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Inner Products on Discrete Morrey Spaces

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Abstract. The discrete Morrey space $m_{u,p}$ is a generalization of the p-summable sequence space ℓ^p. It is known that the discrete Morrey space is a normed space. Furthermore, for $p \neq 2$, the space $m_{u,p}$ equipped with the usual norm is not an inner product space. In this paper, we shall show that this space is actually contained in an inner product space. That means this space equipped with the inner product is an inner product space. The relationship between a standard norm on $m_{u,p}$ and the inner product is studied.

2020 Mathematics Subject Classifications: 46A45

Key Words and Phrases: Discrete Morrey Space, inner product, normed space, p-summable space

1. Introduction

The Discrete Morrey Space $(m_{u,p})$ is defined as follows. Let $m \in \mathbb{Z}$, $N \in \omega := \mathbb{N}\bigcup\{0\}$, and we write $S_{m,N}; = \{m-N, \ldots, m, \ldots, m+N\}$. Hence, $|S_{m,N}| = 2N+1$. Now, let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $1 \leq p \leq u \leq \infty$. The discrete Morrey space, denoted by $m_{u,p}$, is the set of sequences $\lambda = (\lambda_k)_{k\in\mathbb{Z}}$ taking values in K such that

$$\|\lambda\|_{m_{u,p}} := \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{1}{p}} < \infty.$$

Clearly $m_{u,p}$ is a vector space. We remark that when $u = p$, we have $m_{u,p} = \ell^p$, the space of p-summable sequences with integer indices.
We also have some notes among $m_{u,p}$ as follows.

Theorem 1. *[5] For $1 \leq p \leq u < \infty$, the space $(m_{u,p}, \|\cdot\|_{m_{u,p}})$ is a normed space. Moreover, the space is a Bannach space.*

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Theorem 2. *[5] For* $1 \leq p \leq u < \infty$*, we have* $m_{u,p} \subset \ell^p$ *and* $\|\lambda\|_{m_{u,p}} \leq \|\lambda\|_{\ell^p}$ *for every* $\lambda \in m_{u,q}$.

Many mathematicians had discussed about the discrete Morrey space (see [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [20], [21], and [22]). We have known that the space $(m_{u,p}, \|\cdot\|_{m_{u,p}})$ is a normed space. However, for $p \neq 2$, the discrete Morrey space is not an inner product space, because if we take $u = p$ then the norm $\|\cdot\|_{m_{u,p}}$ is equal to the norm $\|\cdot\|_{\ell^p}$ which do not satisfy the parallelogram's law (see [15]).
In this paper, we will show that we can define an inner product on the discrete Morrey space. So, this is the first time to say that the discrete Morrey space is an inner product space. We also discuss the relationship between a standard norm and the inner product on the space.
To define an inner product on $m_{u,p}$, the first, we will show for $p = 2$, the space $m_{u,p}$ or $m_{u,2}$ is an inner product space. Then we begin try to define an inner product on $m_{u,p}$ for $p > 2$. The next, we construct an inner product on $m_{u,p}$ for $p < 2$.
Throughout the paper, we assume that X is a real vector space, as in [16], the norm on X is a mapping $\|\cdot\| : X \to \mathbb{R}^+$ such that for all vector $x, y \in X$ and a scalar $a \in \mathbb{R}$ we have:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

(ii) $\|ax\| = |a|\|x\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

The inner product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that for all vectors $x, y, z \in X$ and a scalar $a \in \mathbb{R}$ we have:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

(ii) $\langle x, y \rangle = \langle y, x \rangle$

(iii) $\langle ax, y \rangle = a\langle x, y \rangle$

(iv) $\langle x + y, z \rangle \leq \langle x, z \rangle + \langle y, z \rangle$

Note: The space X which is equipped by a norm $\|\cdot\|$, $(X, \|\cdot\|)$, is called a normed space. The space X which is equipped by an inner product $\langle \cdot, \cdot \rangle$, $(X, \langle \cdot, \cdot \rangle)$, is called an inner product space.

2. Result

2.1. Inner Product on $m_{u,p}$ for $p = 2$

In $m_{u,2}$, the norm of $\lambda = (\lambda_k)_{k \in \mathbb{Z}} \in m_{u,2}$ is defined as

$$\|\lambda\|_{m_{u,2}} := \sup_{m \in \mathbb{Z}, N \in \omega} |S_{m,N}|^{\frac{1}{u} - \frac{1}{2}} \Big(\sum_{k \in S_{m,N}} |\lambda_k|^2 \Big)^{\frac{1}{2}}.$$

We can observe that the norm satisfies the parallelogram's law

$$\|\lambda+\mu\|_{m_{u,2}}^2+\|\lambda-\mu\|_{m_{u,2}}^2=\|\lambda\|_{m_{u,2}}^2+\|\mu\|_{m_{u,2}}^2$$

for every $\lambda=(\lambda_k)_{k\in\mathbb{Z}},\mu=(\mu_k)_{k\in\mathbb{Z}}\in m_{u,2}$. That means, the norm is formed from an inner product. So, we know that $m_{u,2}$ is an inner product space, equipped with the inner product

$$\langle\lambda,\mu\rangle_{m_{u,2}}:=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\lambda_k\mu_k\Big).$$

then $\|\lambda\|_{m_{u,2}}^2=\langle\lambda,\lambda\rangle_{m_{u,2}}$.

Proposition 1. *In the space $m_{u,2}$, the mapping $\langle\cdot,\cdot\rangle_{m_{u,2}}$ defines an inner product on $m_{u,2}$.*

Proof. We will show that the mapping $\langle\cdot,\cdot\rangle_{m_{u,2}}$ satisfy all conditions of an inner product on $m_{u,2}$.
(i) Clear, for every $\lambda=(\lambda_k)_{k\in\mathbb{Z}}\in m_{u,2}$, we get $\langle\lambda,\lambda\rangle_{m_{u,2}}=\|\lambda\|_{m_{u,2}}^2\geq 0$.
If $\lambda=0$ then $\langle\lambda,\lambda\rangle_{m_{u,2}}=0$. And, if $\langle\lambda,\lambda\rangle_{m_{u,2}}=0$ then $\|\lambda\|_{m_{u,2}}^2=0$, So, it must be $\lambda=0$.

(ii) Given $\lambda=(\lambda_k)_{k\in\mathbb{Z}},\mu=(\mu_k)_{k\in\mathbb{Z}}\in m_{u,2}$. Then we have

$$\begin{aligned}\langle\lambda,\mu\rangle_{m_{u,2}}&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\lambda_k\mu_k\Big)\\&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\mu_k\lambda_k\Big)\\&=\langle\mu,\lambda\rangle_{m_{u,2}}\end{aligned}$$

(iii) Given $a\in\mathbb{R}$ and $\lambda=(\lambda_k)_{k\in\mathbb{Z}},\mu=(\mu_k)_{k\in\mathbb{Z}}\in m_{u,2}$. We get

$$\begin{aligned}a\langle\lambda,\mu\rangle_{m_{u,2}}&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}a\lambda_k\mu_k\Big)\\&=a\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\lambda_k\mu_k\Big)\\&=a\langle\lambda,\mu\rangle_{m_{u,2}}\end{aligned}$$

(iv) Give $\lambda=(\lambda_k)_{k\in\mathbb{Z}},\gamma=(\gamma_k)_{k\in\mathbb{Z}},\mu=(\mu_k)_{k\in\mathbb{Z}}\in m_{u,2}$. Thus

$$\begin{aligned}\langle\lambda+\gamma,\mu\rangle_{m_{u,2}}&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}(\lambda_k+\gamma_k)\mu_k\Big)\\&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\lambda_k\mu_k+\sum_{k\in S_{m,N}}\gamma_k\mu_k\Big)\\&=\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\lambda_k\mu_k\Big)\end{aligned}$$

$$
\begin{aligned}
&+\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}}\gamma_k\mu_k\Big)\\
=\;&\langle\lambda,\mu\rangle_{m_{u,2}}+\langle\gamma,\mu\rangle_{m_{u,2}}
\end{aligned}
$$

■

Remark 1. *The space $(m_{u,2},\|\cdot\|_{m_{u,2}})$ is an inner product space. Moreover, by Theorem 1, the space is a complete. Accordingly, $\big(m_{u,2},\langle\cdot,\cdot\rangle_{m_{u,2}}\big)$ is a Hilbert space.*

For $p\neq 2$, the space $m_{u,p}$ is not an inner product space. Because the norm is not satisfy the parallelogram's law. So, in the next section, we will try to define an inner product on $m_{u,p}$ for $p\neq 2$.

2.2. Inner Product on $m_{u,p}$ for $2<p<\infty$

In this section, we let $2<p\le u<\infty$, unless otherwise stated. We will define an inner product on the space $m_{u,p}$. First, before we define an inner product on the space $m_{u,p}$, we observe that $m_{u,p}\subset m_{u,2}$ (as set). Indeed, if $\lambda=(\lambda_k)_{k\in\mathbb{Z}}$ is a sequence in $m_{u,p}$, then by Holder's inequality, we have

$$
\begin{aligned}
\sum_{k\in S_{m,N}}|\lambda_k|^2 &\le \Big(\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{2}{p}}\Big(\sum_{k\in S_{m,N}}1\Big)^{1-\frac{2}{p}}\\
&= |S_{m,N}|^{1-\frac{2}{p}}\Big(\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{2}{p}}\\
&= |S_{m,N}|\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{2}{p}}.
\end{aligned}
$$

Thus

$$
\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}|\lambda_k|^2\le\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{2}{p}}.
$$

So, taking the square roots of both sides, we get

$$
\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}|\lambda_k|^2\Big)^{\frac{1}{2}}\le\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{1}{p}}
$$

or

$$
|S_{m,N}|^{-\frac{1}{2}}\Big(\sum_{k\in S_{m,N}}|\lambda_k|^2\Big)^{\frac{1}{2}}\le|S_{m,N}|^{-\frac{1}{p}}\Big(\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{1}{p}}.
$$

Then

$$
\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{\frac{1}{u}-\frac{1}{2}}\Big(\sum_{k\in S_{m,N}}|\lambda_k|^2\Big)^{\frac{1}{2}}\le\sup_{m\in\mathbb{Z},N\in\omega}|S_{m,N}|^{\frac{1}{u}-\frac{1}{p}}\Big(\sum_{k\in S_{m,N}}|\lambda_k|^p\Big)^{\frac{1}{p}}.
$$

So, we have

$$\|\lambda\|_{m_{u,2}} \leq \|\lambda\|_{m_{u,p}}$$

which means that λ is in $m_{u,2}$.
Thus, we realize that $m_{u,p}$ can actually be considered as a subspace of $m_{u,2}$, equipped with the inner product

$$\langle\lambda,\mu\rangle_{m_{u,2}} := \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{u}-\frac{1}{2}\right)}\Big(\sum_{k\in S_{m,N}} \lambda_k\mu_k\Big)$$

and the norm

$$\|\lambda\|_{m_{u,2}} := \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{\frac{1}{u}-\frac{1}{2}}\Big(\sum_{k\in S_{m,N}} |\lambda_k|^2\Big)^{\frac{1}{2}}$$

for every $\lambda,\mu \in m_{u,p}$.
A more general result is formulated in the following proposition.

Proposition 2. *For $1 \leq q \leq p \leq u < \infty$ then $m_{u,p} \subset m_{u,q}$ and $\|\lambda\|_{m_{u,q}} \leq \|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.*

Proof. Let $1 \leq q \leq p \leq u < \infty$. For every $\lambda = (\lambda_k)_{k\in\mathbb{Z}} \in m_{u,p}$, we have

$$\begin{aligned}\sum_{k\in S_{m,N}} |\lambda_k|^q &\leq \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)^{\frac{q}{p}}\Big(\sum_{k\in S_{m,N}} 1\Big)^{1-\frac{q}{p}} \\ &= |S_{m,N}|^{1-\frac{q}{p}}\Big(\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)^{\frac{q}{p}} \\ &= \big|S_{m,N}\big|\Big(\frac{1}{\big|S_{m,N}\big|}\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)^{\frac{q}{p}}.\end{aligned}$$

Taking the q-roots of both sides, multiplying by $|S_{m,N}|^{\frac{1}{u}}$ of both sides, and taking supremum of both sides we get $\|\lambda\|_{m_{u,q}} \leq \|\lambda\|_{m_{u,p}}$, which tell us $m_{u,p} \subset m_{u,q}$.
∎

Corollary 1. *For $2 \leq p \leq u < \infty$ then $m_{u,p} \subset m_{u,2}$ and $\|\lambda\|_{m_{u,2}} \leq \|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.*

Proof. Applying Proposition 2, take $q = 2$, then the corollary is proved.

Remark 2. *For $2 < p \leq u < \infty$, the space $\big(m_{u,p}, \|\cdot\|_{m_{u,2}}\big)$ is an inner product space.*

For $2 < p < \infty$, we finish to define an inner product on $m_{u,p}$, in the next section, we will try to define an inner product on $m_{u,p}$ for $1 \leq p < 2$.

2.3. Inner Product on $m_{u,p}$ for $1 \leq p < 2$

In this section, we let $1 \leq p < 2$. First for all, we begin to discuss the following proposition.

Proposition 3. *For $1 \leq p \leq u < 2$ and $v = \frac{2up}{2p-2u+up}$, then $m_{u,p} \subset m_{v,2}$ and $\|\lambda\|_{m_{v,2}} \leq \|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.*

Proof. Let $\lambda = (\lambda_k)_{k\in\mathbb{Z}} \in m_{u,p}$. Then

$$\begin{aligned}
\|\lambda\|^2_{m_{v,2}} &= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{v}-\frac{1}{2}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^2 \Big) \\
&= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{v}-\frac{1}{2}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^{2-p} |\lambda_k|^p \Big) \\
&\leq \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{v}-\frac{1}{2}\right)} \Big(\sup_{k\in S_{m,N}} |\lambda_k|^{2-p} \sum_{k\in S_{m,N}} |\lambda_k|^p \Big) \\
&\leq \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{v}-\frac{1}{2}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{2-p}{p}} \big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big) \\
&= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{2\left(\frac{1}{v}-\frac{1}{2}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{2}{p}}
\end{aligned}$$

Take the square roots of both sides, we get

$$\|\lambda\|_{m_{v,2}} \leq \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{\frac{1}{v}-\frac{1}{2}} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{1}{p}}$$

Because $v = \frac{2up}{2p-2u+up}$ then we find

$$\begin{aligned}
\frac{1}{v} - \frac{1}{2} &= \frac{1}{\left(\frac{2up}{2p-2u+up}\right)} - \frac{1}{2} \\
&= \frac{2p-2u+up}{2up} - \frac{1}{2} \\
&= \frac{2p-2u+up}{2up} - \frac{up}{2up} \\
&= \frac{2p-2u}{2up} \\
&= \frac{p-u}{up} \\
&= \frac{1}{u} - \frac{1}{p}
\end{aligned}$$

Because $1 \le p \le u < 2$, then $2 \le v$ and $0 < \frac{1}{v} \le \frac{1}{2}$. And also, we get

$$\|\lambda\|_{m_{v,2}} \le \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)^{\frac{1}{p}} = \|\lambda\|_{m_{u,p}}$$

Consequently, we have

$$\|\lambda\|_{m_{v,2}} \le \|\lambda\|_{m_{u,p}}$$

Since $\lambda \in m_{u,p}$ is arbitrary, then from the inequality, every element of $m_{u,p}$ is also in $m_{v,2}$. It means $m_{u,p} \subset m_{v,2}$.

■

Remark 3. *For $1 \le p \le u < 2$ and $v = \frac{2up}{2p-2u+up}$, the space $\big(m_{u,p}, \|\cdot\|_{m_{v,2}}\big)$ is an inner product space.*

A more general result of Proposition 3 is formulated in the following theorem. But, before discussing the theorem, we need to introduce the following proposition.

Proposition 4. *If $1 \le p \le u \le q < \infty$ such that $\frac{1}{p} - \frac{1}{u} < \frac{1}{q}$, then there is $v > 0$ such that $q \le v$ and $\frac{1}{v} - \frac{1}{q} = \frac{1}{u} - \frac{1}{p}$. That is, $v = \frac{upq}{pq-uq+up}$.*

Proof. Let $1 \le p \le u \le q < \infty$ and $\left|\frac{1}{u} - \frac{1}{p}\right| < \frac{1}{q}$. Take $v = \frac{upq}{pq-uq+up}$. Then

$$\frac{1}{v} - \frac{1}{q} = \frac{1}{\left(\frac{upq}{pq-uq+up}\right)} - \frac{1}{q} = \frac{pq-uq+up}{upq} - \frac{1}{q} = \frac{pq-uq}{upq} = \frac{1}{u} - \frac{1}{p}.$$

Because $p \le u$ then $\frac{1}{u} \le \frac{1}{p}$. Hence, $\frac{1}{v} \le \frac{1}{q}$ and $q \le v$. Since $\frac{1}{p} - \frac{1}{u} < \frac{1}{q}$ then $\frac{1}{q} - \frac{1}{v} < \frac{1}{q}$. So, $\frac{1}{v} > 0$ and $v \le 0$.

■

Theorem 3. *For $1 \le p \le u \le q < \infty$ and $v = \frac{upq}{pq-uq+up}$, then $m_{u,p} \subset m_{v,q}$ and $\|\lambda\|_{m_{v,q}} \le \|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.*

Proof. Let $\lambda = (\lambda_k)_{k\in\mathbb{Z}} \in m_{u,p}$. Then

$$\begin{aligned}
\|\lambda\|^q_{m_{v,q}} &= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{q\left(\frac{1}{v}-\frac{1}{q}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^q\Big) \\
&= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{q\left(\frac{1}{v}-\frac{1}{q}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^{q-p}|\lambda_k|^p\Big) \\
&\le \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{q\left(\frac{1}{v}-\frac{1}{q}\right)} \Big(\sup_{k\in S_{m,N}} |\lambda_k|^{q-p} \sum_{k\in S_{m,N}} |\lambda_k|^p\Big) \\
&\le \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{q\left(\frac{1}{v}-\frac{1}{q}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)^{\frac{q-p}{p}} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p\Big)
\end{aligned}$$

$$= \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{q\left(\frac{1}{v}-\frac{1}{q}\right)} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{q}{p}}$$

Take the q-roots of both sides and apply Proposition 5, we get

$$\|\lambda\|_{m_{v,q}} \le \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{\frac{1}{v}-\frac{1}{q}} \Big(\sum_{k\in S_{m,N}} |\lambda_k|^p \Big)^{\frac{1}{p}} = \|\lambda\|_{m_{u,p}}$$

Since $\lambda \in m_{u,p}$ is arbitrary, then from the inequality, we get $m_{u,p} \subset m_{v,q}$.

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3. Furthermore Results

Now, we can define two norms in $m_{u,p}$, the usual norm $\|\cdot\|_{m_{u,p}}$ and another new norm ($\|\cdot\|_{m_{u,2}}$ if $2 < p \le u < \infty$, or $\|\cdot\|_{m_{v,2}}$ if $1 \le p \le u < 2$ with $v = \frac{2up}{2p-2u+up}$). One might ask whether $\|\cdot\|_{m_{v,2}}$ is equivalent to $\|\cdot\|_{m_{u,p}}$. The answer is negative. We already have $\|\lambda\|_{m_{v,2}} \le \|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$. The following proposition say that we cannot control $\|\lambda\|_{m_{v,2}}$ and $\|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$ such that the norm $\|\cdot\|_{m_{v,2}}$ is not equivalent to the norm $\|\cdot\|_{m_{u,p}}$.

Proposition 5. *Let $1 \le p \le u < 2$ and $v = \frac{2up}{2p-2u+up}$. There is no constant $C > 0$ such that $\|\lambda\|_{m_{v,2}} \ge C\|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.*

Proof. For each $n \in \mathbb{N}$, take $u = p$ and $\lambda_{(n)} = \left(\frac{1}{k^{\frac{1}{p}+\frac{1}{n}}}\right)$ with

$$\lambda_{k(n)} = \begin{cases} \frac{1}{k^{\frac{1}{p}+\frac{1}{n}}}, & \text{if } k \in \mathbb{N} \\ 0, & \text{if } k \notin \mathbb{N} \end{cases}.$$

Then $v = 2$ and we have

$$\|\lambda_{(n)}\|_{m_{v,2}}^2 = \|\lambda_{(n)}\|_{\ell^2}^2 = \sum_{k\in\mathbb{N}} \frac{1}{k^{\frac{2}{p}+\frac{2}{n}}} \le \sum_{k\in\mathbb{N}} \frac{1}{k^{\frac{1}{p}}} < \infty$$

while

$$\|\lambda_{(n)}\|_{m_{u,p}}^p = \|\lambda_{(n)}\|_{\ell^p}^p = \sum_{k\in\mathbb{N}} \frac{1}{k^{1+\frac{p}{n}}} < \infty$$

We can see that $\|\lambda_{(n)}\|_{m_{v,2}}$ is bounded by a fix number independent of n, while $\|\lambda_{(n)}\|_{m_{u,p}}$ is dependent on n and tends to ∞ as $n \to \infty$. Hence

$$\frac{\|\lambda_{(n)}\|_{m_{v,2}}}{\|\lambda_{(n)}\|_{m_{u,p}}} \to 0$$

as $n \to \infty$. So, there is no constant $C > 0$ such that $\|\lambda\|_{m_{v,2}} \ge C\|\lambda\|_{m_{u,p}}$ for every $\lambda \in m_{u,p}$.

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Proposition 6. *Let $2 < p \le u < \infty$. There is no constant $C > 0$ such that $\|\lambda\|_{m_{u,p}} \le C\|\lambda\|_{m_{u,2}}$ for every $\lambda \in m_{u,p}$.*

Proof. Let $2 < p \le u < \infty$. Take $u = p$. Suppose that a constant exists. Then, for $\lambda_{(n)} = (\ldots, 0, 0, 1, 0, 0, \ldots, 0, 1, 0, \ldots)$, where the first 1 is at $k = 1$ and the second 1 is the $(n+1)^{\text{th}}$-term, we have

$$2^{\frac{1}{p}} \le C \frac{1}{(2n+1)}^{\left(\frac{1}{2}-\frac{1}{p}\right)} \sqrt{2}.$$

But this cannot be true, since $\frac{1}{(2n+1)}^{\left(\frac{1}{2}-\frac{1}{p}\right)} \to 0$ as $n \to \infty$.

■

Remark 4. *Proposition 5 and 6 say that in the discrete Morrey space, the usual norm and the second norm are not equivalent.*

Let $1 \le p \le u < 2$ and $v = \frac{2up}{2p-2u+up}$. As we have seen in the previous section, every sequence $\lambda \in m_{u,p}$ has $\|\lambda\|_{v,2} < \infty$. This suggests that $m_{u,p} \subset m_{v,2}$. We shall now discuss some properties of this space. First, we have the following proposition, which describes the relationship between $m_{u,p}$ and $m_{v,2}$.

Proposition 7. *As a set, we have $m_{u,p} \subset m_{v,2}$ and the inclusion is strict.*

Proof. Let $1 \le p \le u < 2$, $v = \frac{2up}{2p-2u+up}$, and $\lambda \in m_{v,2}$. It follows from Corollary 3 that

$$\|\lambda\|_{m_{v,2}} \le \|\lambda\|_{m_{u,p}}$$

which means that $\lambda \in m_{v,2}$.
To show that the inclusion is strict, we need to find $\lambda = (\lambda_k)_{k\in\mathbb{Z}}$ such that $\|\lambda\|_{m_{v,2}} < \infty$ but $\|\lambda\|_{m_{u,p}} = \infty$. Choose $u = p$ and $\lambda = (\lambda_k)_{k\in\mathbb{Z}}$ with

$$\lambda_k = \begin{cases} \left(\frac{1}{k}\right)^{\frac{1}{p}}, & \text{if } k \neq 0 \\ 0, & \text{if } k = 0 \end{cases}.$$

Hence

$$\|\lambda\|_{m_{v,2}} = \|\lambda\|_{m_{2,2}} = \Big(\sum_{k\in S_{m,N}} \big|\big(\frac{1}{k}\big)^{\frac{1}{p}}\big|^2\Big)^{\frac{1}{2}} = \Big(2\sum_{k\in\mathbb{N}} \big(\frac{1}{k}\big)^{\frac{2}{p}}\Big)^{\frac{1}{2}} < \infty$$

while

$$\|\lambda\|_{m_{u,p}} = \|\lambda\|_{m_{p,p}} = \Big(\sum_{k\in S_{m,N}} \big|\big(\frac{1}{k}\big)^{\frac{1}{p}}\big|^p\Big)^{\frac{1}{p}} = \Big(2\sum_{k\in\mathbb{N}} \big(\frac{1}{k}\big)\Big)^{\frac{1}{p}} = \infty.$$

This means that λ is in $m_{v,2}$ but not in $m_{u,p}$.

■

Proposition 8. *The space $\left(m_{v,2}, \|\cdot\|_{m_{v,2}}\right)$ is complete. Accordingly, $\left(m_{v,2}, \langle\cdot,\cdot\rangle_{m_{v,2}}\right)$ is a Hilbert space.*

Proof. Based on Proposition 1 and Theorem 1, the space $m_{v,2}$ is a complete. So, $\left(m_{v,2}, \langle\cdot,\cdot\rangle_{m_{v,2}}\right)$ is a Hilbert space.

■

4. Concluding Remarks

We have shown the space $m_{u,p}$ can be equipped with an inner product and its induced norm. So, we can define two norms in $m_{u,p}$, the standard norm $\|\cdot\|_{m_{u,p}}$ and another new norm ($\|\cdot\|_{m_{u,2}}$ if $2 < p \leq u < \infty$, or $\|\cdot\|_{m_{v,2}}$ if $1 \leq p \leq u < 2$ with $v = \frac{2up}{2p-2u+up}$). But, we have to know that the two norms are not equivalent. Using the inner product, one may define angles on $m_{u,p}$, define orthogonality on $m_{u,p}$, carry out the Gram-Schmidt process to get an orthogonal set (see [3]), define the volume of an n-dimensional parallelepiped on $m_{u,p}$, define an n-inner product on $m_{u,p}$, discuss the concept of convergence for n-dimensional subspace of $m_{u,p}$ (see [19], [18], and [17]), and so on.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication

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