.$$

Hence, since R is a δ-prime ring, either $\delta(u) = 0$, then $u = 0$ and this contradicts the assumption, or $[v, u] = 0$ for all $u, v \in U$, therefore U is commutative. So we get $UC(R) = 0$, then $C(R) = 0$, hence R is commutative.

4. Reverse Derivation on δ- Ideal

Through this section, we will prove several lemmas arriving to the extension of lemma (1), presented by the main theorem.

Lemma 7. *Let R be a δ-prime ring and let $\delta \neq 0$ be a reverse derivation on R. If $u[\delta^n(u), R] = 0, \forall u \in R$ or $[\delta^n(v), R]u = 0, \forall u, v \in R,\ n \in \mathbb{Z}^+$. Then either $u = 0$ or $v \in Z(R)$.*

proof. Assume $u, v \in R$ and $n \in \mathbb{Z}^+$, then from the assumption

$$u[\delta^n(u), R] = 0,$$

this equivalents to

$$u\partial_{\delta^n(v)}(R) = 0, \tag{4}$$

then

$$0 = u\partial_{\delta^n(v)}(ab) = u\partial_{\delta^n(v)}(b)a + ub\partial_{\delta^n(v)}(a),\ \forall a, b \in R$$

Now from (4)

$$ub\partial_{\delta^n(v)}(a) = 0,$$

This means

$$uR[\delta^n(v), a] = 0.$$

Consequently,

$$uR\delta^k([\delta^n(v), a]) = 0.$$

What forces that $u = 0$ or $[\delta^n(v), a] = 0$. Hence $v \in Z(R)$.

Lemma 8. *Let R be a δ-prime ring and let $U \neq \{0\}$ be a right δ-ideal, which δ is a reverse derivation on R. If U is commutative, then R is commutative.*

proof. Assume that $u \in U$. Since U is commutative, then $\partial_u(U) = [u, U] = 0$. Now by lemma (5), $\partial_u(R) = 0$, then $u \in Z(R), \forall u \in U$, hence $U \subseteq Z(R)$. Thus $C(R) = 0$, this means R is commutative.

Lemma 9. *Let R be a δ-prime ring and $\delta \neq 0$ a reverse derivation on R. If $[v, \delta^n(u)v] \in Z(R), \forall u, v \neq 0 \in R, n \in \mathbb{Z}^+$, then $u \in Z(R)$.*

proof. Assume $a \in R$, then we get

$$0 = [\delta^n(u)v, a] = \delta^n(u)[v, a] + [\delta^n(u), a]v = [\delta^n(u), a]v.$$

Then, by lemma (7), $u \in Z(R)$.

Lemma 10. *Let R be a δ-prime ring of char$R \neq 2$ and let U be a δ-ideal of R, which $\delta \neq 0$ is. If*

$$[a, \delta(a)] \in Z(R)\ \forall a \in U, \tag{5}$$

then $[a, \delta(a)] = 0$.

proof. Assume that $a, b \in U$. Now by linearlizing the identity (5), we see

$$[a + b, \delta(a + b)] = [a, \delta(a)] + [a, \delta(b)] + [b, \delta(a)] + [b, \delta(b)],$$

$$= [a, \delta(b)] + [b, \delta(a)],$$

then, from (5)

$$[a, \delta(b)] + [b, \delta(a)] \in Z(R). \tag{6}$$

Now replace b by a^2 in (6), we have

$$[a, \delta(a^2)] + [a^2, \delta(a)] = [a, \delta(a)a + a\delta(a)] + [a^2, \delta(a)].$$

Then,

$$[a, \delta(a^2)] + [a^2, \delta(a)] = 4a[a, \delta(a)] \in Z(R).$$

Hence,

$$a[a, \delta(a)] \in Z(R). \tag{7}$$

Therefore,

$$[a[a, \delta(a)], \delta(a)] = 0$$

$$a[a, \delta(a)]\delta(a) - \delta(a)a[a, \delta(a)] = 0$$

$$[a, \delta(a)](a\delta(a) - \delta(a)a) = 0$$

$$[a, \delta(a)]^2 = 0.$$

From (5),

$$\delta([a, \delta(a)]) \in Z(R)$$

and

$$\delta([a, \delta(a)]) = [\delta(a), \delta(a)] + [a, \delta^2(a)] = [a, \delta^2(a)].$$

We obtain,

$$\delta([a, \delta^2(a)]) \in Z(R).$$

Now

$$\delta([a, \delta^2(a)]) = [\delta(a), \delta^2(a)] + [a, \delta^3(a)] = [a, \delta^3(a)].$$

Then,

$$[a, \delta^3(a)] \in Z(R). \tag{8}$$

Now, using the induction on a number n, so we have

$$[a, \delta^n(a)] \in Z(R). \tag{9}$$

replacing b in (6) by $a\delta^n(a)$, we have

$$[a, \delta(a\delta^n(a))] + [a\delta^n(a), \delta(a)] \in Z(R).$$

Then,

$$[a, \delta(a\delta^n(a))] + [a\delta^n(a), \delta(a)] =$$

$$[a, \delta^{n+1}(a)a + \delta^n(a)\delta(a)] + [a\delta^n(a), \delta(a)] =$$

$$[a, \delta^{n+1}(a)a] + [a, \delta^n(a)\delta(a)] - [\delta(a), a\delta^n(a)] =$$

$$[a, \delta^{n+1}(a)]a + \delta^n(a)[a, \delta(a)] + [a, \delta^n(a)]\delta(a) - a[\delta(a), \delta^n(a)] - [\delta(a), a]\delta^n(a) = S.$$

Then,

$$\begin{aligned} 0 = [S, \delta^n(a)] =& [[a, \delta^{n+1}(a)]a, \delta^n(a)]+ \\ & [\delta^n(a)[a, \delta(a)], \delta^n(a)]+ \\ & [[a, \delta^n(a)]\delta(a), \delta^n(a)]- \\ & [a[\delta(a), \delta^n(a)], \delta^n(a)]- \\ & [[\delta(a), a]\delta^n(a), \delta^n(a)]. \end{aligned} \tag{10}$$

Now substituting instead b in (6) by $a^2\delta^n(a)$, then we get

$$[a, \delta(a^2\delta^n(a))] + [a^2\delta^n(a), \delta(a)] \in Z(R).$$

Then,

$$[a, \delta(a^2\delta^n(a))] + [a^2\delta^n(a), \delta(a)] =$$

$$[a, \delta(a)a\delta^n(a))] + [a, a\delta(a)\delta^n(a)] + [a, a^2\delta^{n+1}(a)] - [\delta(a), a^2\delta^n(a)] =$$

$$4[a, \delta(a)]a\delta^n(a) + [a, \delta^n(a)]a\delta(a) + [a, \delta^n(a)]\delta(a)a + [a, \delta^{n+1}(a)]a^2 - [\delta(a), \delta^n(a)]a^2 = M.$$

Now, multiply M by $[a, \delta^n(a)]$ and in view of (10), w have

$$[a, \delta^n(a)]^2 a\delta(a) + [a, \delta^n(a)]^2\delta(a)a - [\delta(a), \delta^n(a)][a, \delta^n(a)]a^2.$$

Then, by continue the processes, we obtain

$$[a, \delta^n(a)]^3\delta(a) = 0,$$

and so

$$[a, \delta^{n+1}(a)]^4\delta(a) = 0,$$

then,

$$[a, \delta^{n+1}(a)]^4 R = 0,$$

This means

$$B = \sum_{n=1}^{\infty} \sum_{a \in U} [a, \delta^n(a)]R$$

is a sum of nilpotent ideals and so U is a nil ideal, then $B = 0$. this means $[a, \delta(a)] = 0$.

Lemma 11. *Let R be δ- prime ring of char$R \neq 2$ and let δ be a reverse derivation, such that $[a, \delta(a)] \in Z(R), \forall a \in R$. Then R is commutative.*

proof. Recall that $U = [R, R]$ is a Lie ideal of a prime ring R. Moreover,

$$\delta([R, R]) \subseteq [R, R].$$

Since every ideal in δ- prime ring is a δ- ideal, Now, if $U = [R, R]$ is commutative, then, $C(R)$ is a nil ideal (see [5], Lemma 1.7). Hence $C(R) = 0$ and R is commutative. Therefore, by using lemma (4), $U = [R, R]$ contains a δ- ideal of R. Thus, by (6), this implies $[a, \delta(a)] \in Z(R)$, $\forall a \in R$, this gives $\delta(U) \in Z(R)$, then for all $a \in U$, $([a, \delta(a)] = 0$, and based on lemma (8) R will be commutative.

Now, The proof of the next and main theorem is just a generalization of the lemmas (2) and (3) which is represented as the following:

Theorem 1. *Let R be δ- prime ring of char$R \neq 2$ and $U \neq \{0\}$ be a δ- ideal, which $\delta \neq 0$ is a reverse derivation. if $[u, \delta(u)] \in Z(R), \forall u \in U$. Then R is commutative.*

proof. From $[u, \delta(u)] \in Z(R)$, then, by lemma (6), we have

$$[u, \delta(u)] = 0.$$

Then using lemma (8) and since R is δ- prime ring, which $[u, \delta(u)] = 0$. then R is commutative.

References

[1] Ahmad Al Khalaf, Iman Taha, and Orest D Artemovych. Commutators in semiprime gamma rings. *Asian-European Journal of Mathematics*, 13(04):2050078, 2020.

[2] A Alkhalaf, O Artemovych, and I Taha. Derivations in differentially prime rings. *Journal of Algebra and Its Applications*, 17(07):1850129, 2018.

[3] A Alkhalaf, O Artemovych, I Taha, and A Aljouiiee. Derivations of differentially semiprime rings. *Asian-European Journal of Mathematics*, 12(05):1950079, 2019.

[4] O Artemovych and M Lukashenko. Lie and jordan structures of differentially semiprime rings. *Algebra and Discrete Mathematics*, 20(1), 2015.

[5] H Bell and A Klein. Combinatorial commutativity and finiteness conditions for rings. *Communications in Algebra*, 29(7):2935–2943, 2001.

[6] M Brešar. On a generalization of the notion of centralizing mappings. *Proceedings of the American Mathematical Society*, 114(3):641–649, 1992.

[7] M Brešar. Centralizing mappings and derivations in prime rings. *J. Algebra*, 156(2):385–394, 1993.

[8] Matej Brešar and Joso Vukman. On some additive mappings in rings with involution. *aequationes mathematicae*, 38(2):178–185, 1989.

[9] Mikhail A Chebotar and Pjek-Hwee Lee. Prime lie rings of derivations of commutative rings. *Communications in Algebra®*, 34(12):4339–4344, 2006.

[10] I Herstein. On the lie and jordan rings of a simple associative ring. *American Journal of Mathematics*, 77(2):279–285, 1955.

[11] I Herstein. Topics in ring theory. e university of chicago press. *Chicago, IL*, 1965.

[12] I Herstein. *Noncommutative rings*, volume 15. American Mathematical Soc., 1994.

[13] Y Hirano, A Kaya, and H Tominaga. On a theorem of mayne. *Mathematical Journal of Okayama University*, 25(2):125–132, 1983.

[14] Y Hirano, H Tominaga, and A Trzepizur. On a theorem of posner. *Mathematical Journal of Okayama University*, 27(1):19–23, 1985.

[15] M Hongan and A Trzepizur. On generalization of a theorem of posner. *Mathematical Journal of Okayama University*, 27(1):19–23, 1985.

[16] CR Jordan and DA Jordan. Lie rings of derivations of associative rings. *Journal of the London Mathematical Society*, 2(17):33–41, 1978.

[17] David Alan Jordan. Noetherian ore extensions and jacobson rings. *Journal of the London Mathematical Society*, 2(3):281–291, 1975.

[18] Ahmad Al Khalaf, Orest D Artemovych, and Iman Taha. Rings with simple lie rings of lie and jordan derivations. *Journal of Algebra and Its Applications*, 17(04):1850078, 2018.

[19] Tsiu-Kwen Lee and Kun-Shan Liu. The skolem–noether theorem for semiprime rings satisfying a strict identity. *Communications in Algebra®*, 35(6):1949–1955, 2007.

[20] Chia-Hsin Liu. Semiprime lie rings of derivations of commutative rings. *Contemporary Mathematics*, 420:259, 2006.

[21] J Mayne. Centralizing automorphisms of prime rings. *Canadian Mathematical Bulletin*, 19(1):113–115, 1976.

[22] J Mayne. Ideals and centralizing mappings in prime rings. *Proceedings of the American Mathematical Society*, 86(2):211–212, 1982.

[23] J Mayne. Centralizing mappings of prime rings. *Canadian Mathematical Bulletin*, 27(1):122–126, 1984.

[24] R Miers. Centralizing mappings of operator algebras. *Journal of Algebra*, 59(1):56–64, 1979.

[25] DS Passman. Simple lie algebras of witt type. *Journal of Algebra*, 206(2):682–692, 1998.

[26] E Posner. Derivations in prime rings. *Proceedings of the American Mathematical Society*, 8(6):1093–1100, 1957.

[27] Mohammad Samman and Nouf Alyamani. Derivations and reverse derivations in semiprime rings. In *International Mathematical Forum*, volume 2, pages 1895–1902, 2007.

[28] Iman Taha, Rohaidah Masri, Ahmad Alkhalaf, and Rawdah Tarmizi. Derivations in differentially δ-prime rings. *European Journal of Pure and Applied Mathematics*, 15(2):454–466, 2022.

[29] J Vukman. Commuting and centralizing mappings in prime rings. *Proceedings of the American Mathematical Society*, 109(1):47–52, 1990.