EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 62-70
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Results about C-κ-normality and C-mild normality

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Abstract. A topological space X is *C-κ-normal* (*C-mildly normal*) if there exist a κ-normal (mildly normal) space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. We present new results about those two topological properties and use a discrete extension space to solve open problems regarding C_2-paracompactness and α-normality.

2020 Mathematics Subject Classifications: 54D15, 54C10

Key Words and Phrases: κ-normal, normal, mildly normal, compact, C-normal, $C-\kappa$-normal, C-mildly normal, minimal Hausdorff, discrete extension, epinormal, epi-mildly normal, α-normal, C_2-paracompact

1. Preliminaries

In the present work, we give some new results about C-κ-normality and C-mild normality [2] and use the discrete extension space to answer the open problems "*Is C_2-paracompactness hereditary with respect to closed subspaces?*" [5] And "*Is α-normality preserved by the discrete extension?*" [3].

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. Two subsets A and B of a space X are called *separated* if there are two disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$. A space X is regular if for any closed subset E of X and for any element $x \in X \setminus E$ we have $\{x\}$ and E can be separated. A T_3 space is a T_1 regular space, a normal space is a space where any two disjoint closed subsets can be separated, a T_4 space is a T_1 normal space, and a Tychonoff space ($T_{3\frac{1}{2}}$) is a T_1 completely regular space. We do not

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4607

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assume T_2 in the definition of compactness, countable compactness, local compactness, and paracompactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 and the first uncountable ordinal is ω_1.

A subset A of a space X is called *a closed domain* [12], called also *regularly closed* [23], *κ-closed* [14], if $A = \overline{\text{int}A}$. On 1972, Š̌čepin introduced the notion of κ-normality [22]. A space X is *κ-normal* if X is regular and any two disjoint closed domains can be separated. About the same time, Singal defined the notion of mild normality [23]. A space X is *mildly normal* if any two disjoint closed domains can be separated.

We begin by recalling the following definitions.

Definition 1. [15] A space (X , $\mathcal{T}$) is called *epi-mildly normal* if there exists a coarser topology $\mathcal{T}'$ on X such that (X , $\mathcal{T}'$) is Hausdorff (T_2) mildly normal .

Definition 2. [4] A topological space (X , $\mathcal{T}$) is called *epi-normal* if there is a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that (X , $\mathcal{T}'$) is T_4.

In a personal contact, Arhangel'skii intoduced in 2012 to Kalantan the following definition:

Definition 3. *(Arhangel'skii) A topological space X is called C-κ-normal* if there exist a κ-normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

Definition 4. [2] A topological space X is called *C-mildly normal* if there exist a mildly normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

In [2], the following theorem was proved.

Theorem 1. *If X is C-mildly normal (C-κ-normal) Fréchet space and $f : X \longrightarrow Y$ is a witness of the C-mild normality (C-κ-normality) of X, then f is continuous.*

2. Main Results and Examples

Recall that a topological space X is called *almost compact* [19] if each open cover of X has a finite subfamily such that the closures of whose members covers X. A space X is said to be *almost regular* [23] if for any closed domain subset A and any $x \notin A$, there exist two disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. A technique which is useful in the theory of coarser topologies is the semiregularization. The topology on X generated by the family of all open domains is denoted by $\mathcal{T}_s$. The space (X, $\mathcal{T}_s$) is called the semiregularization of X. A space (X , $\mathcal{T}$) is semi-regular if $\mathcal{T}$=$\mathcal{T}_s$.

Theorem 2. *Let X be an almost regular Hausdorff space. If X is mildly normal then X is C-κ-normal.*

Proof. Since $(X, \mathcal{T})$ is an almost regular Hausdorff space, then $(X, \mathcal{T}_s)$ is a Hausdorff regular space [20]. Since X is mildly normal space we get $(X, \mathcal{T}_s)$ is mildly normal [15]. Then the identity function $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}_s)$ is a continuous bijective function. If C is any compact subspace of $(X, \mathcal{T})$, then the restriction of the identity function from C onto $id_X(C)$ is continuous and "*every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.*" [12, Theorem 3.1.13]. So X is C-κ-normal space.

Theorem 3. *If $(X, \mathcal{T})$ is almost regular almost compact space and $\mathcal{T}_s$ is T_1, then $(X, \mathcal{T})$ is C-κ-normal (C-mildly normal).*

Proof. Since $(X, \mathcal{T})$ is an almost regular space, $(X, \mathcal{T}_s)$ is regular space [20]. Hence $(X, \mathcal{T}_s)$ is T_3. Moreover, the coarser topology of an almost compact space is an almost compact space. So $\mathcal{T}_s$ is almost compact. But every almost regular almost compact space is mildly normal [23]. Thus $\mathcal{T}_s$ is regular mildly normal. Therefore by using the same argument of the proof of theorem 2 we conclude that $(X, \mathcal{T})$ is C-κ-normal (C-mildly normal).

Theorem 4. *C-κ-normality (C-mild normality) is an additive property.*

Proof. Let X_α be a C-κ-normal (C-mildly normal) space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is C-κ-normal (C-mildly normal). For each $\alpha \in \Lambda$, pick a κ-normal (mildly normal) space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that ${f_\alpha}_{|_{C_\alpha}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each compact subspace C_α of X_α. Since regularity is additive [12, Theorem 2.2.7], then $(Y_\alpha, \oplus_{\alpha\in\Lambda} \mathcal{T}'_\alpha)$ is a regular space. On the other hand, mild normality is an additive property because each factor is open-and-closed in $\oplus_{\alpha\in\Lambda} X_\alpha$ and the intersection of any closed domain in $\oplus_{\alpha\in\Lambda} X_\alpha$ with each factor X_α will be a closed domain in X_α. Then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is κ-normal (mildly normal). Consider the function sum [12, Exercises 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is compact in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)|_C$ is a homeomorphism because ${f_\alpha}_{|_{C\cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$.

Recall that a topology $\mathcal{T}$ on a non-empty set X is said to be *minimal Hausdorff* if $(X, \mathcal{T})$ is Hausdorff and there is no Hausdorff topology on X strictly coarser than $\mathcal{T}$, see [7, 8]. It was proved that "*if the product space is minimal Hausdorff, then each factor is minimal Hausdorff*" [7], In [13] the converse of the previous statement was proved. Namely, "*the product of minimal Hausdorff spaces is minimal Hausdorff*". In the next theorem we will use the following theorem: "*A minimal Hausdorff space is compact if and only if it is completely Hausdorff ($T_{2\frac{1}{2}}$)*"[21, Theorem 1.4]. We conclude the following theorems.

Theorem 5. *Let X and Y be minimal Hausdorff spaces, if X and Y are κ-normal spaces then $X \times Y$ is κ-normal.*

Proof. Since X and Y are κ-normal, they are regular Hausdorff spaces, which implies T_3, hence $T_{2\frac{1}{2}}$. Since the $T_{2\frac{1}{2}}$ is multiplicative, the product space is $T_{2\frac{1}{2}}$. So $X \times Y$ is $T_{2\frac{1}{2}}$ minimal Hausdorff space which implies that $X \times Y$ is T_2 compact, hence T_4 and thus κ-normal.

Now, we give the following characterization in the class of minimal Hausdorff spaces.

Theorem 6. *Let X be a minimal Hausdorff Fréchet space. The following are equivalent.*

(i) X is C-κ-normal.

(ii) X is locally compact.

(iii) X is compact

(iv) X is T_4.

(v) X is epinormal, hence epi-mildly normal.

Proof. (1) $\Rightarrow$ (2) Since X is C-κ-normal Fréchet space, X is $T_{2\frac{1}{2}}$ see [2], By Theorem "*A minimal Hausdorff space is compact if and only if it is completely Hausdorff* ($T_{2\frac{1}{2}}$)" [21, Theorem 1.4], gives that X is T_2 compact, hence locally compact.

(2) $\Rightarrow$ (3) Since any T_2 locally compact space is Tychonoff and hence $T_{2\frac{1}{2}}$, we obtain X is compact.

(3) $\Rightarrow$ (4) Any T_2 compact space is T_4.

(4) $\Rightarrow$ (5) Any T_4 is epinormal, hence epi-mildly normal.

(5) $\Rightarrow$ (1) Any epinormal space is C-κ-normal [2].

From the above theorem, we conclude the following corollary

Corollary 1. *In class of minimal Hausdorff, any Frėchet C-κ-normal space is κ-normal.*

Since κ-normality is not hereditary [16], it seems to us that both C-mild normality and C-κ-normality are not hereditary, but we still could not find a counterexample.

The question "*Is there a Tychonoff space which is not C-κ-normal (C-mildly normal) ?*" We answer this in the class of minimal Tychonoff spaces by using theorem "*All minimal completely regular spaces are compact*", [7], hence T_4. So we get the following corollary.

Corollary 2. *Any minimal Tychonoff space is C-κ-normal.*

We know Tychonoff spaces which are not κ-normal (mildly normal). This spaces turn out to be C-κ-normal (C-mildly normal) see [2, example 3], and also see example 5 below.

Let M be a non-empty proper subset of a topological space $(X, \mathcal{T})$. Define a new topology $\mathcal{T}_{(M)}$ on X as follows: $\mathcal{T}_{(M)} = \{U \cup K : U \in \mathcal{T} \text{ and } K \subseteq X \setminus M\}$. $(X, \mathcal{T}_{(M)})$ is called a *discrete extension* of $(X, \mathcal{T})$ and we denote it by X_M see [12, Example 5.1.22].

In general, C-κ-normality is not preserved by a discrete extension space. Here is an example of C-κ-normal space whose a discrete extension space is not C-κ-normal.

Example 1. *Consider* $(\mathbb{R}, \mathcal{I})$ *where* $\mathcal{I}$ *is the indiscrete topology. Let* $M = \mathbb{R} \setminus \{1, 2, 3\}$*. We have* $1 \notin M$ *and* M *is closed in* $\mathbb{R}_M$*. The only open set in* $\mathbb{R}_M$ *containing* M *is* $\mathbb{R}$*. But* $\mathbb{R} \cap \{1\} \neq \emptyset$*. Thus* $\mathbb{R}_M$ *is not regular. So,* $\mathbb{R}_M$ *is not* C*-*κ*-normal because it is a compact non-regular space [2].* ■

Recall that a topological space X is called C_2*-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$ [17].

From definition since any T_2 paracompact space is T_4, any C_2-paracompact space is C-κ-normal. The converse is not true, we did show in [2] that example 2 below is a C-κ-normal and it was shown in [5, example 4] it is not C_2-paracompact.

By using the discrete extension space, we answer the following open problem : "*Is* C_2*-paracompactness hereditary with respect to closed subspaces?*" [5]. The answer is negative even for open subspaces and here is a counterexample.

Example 2. *Consider the infinite Tychonoff product space* $G = D^{\omega_1} = \prod_{\alpha \in \omega_1} D$*, where* $D = \{0, 1\}$ *considered with the discrete topology. Let* H *be the subspace of* G *consisting of all points of* G *with at most countably many non-zero coordinates. Put* $M = G \times H$*. Raushan Buzyakova proved that* M *cannot be mapped onto a normal space* Z *by a bijective continuous function [9, example 4] result and the fact that* M *is a* k*-space, we conclude that* M *is a Tychonoff space which is not* C_2*-paracompact [5, example 4] . Let* X *be any compactification of* M *and consider the discrete extension space* X_M *of* X*. By Theorem "Every lower compact space is* C_2*-paracompact"* [17, theorem 2.20] , X_M is C_2-paracompact. Since M as a subspace of X_M is the same as a subspace of X and M is closed-and-open in X_M, we get that C_2-paracompactness is not hereditary with respect to both closed and open subspaces . ■

Recall that a space X is called α*-normal* if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B [6].

We answer the following open problem : "*Is* α*-normality preserved by the discrete extension?*" [3]. The answer is no and here is an example of an α-normal space whose a discrete extension space is not α-normal.

Example 3. *Let* $M = ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_1, \omega_0 \rangle\}$ *is a Tychonoff Plank space see [24, example 87] we know that* M *is a Tychonoff non* α*-normal space [6], take the compactification* X *of* M *then it is* α*-normal being* T_2 *compact space. Consider the discrete extension* X_M*. Observe that* M *is closed in* X_M*. Since* α*-normality is hereditary with respect to closed subspaces [6], we conclude that* X_M *cannot be* α*-normal.* ■

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The following example answers three kinds of invariants. We used two well-known spaces, the Alexandroff duplicate space and the closed extension space.

Let X be any T_1 topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$

in X' by x' and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of X* [11].

Example 4. *Consider the Alexandroff duplicate space $A(\mathbb{R})$ of $\mathbb{R}$ with its usual metric topology. It is C_2-paracompact [5], hence C-κ-normal. Now, let $i = \sqrt{-1} \notin \mathbb{R}$ and put $X = \mathbb{R} \cup \{i\}$. Let $\mathcal{T}$ be the closed extension topology on X generated from $\mathbb{R}$ with its usual metric topology and i. So, $\mathcal{T}= \{\emptyset\} \cup \{W \cup \{i\} : W \subseteq \mathbb{R}; W$ is open in the usual metric topology $\}$.*
$(X, \mathcal{T})$ is not C-normal see [1]. So it is not not C-κ-normal because it is Fréchet being first countable, Lindelöf space, which is not C-normal [2, theorem0.10]. Define $g : A(\mathbb{R}) \longrightarrow X$ by

$$g(x) = \begin{cases} i & ; \text{if } x \in \mathbb{R}' \\ x & ; \text{if } x \in \mathbb{R} \end{cases}$$

g is an open onto function. Thus C-κ-normality is neither invariant, open invariant, nor quotient invariant. ■

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Since κ-normality is not multiplicative, it seems to us that both C-mild normality and C-κ-normality are not multiplicative, but we still could not find a counterexample. We know the example of two linearly ordered topological spaces whose product is not κ-normal (mildly normal) was given in [14]. This space turns out to be C-κ-normal. Here is an example.

Example 5. *We will define a Hausdorff compact linearly ordered space Y such that $\omega_1 \times Y$ is C-κ-normal. Let $\{y_n : n < \omega_0\}$ be a countably infinite set such that $\{y_n : n < \omega_0\} \cap (\omega_1 + 1) = \emptyset$. Let $Y = \{y_n : n < \omega_0\} \cup (\omega_1 + 1)$. Let $\mathcal{T}$ be the topology on Y generated by the following neighborhood system: For an $\alpha \in \omega_1$, a basic open neighborhood of α is the same as in ω_1 with its usual order topology. For $n \in \omega_0$, a basic open neighborhood of y_n is $\{y_n\}$. A basic open neighborhood of ω_1 is of the form $(\alpha, \omega_1] \cup \{y_n : n \geq k\}$ where $\alpha < \omega_1$ and $k \in \omega_0$. In other words, $\{y_n : n < \omega_0\}$ is a sequence of isolated points which converges to ω_1. Note that if we define an order $<$ on Y as follows: For each $n \in \omega_0$, $\omega_1 < y_{n+1} < y_n$, and $<$ on $\omega_1 + 1$ is the same as the usual order on $\omega_1 + 1$, then $(Y, \mathcal{T})$ is a linearly ordered topological space. It was shown in [14] that $(Y, \mathcal{T})$ is a Hausdorff compact space, hence it is mildly normal. Also, it is well known that ω_1 is a Hausdorff normal space and hence mildly normal. But $\omega_1 \times Y$ is not mildly normal [14]. A similar proof as in [15] shows that $\omega_1 \times Y$ is C-κ-normal.*■

Here are cases when the product of two C-κ-normal spaces will be C-κ-normal.

Since the product of ordinals is always κ-normal (mildly normal) [18], we conclude the following theorem.

Theorem 7. *The product of ordinals is C-κ-normal (C-mildly normal).*

Theorem 8. *If X and Y are minimal Hausdorff Fréchet C-κ-normal, then $X \times Y$ is C-κ-normal.*

Proof. Since X and Y are minimal Hausdorff Fréchet C-κ-normal, X and Y are $T_{2\frac{1}{2}}$ [2]. Since the $T_{2\frac{1}{2}}$ is multiplicative, the product space is $T_{2\frac{1}{2}}$. So $X \times Y$ is $T_{2\frac{1}{2}}$ minimal Hausdorff space implies that $X \times Y$ is T_2 compact, hence C-κ-normal.

Theorem 9. *If X is Fréchet and countably compact C-κ-normal space, and Z is T_2 paracompact first countable space then $X \times Z$ is C-κ-normal.*

Proof. Let Y be a κ-normal space, $f : X \longrightarrow Y$ be a bijective function such that the restriction on any compact subspace is a homeomorphism. Now, X is Fréchet gives that f is continuous, see Theorem 1. Since X is countably compact and f continuous surjective, we have Y is countably compact κ-normal . Since a product of a countably compact κ-normal space with a paracompact first countable space is κ-normal , $Y \times Z$ is κ-normal [14]. Now, define $g : X \times Z \longrightarrow Y \times Z$ by $g(\langle x, i\rangle) = \langle f(x), i\rangle$. Then g is a bijective function and $g = f \times id_Z$, where id_Z is the identity function on Z. Let C be any compact subspace of $X \times Z$. Then $C \subseteq p_1(C) \times p_2(C)$, where p_1 and p_2 are the usual projection functions. $p_1(C)$ is a compact subspace of X and $p_2(C)$ is a compact subspace of Z, thus $p_1(C) \times p_2(C)$ is a compact subspace of $X \times Z$. Now, $f_{|_{p_1(C)}} : p_1(C) \longrightarrow f(p_1(C))$ is a homeomorphism and $id_{Z_{|_{p_2(C)}}} : p_2(C) \longrightarrow p_2(C)$ is a homeomorphism. Thus $(f \times id_Z)_{|(p_1(C)\times p_2(C))} : p_1(C) \times p_2(C) \longrightarrow fp_1(C) \times p_2(C)$ is a homeomorphism. We conclude that $g_{|_C} : C \longrightarrow g(C)$ is a homeomorphism because

$$g_{|_C} = ((f \times id_Z)_{|_{p_1(C)\times p_2(C)}})_{|_C}.$$

Recall that a space X is *Dowker* if X is T_4 and $X \times \mathbb{I}$ is not normal, where $\mathbb{I}$ is the closed unit interval considered with its usual metric topology, [12]. Dowker, in [10], stated the following theorem: "*A space X is normal and countably paracompact if and only if $X \times \mathbb{I}$ is normal*". Here is a C-κ-normal version, one direction of the Dowker's theorem. If C-κ-normality is hereditary with respect to closed spaces, then the converse will be true.

Theorem 10. *If X is T_1 Fréchet C-κ-normal Lindelöf space , then $X \times \mathbb{I}$ is C-κ-normal space.*

Proof. Let X be a T_1 Fréchet C-κ-normal Lindelöf space. Pick a witness function f and a κ-normal space Y. Then by Theorem 1 the witness function $f : X \longrightarrow Y$ is continuous and the witness space Y is T_3, [2]. Since X is Lindelöf and T_3, we get Y is paracompact, and hence T_4. So, by Dowker's theorem $Y \times \mathbb{I}$ is T_4. By a similar argument as in the proof of Theorem9, we can prove that $X \times \mathbb{I}$ is a C-κ-normal space.

Recall that a space X is called *nearly compact* [24] if each open cover of X has a finite subfamily the interiors of the closures of whose members covers X.

Theorem 11. *Let X, Y are Hausdorff nearly compact C-κ-normal space, then $X \times Y$ is a C-κ-normal space.*

Proof. Since $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are nearly compact Hausdorff spaces, we get $(X, \mathcal{T}_s)$ and $(Y, \mathcal{T}'_s)$ are Hausdorff compact spaces [20], and $(X, \mathcal{T}_s) \times (Y, \mathcal{T}'_s)$ is a T_2 compact topological space which is coarser than the topology on $X \times Y$. Thus, $X \times Y$ is epinormal and hence it is C-κ-normal [2].

The following problems are still open:

(i) Does there exist a Tychonoff space which is not C-κ-normal? Observe that such a space is not in the class of minimal Hausdorff space, not in the class of minimal T_3 spaces, not locally compact, not submetrizable, not C-normal, a space can not be ordinal, can not epinormal, not Lindelöf. Observe also that the existence of such a space, will show that C-κ-normal is not hereditary just by taking a compactification of it.

(ii) Is C-κ-normality (C-mild normality) multiplicative?

References

[1] D Abuzaid, S Al-Qarhi, and L Kalantani. Closed extension topological spaces. *European Journal of Pure and Applied Mathematics (EJPAM)*, 15(2):672–680, 2022.

[2] A Al-Awadi, L Kalantan, and S Thabit. c-κ-normal and c-mildly normal topological properties. *to appear.*

[3] A Alawadi, L Kalantan, and M Saeed. On the discrete extension spaces. *Journal of Mathematical Analysis*, 9(2):150–157, 2018.

[4] S AlZahrani and L Kalantan. Epinormality. *Journal of Nonlinear Sciences* & *Applications*, 9(9):5398–5402, 2016.

[5] H Alzumi, L Kalantan, and M M Saeed. Results on c_2-paracompactness. *European Journal of Pure and Applied Mathematics.*, 14(2):351–357, 2021.

[6] A V Arhangel'skii and L D Ludwig. On α-normal and β-normal spaces. *Commentationes Mathematicae Universitatis Carolinae.*, 42(3):507–519, 2001.

[7] M P Berri. Minimal topological spaces. *Transactions of the American Mathematical Society.*, 108(1):97–105, 1963.

[8] N Bourbaki. *General Topology.* Springer-Verlag, 1995.

[9] R Z Buzyakova. An example of a product of two normal groups that cannot be condensed onto a normal space. *Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika.*, 3:59–59, 1997.

[10] C H Dowker. On countably paracompact spaces. *Canadian Journal of Mathematics.*, 3:219–224, 1951.

[11] R Engelking. On the double circumference of alexandroff. *Bull. Acad. Pol. Sci. Ser. Astron. Math. Phys.*, 16(8):629–634, 1968.

[12] R Engelking. *General Topology.* PWN, Warszawa, 1977.

[13] S Ikenage. Product of minimal topological spaces. *Proceedings of the Japan Academy.*, 40(5):329–331, 1964.

[14] L Kalantan. Results about κ-normality. *Topology and its Applications*, 125(1):47–62, 2002.

[15] L Kalantan and I Alshammari. Epi-mild normality. *Open Mathematics.*, 16(1):1170–1175, 2018.

[16] L Kalantan and N Kemoto. Mild normality in products of ordinals. *Houston Journal of Mathematics.*, 29(4):937–947, 2003.

[17] L Kalantan, M M Saeed, and H Alzumi. c-paracompactness and c_2-paracompactness. *Turkish Journal of Mathematics.*, 43(1):9–20, 2019.

[18] L Kalantan and P Szeptycki. κ-normality and products of ordinals. *Topology and its Applications*, 123(3):537–545, 2002.

[19] P T Lambrinos. On almost compact and nearly compact spaces. *Rendiconti del Circolo Matematico di Palermo*, 24(1):14–18, 1975.

[20] M Mršević, I L Reilly, and M K Vamanamurthy. On semi-regularization topologies. *Journal of the Australian Mathematical Society*, 38(1):40–54, 1985.

[21] J R Porter and R M Stephenson. Minimal hausdorff spaces—then and now. *In Handbook of the History of General Topology*, pages 669–687, 1998.

[22] E V Shchepin. Real functions and spaces that are nearly normal. *Sibirskii Matematicheskii Zhurnal*, 13(5):1182–1196, 1972.

[23] M K Singal and A R Singal. Mildly normal spaces. *Kyungpook Mathematical Journal*, 13(1):29–31, 1973.

[24] L Steen and J A Seebach. *Counterexamples in Topology.* Dover Publications INC, USA, 1995.